The obtained equation is applicable to the presently known electric monopole transitions between a 7.68 mev state and the ground state in C^{12} and between a 6.06 mev state and the ground state in O^{16} [2]. In both cases the transition is in the main accompanied by the emission of electron-positron pairs. The Born approximation used in the calculation should be correct, because C^{12} and O^{16} are light nuclei, and the transition energy is considerable. The relative probability of bremsstrahlung will be

$$dW^{(0)}_{\pi,\gamma} = \frac{e^2 \cdot 10^{-3}}{(2\pi)^2 I(E)} F(E;\ W,\ \omega)\, dW\, d\omega, \qquad (3)$$

$$I(E) = \int_0^{E-2} [E - 2 + (E-2)x - x^2] \sqrt{(E-1-x)^2 - 1}\, \sqrt{(x+1)^2 - 1}\, dx$$

($e^4 Q_0^2 I(E) / 9(2\pi)^3$ is the known[3] expression for the total probability of the formation of an electron-positron pair in an electric monopole transition).

Numerical integration of Eq. (3) permits one to obtain the energy spectrum of the γ rays and a total relative probability of the process. For O^{16} these are

$$N_1(\omega) = 2.64 \cdot 10^{-8} \int_0^{9.86-\omega} F(E = 11.86;\ W,\ \omega)\, dW,\ N_1 = \int_\varepsilon^{9.86} N_1(\omega)\, d\omega = 3.30 \cdot 10^{-3}$$

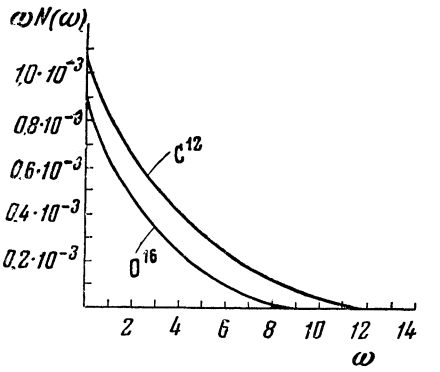
and for C^{12}

$$N_2(\omega) = 0.79 \cdot 10^{-8} \int_0^{13.03-\omega} F(E = 15.03;\ W;\ \omega)\, dW;$$

$$N_2 = \int_\varepsilon^{13.03} N_2(\omega)\, d\omega = 3.96 \cdot 10^{-3},$$

where ϵ is a small constant.

The graphs show functions $\omega N_1(\omega)$ and $\omega N_2(\omega)$ (in units of μc^2). The values N_1 and N_2 present the ratio of the number of γ quanta emitted simultaneously with electron-positron pairs to the number of pairs not accompanied by bremsstrahlung in unit time. They are equal in order of magnitude (10^{-3}) to the integral relative probability of bremmstrahlung in an ordinary conversion for an electric monopole nuclear transition. We also evaluated the contribution to the continuous spectrum of γ rays by a process in which the energy of the $0^+ \to 0^+$ transition goes into the formation of a conversion electron and a γ quantum, with the γ quantum being emitted by the nucleus itself. As might have been expected, the role played by this process is insignificantly small, especially in the region of small γ ray energies. Its total contribution is about 10^4 times less than those made by the process examined here and the one dealt with in the investigation conducted by I. S. Shapiro and this author[1].



In conclusion I express my sincere gratitude to I. S. Shapiro for the interest he has shown in this investigation and to V. V. Turevtsev, who called my attention to this effect.

* In this paper a factor 1/2 must be introduced in the equation for the matrix element and correspondingly, a factor 1/4 in the probability equations. Then the order of magnitude of the integral relative probability must be 10^{-3}.

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The Numerical Calculation of Path Integrals

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IT is known that we can write down the answers to many physical problems in the form of so-called paths integrals – integrals in function

spaces. However, the actual calculation of a path integral produces a number of difficulties; only rarely can the analytic contributions be found numerically. Therefore the opinion that these are useless in practice is given repeatedly.

However, we should expect that as soon as methods of numerical path integration are worked out, the solutions of problems by such means will provide practical interest. In the present work, the following course of calculations is proposed. The path integral, described as an integral over a certain measure, is approximated by a finite measure Stieltjes integral of sufficiently high multiplicity, which is then calculated in ordinary fashion. This course of action is shown to be entirely acceptable, since the calculation of multiple integrals with the aid of modern computing machine techniques does not present any great difficulties.

By way of an example of such a calculation, we determine in the present note thevalue E_0 of the lowest energy level of the polaron – a slow electron in an ionic crystal. The theory of polarons has been treated by Pekar[1], who, in particular, calculated the value of E_0 in the adiabatic approximation. Feynman[2] has shown that the determination of E_0 can be put in the form of the calculation of the asymptotic form of the path integral

$$K = \int \exp S\, Dx(t), \tag{1}$$

where the functional S depends on the coupling parameter α:

$$S = -\frac{1}{2}\int_0^T \left(\frac{dx}{dt}\right)^2 dt + \frac{\alpha}{2\sqrt{2}} \int_0^T\int_0^T |x(t) - x(s)|^{-1} e^{-|t-s|}\, dt\, ds, \tag{2}$$

$$\alpha = {}^1\!/_2\,(1/\varepsilon_\infty - 1/\varepsilon)\,(e^2/\hbar\omega)\,(2m\omega/\hbar)^{1/2}.$$

here ϵ and ϵ_∞ are the static and high frequency values of the dielectric constant. Then for $T \to \infty$, $K(T) \sim \exp\{-E_0 T\}$. For the numerical determination of the value of E_0 as a function of α, Feynman made use of variational methods. The approximated expansion of E_0 as a function of α that he obtained are plotted in Ref. 2. For large α, the adiabatic approximation of Pekar[1] is very satisfactory.

