During the transition through resonance, a^2 contains a constant term, an oscillating term and a slowly increasing term. The constant term changes only the magnitude ϵ_0, and the oscillating term has no effect. As far as the slowly increasing term is concerned, when the inequality $\alpha(\nu - k/M) < 0$ is satisfied, the increase of a^2 lead to the situation where the ratio $(\nu - k/M + \alpha a^2)/\Omega$ remains constant as Ω decreases. This will lead to particle loss.

For this not to happen, it is obviously sufficient that the following inequality be satisfied.

$$n d\Omega/d\theta \gg \alpha(da^2/d\theta)_{max},$$

where $(da^2/d\theta)_{max}$ has to be taken in a region of monotonic increase of a, according to (27). In this region,

$$\frac{dC}{d\theta} \approx \frac{dS}{d\theta} = \frac{dS}{du}\frac{du}{d\theta} \approx 0.7\,\frac{1}{2\pi}\left(\frac{\epsilon_0\,dT/dN}{MT}\right)^{1/2}.$$

Therefore

$$\alpha\left(\frac{da^2}{d\theta}\right)_{max} \approx 3.6\pi\alpha h^2 J_n^2\left(n\,\frac{\epsilon_1}{\epsilon_2}\right)\left(\frac{MT}{\epsilon_0\,dT/dN}\right)^{1/2}.$$

And we finally get the safety factor condition

$$150\alpha h^2 J_n^2\left(n\,\frac{\epsilon_1}{\epsilon_0}\right)\left(\frac{MT}{\epsilon_0\,dT/dN}\right)^{3/2} \ll 1 \qquad (37)$$

or

$$\frac{\alpha\,(\Delta r)^2_{max}}{|\varphi|^2_{max}}\left(\frac{MT}{\epsilon_0\,dT/dN}\right)^{1/2} \ll 1. \qquad (38)$$

This condition is not difficult to satisfy. It is automatically satisfied for the usual specifications on $\partial^3 H/\partial r^3$ and $(\Delta r)_{max}$

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1 Hammer, Pidd and Terwilliger, Rev. Sci. Instr. 26, 555 (1955).

On the Construction of the Scattering Matrix. II.
The Theory with Non–Local Interaction

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(Submitted to JETP editor July 28, 1955)
J. Exptl. Theoret. Phys. (U.S.S.R.) 32, 87–98 (January, 1957)

N. N. Bogoliubov's method for constructing the scattering matrix is generalized to the case of a theory with non–local interaction. For such a theory, a scattering matrix is constructed which satisfies the physically necessary requirements.

1. INTRODUCTION

ATTEMPTS, having their origin in the "struggle with divergences", to avoid the use of point interactions in the quantum theory of fields and to replace it by an extended interaction, are as old as quantum electrodynamics itself.[1-3] However, elaborate investigations of such theories,[4-5] undertaken within the framework of the description of a many–electron system by means of the many–time formalism or the Tomonoga–Schwinger equation in the interaction representation, have shown that the introduction of a form factor violates the conditions for solvability of the equations of motion, since the Hamiltonians at points with space–like separation no longer commute. Consequently, the non–local theory is incompatible with the Hamiltonian method. The physical reason for this is that the introduction of a form factor actually results in propagation of signals (at least, in the small) with super–light velocity. Thus the requirement that there exist a wave function describing the state of the system at a definite time loses its meaning.

In the hope of avoiding the difficulties of the Hamiltonian method, attempts have been made to go directly to the Euler–Lagrange integro–differential equations which follow from the variational principle with non–local interaction.[6] In quantum theory this procedure leads to the considera-

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tion of the equation of motion of the field operators in the Heisenberg representation. More recently, there has been undertaken a further development of one of the variants of such a theory[7-8], which is characterized by the action function for the interaction:

$$S_{int} = \int dx'\, dx''\, dx''' \, \Lambda \,(x', x'', x''') \qquad (1)$$

with the "Lagrangian"

$$\Lambda \,(\xi) = \Lambda \,(x', x'', x''') \qquad (2)$$

$$= gF \,(x', x'', x''')\bar{\psi} \,(x')\, u \,(x'')\, \psi \,(x'''),$$

where $\bar{\psi}_{/}$, ψ and u are the Heisenberg operators of Dirac and scalar fields, $F \,(x', x'', x''')$ is the form factor, and ξ denotes the triplet of points x'. x'' and x''' (in the sequel $d\xi$ will denote the product $dx'\, dx''\, dx'''$).

However, more detailed investigation[9-11] has shown that Bloch's proof[8] of the unitary of the S—matrix is incorrect, and that with this method for constructing a non—local theory one obtains (except, possibly, for a very restricted class of special Lagrangians) a non—unitary S—matrix, which is physically inadmissible. One may surmise that the difficulties with unitarity which occur in the Heisenberg representation are a direct consequence of the failure to satisfy the conditions for solvability of the Tomonaga- Schwinger equation in the interaction representation.

This idea prompts one to introduce the non—local interaction not into the theory of the equation of motion, but rather into the theory of the S—matrix, whose framework is much broader, and in which the formulation of the problem is more natural for a non—local theory. The present paper is devoted to the generalization to a non—local theory of Bogoliubov's method. [12,13] * where it will be convenient to use the form of the theory presented in Ref. 14.** The starting point for the construction will be the physically obvious requirements I, A-D which are imposed on the S—matrix.

As in the local case, in order to give an explicit description of the operation of switching on and switching off the interaction, we replace (cf. B. S. and I) the actual Lagrangian (2)*** by

*Cited as B.S. in the sequel. We shall use the notation of this and the next paper.

**The starting point for the construction will be the physically obvious.

***For purposes of concreteness, we shall consider the non—local theory with Lagrangian (2). However, all the results will, of course, be valid in a theory with any Lagrangian of similar type.

$g \,(\xi)\, \Lambda \,(\xi)$ where $g \,(\xi)$ here is a function of the three space—time points x', x'' and x'''. The full switching on of the interaction over all space will, of course, correspond to $g \,(\xi) = 1$. The expansion of $S \,(g)$ in series can be written in the form:

$$S \,(g) = 1 \qquad (3)$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n!} \int S_n \,(\xi_1, \ldots, \xi_n)\, g \,(\xi_1) \ldots g \,(\xi_n)\, d\xi_1 \ldots d\xi_n.$$

It is clear that the conditions of correspondence to classical theory and relativistic invariance (B. S. 3.17 and 4.4) are taken over into the non—local theory with practically no change:

$$S_1 \,(\xi) = i\Lambda \,(\xi), \qquad (4)$$

$$U_L S_n \,(L\xi_1, \ldots, L\xi_n)\, U_L^+ = S_n \,(\xi_1, \ldots, \xi_n). \qquad (5)$$

(In the last formula, $L\xi$ in an obvious way denotes the aggregate of the three points $L x'$, $L x''$ and $L x'''$.) No new points come up in the formulation of the condition of unitarity; one can immediately write:

$$S_n \,(\xi_1, \ldots, \xi_n) + S_n^+ \,(\xi_1, \ldots, \xi_n) \qquad (6)$$

$$+ \sum_{k=1}^{n-1} P \,(\xi_1, \ldots, \xi_k \,/\, \xi_{k+1}, \ldots, \xi_n)$$

$$\times S_k \,(\xi_1, \ldots, \xi_k)\, S_{n-k}^+ \,(\xi_{k+1}, \ldots, \xi_n) = 0.$$

The situation is different for the causality condition. Formally it is impossible in a non—local theory to satisfy the causality condition in its classical sense, because of the very basic physical ideas——the presence of a form factor will always lead to propagation of the interaction, at least in the small, with a velocity greater than the velocity of light. Physically, this is not inadmissible since,[15] so long as such violations are limited to regions of the order of "elementary lengths", they will be unobservable in a reasonable theory, and will mean only that the "mathematical" points x' . . . which serve as variables of integration do not signify physical points.

In fact, the introduction of a form factor in the interaction Lagrangian (2) means physically that we drop the usual picture of point elementary particles, and go over to a picture of extended elementary particles which are, so to speak, smeared

out in space and time. But as soon as we drop the notion of point elementary particles, we are left without any way of assigning a physical meaning to the assertion: "something happens at a definite space—time point " , and all physical quantities and concepts must now refer, not to points, but to (small) space—time regions. The use of "points" $x'\ldots$ is only as a mathematical tool, and it would be altogether unnatural to require the ful—fillment of any physical conditions with regard to them, such as, for example, that there be no propa—gation of the interaction from the point x to point y if y does not occur later than x.