For our purposes, Eq. (1) has been transformed. The exponential factor

$$\exp\left\{-{}^1\!/_2\int_0^T (dx/dt)^2\, dt\right\}$$

is put under the differential sign, and then the integral is described in the form of the mean value of a certain functional in Wiener measure:

$$K(T) = \int \exp\left\{\frac{\alpha}{2\sqrt{2}} \times \int_0^T\int_0^T |x(t) - x(s)|^{-1} e^{-|t-s|}\, dt\, ds\right\} d_W x(t) \tag{3}$$

$$= \mathfrak{M}_W\left[\exp\left\{\frac{\alpha}{2\sqrt{2}} \times \int_0^T\int_0^T |x(t) - x(s)|^{-1} e^{-|t-s|}\, dt\, ds\right\}\right],$$

$$E_0(\alpha) = -\lim_{T\to\infty} \frac{1}{T} \ln K(T) \tag{4}$$

It is not difficult to see that the limit

$$-\lim_{T\to\infty} \frac{1}{T}\, \mathfrak{M}_W[\exp\{\alpha I\}], \tag{5}$$

where

$$I = \frac{1}{\sqrt{2}} \int_0^T d\eta \int_0^\Xi \sqrt{\xi}\, e^{-\xi} |x(t) - x(s)|^{-1} d\sqrt{\xi}, \tag{6}$$

($\xi = t - s$, $\eta = t + s$), , differs from $E_0(\alpha)$ by a sufficiently small amount provided that Ξ is large enough. At the same time the order of the remaining term in (5) is smaller than in (4). Since $\sqrt{\xi}\, e^{-\xi}$ falls off rapidly with increase in ξ, it suffices in approximate calculations to carry out the integration over ξ up to $\Xi \sim 5$-7. The mean value of the functional $e^{\alpha I}$ was calculated by the Monte Carlo method. Two variants were considered, with $T = 4.5$ and $T = 9.0$. In both variants, $\Xi = 5.0$, which gives an error in integral I of about 0.3% in the lower direction. With a step of $\tau = 0.05$ from $t = 0$ to $t = T + \Xi$, we determined the random positions of the three dimensional Brownian trajectory. The region of integration was divided into rectangles, the double integral over each rectangle being calculated according to a simple difference scheme, in which only the value of the function under the integral at the center of the rectangle enters. Then, for the set of values of α, $e^{\alpha I}$ was computed, as was its arithmetic mean over all N trajectories. Thus we calculated (by the Monte

T	4.5	4.5	9.0	9.0
N	300	600	300	400
α	$-E_0(\alpha)$			
0.5	0.5051	0.5048	0.5047	0.5043
1.0	1.0208	1.0195	1.0195	1.0192
2.0	2.0880	2.0819	2.0839	2.0851
4.0	4.3900	4.3587	4.3469	4.3538
5.0	5.6271	5.5787	5.5205	5.5305

Carlo approximation) 190- and 280-fold integrals in place of the path integral. The results, after the production of statistical corrections, are given in Tables 1 and 2.

As a control, we computed the mean value of random quantities and their chief moments, according to which we found the coefficients of expansion of the function $E_0(\alpha)$ in the series $E_0(\alpha) = (\varkappa_1/1!)\alpha + (\varkappa_2/2!)\alpha^2 + \ldots$

T	4.5	4.5	9.0	9.0	exact value according to Ref. 3
N	300	600	300	400	
$-\varkappa_1$	1.0063	0.9999	0.9912	0.9940	1.0000
$-\varkappa_2$	0.0397	0.0370	0.0354	0.0370	0.0252

Both A. S. Frolov and V. Iu. Krylov took part in this research. V. N. Toroptseva and T. I. Frolova programmed the problem. The authors express their gratitude to M. R. Shura-Bura for valued advice on the problems of programming the problem.

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On Quantities with Anomalous Parity and on a Possible Explanation of Parity Degeneracy of K-Mesons

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IT is well known that K-particles may decay into two or three π-mesons. The corresponding functions must have different parity with respect to the transformation of spatial reflection (i.e., the transformation $x_i \to -x_i$, $i = 1, 2, 3$, $t \to t$), in particular: the function corresponding to the θ-decay must be even, while the function corresponding to the τ-decay must be odd. At the same time it has been established that within experimental error the rest-masses of the θ- and τ-mesons coincide. This coincidence of masses is referred to as the parity-degeneracy of K-mesons. Since the situation in connection with this problem is not at all clear, it seems to be useful to investigate the behavior of quantities with respect to reflections (we use here the word quantity to denote a representation of the complete Lorentz group including reflections of both the space and the time coordinates). Such an investigation has a certain intrinsic interest of its own. It turns out that in addition to the well known possible parities with respect to reflections of the space or the time coordinates there exists also one additional possibility which we shall here refer to as anomalous parity.

In quantum mechanics wave functions are determined up to a factor. It is therefore natural to define transformations of quantities with respect to some given group also up to a factor. Well known examples of the appearance of such factors are provided by spinors which change sign as a result of a rotation by 360°, or wave functions of a system of particles obeying Fermi statistics which change sign as a result of an interchange of particles. The corresponding mathematical concepts are well known – they are the so-called projective representations of the group. We say that a projective representation of the group G is given if to each element $g \in G$ there corresponds a linear transformation T_g such that $T_{g_1 g_2} = \alpha(g_1, g_2) T_{g_1} T_{g_2}$, where $\alpha(g_1, g_2)$ is a scalar. It may be shown that the quantities $\alpha(g_1, g_2)$ satisfy the following functional equation:

$$\alpha(g_1 g_2, g_3)\,\alpha(g_1, g_2) = \alpha(g_1, g_2 g_3)\,\alpha(g_2, g_3).$$