It is obvious that, in place of such requirements of formal causality, in a non—local theory we should impose requirements which result in satis—fying the principle of causality for macroscopic distances (compared to the "elementary length" when it has a real physical meaning. The condi—tions which must be imposed on form factors, in order to restrict to macroscopic distances the breakdown of strict causality in expressions where $F(\xi)$ appears once, have been analyzed in detail recently; [16] as a result of this analysis, a set of sufficient conditions were found, which we shall assume to be satisfied.

However, in the expressions for the operator functions S_n in the scattering matrix, the form factor $F(\xi)$ will occur repeatedly. Therefore the criteria found in Ref. 16 are not enough, and we must now, dropping the strict causality condition I—2, formulate a weakened condition which should eliminate the possibility of adding to the break—down of formal causality, due to the presence of the form factor, new violations caused by unsatis—factory construction of the S—matrix. Such a con—dition is found in a natural way if we try to gener—alize the integral causality condition formulated in I to the case of the non—local interaction (1).

2. THE CONDITION OF ALMOST—CAUSALITY

Again, as in I, let us consider two regions, G_1 and G_2, situated so that $G_1 \geq G_2$ and generalize the definition of classes of functions $g_1(x)$ and $g_2(x)$, introduced in I—3, to the case where each of the functions depends, not on one, but on three points x', x'', x'''. We shall define these class—es in the strictest sense, namely we shall require that:

$$g_1(\xi) \neq 0, \quad \text{only if simultaneously}$$
$$g_2(\xi) \neq 0, \quad \text{only if simultaneously}$$

$$x_1' \in G_1, \ x_1'' \in G_1 \text{and} x_1''' \in G_1, \tag{7}$$
$$x_2' \in G_2, \ x_2'' \in G_2 \text{and} x_2''' \in G_2.$$

With such a definition, the principle of causality will require that, despite the presence of the form factor $\dot{F}(\xi)$, the interaction $g_2(\xi) \Lambda(\xi)$ shall act on the system as if the interaction $g_1(\xi) \Lambda(\xi)$ were not present, while the effect of the interaction $g_1(\xi) \Lambda(\xi)$ shall not depend on the specific nature of the interaction $g_2(\xi) \Lambda(\xi)$, but only on the state of the system which develops as a re—sult of the latter's action. In fact, what we want is that formally "acausal" interactions be assoc—iated only with triples of points x', x'', x''' appearing in the argument of a single form factor, i.e., referring to the same "point" ξ, while the possibility of meeting pairs of such points in the arguments of different functions $g_1(\xi)$ and $g_2(\xi)$ is precisely what is excluded by the conditions (7) and $G_1 \gtrsim G_2$.

Considerations which are completely analogous to those used in the derivation of condition (6) of Reference I give, as the mathematical expression

of this requirement, the condition

$$S(g_1 + g_2) = S(g_1) S(g_2), \quad \text{if} \quad G_1 \gg G_2, \tag{8}$$

which we shall name the *condition of almost—causality*, in order to distinguish it from the strict causality conditions which occur in the local the—ory. It satisfies the principle of correspondence to local theory in an obvious fashion — when we make the limiting transition

$$F(x', x'', x''') \rightarrow \delta(x' - x'') \times \delta(x'' - x''')$$

it becomes the integral causality condition I—6 of the local theory.

The fact that the condition of almost—causality is weaker than the integral causality condition I—6 manifests itself graphically in the fact that it is impossible to go over from it to a differential formulation analogous to I—2. Actually, there is no difficulty in deriving, from equation (8), a con—dition analogous to condition I—9 for a class of functions analogous to the special class I—8. How—ever, in the non—local theory it turns out that it

is not possible to approximate an arbitrary function $g(\xi)$ by functions of this special class, since for this purpose we would require the regions G_1 and G_2 to not merely touch, but rather to interpenetrate.

In order now to translate condition (8) into the language of the operator functions $S_n(\xi_1 \ldots \xi_n)$ we must, as in the derivation of I–7 use for $S(g_1)$, $S(g_2)$ and $S(g_1+g_2)$ the expansions (3). After exactly the same algebraic transformations, we arrive at the condition: *

$$S_n(\xi_1, \ldots, \xi_n) \tag{9a}$$

$$= S_l(\xi_1, \ldots, \xi_l) S_{n-l}(\xi_{l+1}, \ldots, \xi_n),$$

$$\text{if } \{\xi_1, \ldots, \xi_l\} \not\succ \{\xi_{l+1}, \ldots, \xi_n\}. \tag{9b}$$

Thus the condition of almost–causality (8) leads, in the same way as the integral causality condition I–6 , to a multiplicative representation for the operator functions S_n of separable (cf. I) aggregates of arguments $(\xi_1 \cdots, \xi_n)$.* We now proceed to investigate the compatibility of this condition with the other conditions, IA–C, which are imposed on the S–matrix.

3. COMPATIBILITY OF CONDITIONS IMPOSED ON THE SCATTERING MATRIX

We note first that if the set of variables $\{\xi_1 \cdots, \xi_n\}$ splits into a sum of two space–like aggregates $\{\xi_1 \cdots, \xi_l\}$ and $\{\xi_{l+1} \cdots, \xi_n\}$, then, as in the

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*The notation $\{\xi_1, \ldots, \xi_l\} \not\succ \{\xi_{l+1}, \ldots, \xi_n\}$

means that $\{x_1', x_1'', x_1''', \ldots, x_l', x_l'', x_l'''\}$

$$\not\succ \{x_{l+1}', \ldots, x_n'''\}.$$

One should also keep in mind the remark (cf. I, footnote 3) concerning the order of enumeration of the variables.

*We note that for separability of the aggregate
$$\{\xi_1, \ldots, \xi_n\}$$
it is not sufficient to have the aggregate
$$\{x_1', x_1'', x_1''', \ldots, x_n', x_n'', x_n'''\}$$
be separable, i.e., to have at least one cut through it; it is neccesary , in addition, that this cut not go through any one of the sets of points $\{x_i', x_i'', x_i'''\}$, referring to the same composite "point" ξ_i.

local case, we obtain from (9) the requirement of commutability of the operator functions of the space–like sets of arguments:

$$S_l(\xi_1, \ldots, \xi_l) S_{n-l}(\xi_{l+1}, \ldots, \xi_n) \tag{10}$$

$$= S_{n-l}(\xi_{l+1}, \ldots, \xi_n) S_l(\xi_1, \ldots, \xi_l),$$

$$\text{if } \{\xi_1, \ldots, \xi_l\} \sim \{\xi_{l+1}, \ldots, \xi_n\}.$$

Like the corresponding condition I–10 of the local theory, (10) will be fulfilled automatically if the elementary commutators (anticommutators) of the free field operators vanish for space–like intervals. The fact that (10) cannot be satisfied, if the elementary commutators differ from zero, even if only for extremely small space–like intervals, is apparent from the fact that (10) must , in particular, be valid when all the arguments in each set $\{\xi_1 \cdots, \xi_l\}$ and $\{\xi_{l+1} \cdots, \xi_n\}$ coincide.

The situation which we have just described can be regarded as the mathematical formulation of the fact that we are dealing with a field theory with non–local interaction: the commutators of free fields must vanish outside the light cone; in this sense we may say that we are dealing with a theory in which the free fields have local character. Therefore condition (10) also has the significance of a condition on the local nature of the theory of the free fields (here we have in mind the fact that the fields must appear in the Lagrangian as a whole, without being split into positive and negative frequency parts).

In I the theorem was proved that, if in a local theory the set of arguments of the operator function S_n separates in several ways, the representations of S_n which, by virtue of I-7, result from these splittings will differ from one another only by a transposition of functions S_ν , $S_{\nu'}$ of space- like sets of arguments. It is easy to see that the proof of this theorem carries over verbatim to the non–local theory, so that it remains valid there. Since, by virtue of (10), operator functions of space–like sets of arguments also commute in the non–local theory, the compatibility of the requirements imposed on each S_n by the condition of almost–causality is proven if this condition can be applied repeatedly.

The situation is similar for the theorems concerning unitarity which were demonstrated in I. In fact, their proofs were based entirely on certain algebraic relations for which the nature of the symbols $x_1 \ldots, x_n$ or $\xi_1 \ldots, \xi_n$ was completely

irrelevant; it was required only that the separation
I—7a follow from the relation I—7b, and this remains
true when we replace all the x_i by ξ_i , because of
(9).

Thus if the set of arguments $\{ \xi_1 \ldots , \xi_n \}$
of the operator function S_n is separable (only in
this case is the condition of almost—causality ap-
plicable to S_n), and if the unitarity-condition is
satisfied for $S_1 \ldots S_{n-1}$ for arbitrary values of
the arguments, then the representation (9) for S_n
which results from the condition of almost—causality
will automatically satisfy the unitarity condition.
So the compatibility of conditions D and C for the
S—matrix is demonstrated.

Since the mutual compatiblity of the remaining
conditions imposed on S_n is obvious from the same
considerations as in the local theory, all four cond-
itions $A—D$ imposed by us on the S—matrix are
consistent with one another.

From this it follows immediately that we can al-
ways find a sequence of operator functions $S_1 \ldots S_n$
all of whose terms will satisfy conditions $A—D$.

In fact, let us assume that the operator functions
$S_1 \ldots S_n$ have been constructed. We shall show
that we can always construct an S_{n+1} which, to-
gether with the already constructed $S_1 \ldots S_n$ will
satisfy all the conditions A-D. The manifold of all
possible values of the arguments $\xi_1 \ldots \xi_{n+1}$
of the operator functions S_{n+1} separates into two
classes: arguments forming an inseparable set, and
arguments forming a separable set.

If a certain set of arguments belongs to the se-
cond class, then from the condition of almost—
causality the corresponding value of S_{n+1} will be
represented as a product of already known opera-
tor functions of lower index, so that according to our
earlier remarks all the conditions $A—D$ will be
satisfied.

If a certain set of arguments belongs to the first
class, then the condition of almost—causality in
general imposes no limitations on the correspond-
ing value of the operator function. As we see from
(6), the unitarity condition uniquely determines
the Hermitean part of S_{n+1} in terms of the already
known $S_1 \ldots S_n$. The anti—Hermitean part of
S_{n+1} remains arbitrary.

So if we are given operator functions $S_1 \ldots S_n$
satisfying $A—D$, we can always construct an
operator function S_{n+1} which together with them
satisfies the same conditions. On the basis of the

principle of complete induction, we then arrive
at the possibility of constructing a sequence of
operator functions $S_1 \ldots S_n \ldots$ satisfying all the
requirements $A—D$, i.e., we get the theorem of the
existence of the scattering matrix (cf. Ref. 6 in I).

4. THE CONSTRUCTION OF THE OPERATOR FUNCTIONS OF THE S—MATRIX

In order to formulate the method of successive
construction of the operator functions of the S—matrix
and to get compact and symmetrical expressions
which appear in it, we shall investigate the struct-
ure of the operator functions S_n $(\xi_1 \ldots , \xi_n)$ in
more detail . From the theorem of the complete
separability of separable aggregates, which was
proven in I, it follows that, for any combination
of arguments of the function S , the points $\xi_1 \ldots, \xi_n$
can always be divided into m $(1 \leq m \leq n)$groups
$\{ \xi_{\lambda_i} \ldots , \xi_{\lambda_{\nu_1}} \} ; \ldots ; \{ \xi_{\lambda_{\nu_i + \ldots + \nu_{m-1} +1}} \ldots ,$
$\xi_{\lambda_m} \}$, which are separated from one another, while
the group of points in each set are inseparable. In
the local theory, this separation led to a division
into m groups of points, such that the points within
each group coincided, while the points in different
groups were distinct. Since we here want to main-
tain the analogy with this formulation, we shall
say that the ν points $\xi_{\lambda_1} \ldots , \xi_{\lambda_\nu}$ of one inse-
parable group "coalesce" and form a composite
pointΞ^ν , consisting not of three, but of 3ν ordi-
ary points $x'_{\lambda_i} \ldots , x'''_{\lambda_\nu}$.

Here we introduce the concept of an almost—local
operator. We shall say that the operator expres-
sion $N_\nu (\xi_{\lambda_i} \ldots , \xi_{\lambda_\nu})$, depending on the field op-
erators at the "points" $\xi_{\lambda_1} \ldots , \xi_{\lambda_\nu}$, is an
almost—local operator if

$$N_\nu (\xi_{\lambda_i}, \ldots , \xi_{\lambda_\nu}) = 0 \qquad (11)$$

for an arbitrary separable set of arguments, and if
the conditions of relativistic invariance, (5) and
(10), are satisfied for N_ν . It is clear that the almost-
local operator is a direct generalization of the con-
cept of quasi—local operator, introduced in B. S.; in
fact, in a local theory the set of arguments $x_1 \ldots x_\nu$
can, as already mentioned, be inseparable if and
only if all the points $x_1 \ldots , x_\nu$ coincide.

Comparing the definition of almost—local opera-

tor with the definition of a composite point, we see that the set of arguments of an almost—local operator always forms a composite point Ξ^ν; we may threfore say that each almost—local operator will depend, not on several composite points (of third order) ξ, but on one composite point Ξ^ν.

$= \{ \xi_{\lambda_1} \cdots, \xi_{\lambda_\nu} \}$ of order 3ν. Since the composite points ξ, consisting of triplets of ordinary points (so that they are composite points of third order) are in no way physically distinguished from composite points of n'th order, the Lagrangian $\Lambda(\xi)$, "depending only on a single point" ξ, loses its special position among almost—local operators.

Using the concept of a composite point, we may say that the n arguments of S_n always break up (uniquely) into m ($1 \leq m \leq n$) mutually separated composite points $\Xi^{\nu_1}_1 \ldots, \Xi^{\nu_m}_m$ ($\nu_1 + \ldots + \nu_m = n$). From the theorem demonstrated in I, concerning the possibility of representing a sum of inseparable sets as an ordered sequence, it will now follow that the "points" $\Xi^{\nu_1}_1 \ldots, \Xi^{\nu_m}_m$ can always (not in general, uniquely) be ordered in time, maintaining the relation (cf. Ref. 6 in I):

$$\Xi^{\nu_{\lambda_1}}_{\lambda_1} \gg \Xi^{\nu_{\lambda_2}}_{\lambda_2} \gg \cdots \gg \Xi^{\nu_{\lambda_m}}_{\lambda_m}. \tag{12}$$

Now applying the condition (9) of almost—causality to each of the sections occurring in (12), we get for S_n the representation

$$S_n(\xi_1, \ldots, \xi_n) \tag{13}$$

$$= S_{\nu_{\lambda_1}}(\Xi^{\nu_{\lambda_1}}_{\lambda_1}) \ldots S_{\nu_{\lambda_m}}(\Xi^{\nu_{\lambda_m}}_{\lambda_m}),$$

similar to the representation I—25 of the local theory. If the "points" $\Xi^{\nu_i}_1 \ldots, \Xi^{\nu_m}_m$ can also be ordered in some other way, we obtain for S_n a representation differing in form from (13). However, because of the uniqueness of the resolution of a separable aggregate into a sum of inseparable aggregates, and the self—consistency of the condition of almost—causality, it can differ from (13) only by transpositions of commuting S_ν's of space—like separated pairs of "points" Ξ^ν, — consequently all such representations will be equivalent; they can threfore be combined into the symmetric expression:

$$S_n(\xi_1, \ldots, \xi_n) \tag{14a}$$

$$= T[S_{\nu_1}(\Xi^{\nu_1}_1) \ldots S_{\nu_m}(\Xi^{\nu_m}_m)],$$

if

$$\{\xi_1, \ldots, \xi_n\} = \Xi^{\nu_1}_1 + \cdots + \Xi^{\nu_m}_m; \tag{14b}$$

$$1 \leqslant m \leqslant n; \quad \Sigma \nu_i = n.$$

Here it is implied that the T-ordering applies only between "points" $\Xi^{\nu_i}_i$ while the operator functions of individual "points" enter as a whole in (14). As a consequence of the theorems concerning separation and ordering of aggregates, which we have just quoted, the meaning of the T-products which occur in (14a) is completely obvious and unique: the expression (14) simply denotes the representation (13) for all possible orderings (12) with the separation (14b); in addition, it shows explicitly the symmetry of S_n with respect to all its arguments.

Formula (14) reduces the problem of determining S_n for arbitrary sets of arguments to the determination of the operator functions S_ν for the individual arguments Ξ^ν which are composite "points", i.e., for the inseparable sets $\{\xi_1 \ldots, \xi_\nu\}$.

We have already noted that the condition of almost—causality in general imposes no limitations on the value of S_ν for such arguments, that the unitarity condition uniquely determines the Hermitean part of such S_ν in terms of the operator functions $S_1 \ldots S_{\nu-1}$ of lower index, while the anti—Hermitean part of such S_ν remains arbitrary [of course, within the limitations of the requirements arising from the conditions of relativistic invariance (5) and (10)]. Since these quantities can be regarded as almost—local operators,

$$\widetilde{N}_\nu(\xi_1, \ldots, \xi_\nu) = \widetilde{N}_\nu(\Xi^\nu) \tag{15}$$

$$= \widetilde{M}_\nu(\Xi) + i\widetilde{\Gamma}_\nu(\Xi^\nu)$$

($\widetilde{M}_\nu$ and Γ_ν are Hermitean), we may say that (14) completely (and uniquely) determines all the operator functions $S_1(\xi) \ldots, S_n(\xi_1 \ldots \xi_n)$ in terms of the sequence of Hermitean almost—local operators

$$\Lambda(\xi) = \Gamma_1(\Xi^1), \tag{16}$$

$$\Gamma_2(\Xi^2), \ldots, \quad \Gamma_n(\Xi^n), \ldots,$$

which are assigned completely arbitrarily, and to-

gether determine the physical system whose theory is being constructed. Thus the problem of construction of the S-matrix in the non-local theory is solved in principle.

The representation (14) for S_n has, however, an essential defect from the practical point of view: it does not give a single expression for the operator function S_n ($\xi_1 \ldots , \xi_n$) for all values of its arguments—for combinations $\xi_1 \ldots , \xi_n$, in which the points ξ are coupled differently, (14) gives for S_n different expressions, which do not automatically go over into one another. Thus, for example, according to (14), we must use for S_2 the expression $T [S_1 (\xi_1) S_1 (\xi_2)]$, if ξ_1 and ξ_2 are unconnected, and the expression $\tilde{N}_2 (\xi_1, \xi_2)$ if they are connected.

In order to eliminate this defect, we must, obviously, represent the value of S_n for a coupled set of arguments as a sum of expressions which continuously extend its meaning for uncoupled arguments plus some additional term which automatically vanishes if the arguments are uncoupled. To do this we must first extend continuously the definition of the T-product, which so far exists only for T-products of functions of uncoupled points Ξ_1, $\ldots \Xi_k$, to the case where the points are coupled, when the T-product in its usual intuitive sense does not exist, and we must take refuge in additional arbitrary conventional definitions.

One possible convention is the definition of the T-product according to Wick's theorem as the sum of all possible normal products with all possible chronological contractions. It is easy to see that this definition is equivalent to the independent time ordering of the individual free field operators appearing in the operator function. In the local theory (cf. I) this definition was a completely natural one, since it led (for suitable choice of the regularization) to the automatic fulfillment of the unitarity condition when the points were coupled. It is easy to see by direct computation that this way of extending the definition of the T-product in the non-local theory leads to violation of unitarity.

Another possibility would be a definition of the T-product which would not lead to a violation of unitarity for coupled points. Such a definition would be preferable from the point of view of the general theory, but it would be much less convenient for calculation, since there would clearly be no analog of Wick's theorem.

We shall therefore assume that we have a de-. finition of the T-product for coupled points which

does not necessarily achieve unitarity, and split up the value of S_n for completely uncoupled arguments $\xi_1 \ldots , \xi_n$ into a sum

$$S_n (\xi_1, \ldots , \xi_n) = S_n^T (\xi_1, \ldots , \xi_n) + N_n (\Xi^n) \quad (17)$$

of quantities S_n^T, consisting of a sum of T-products of operator functions with a smaller number of arguments, extended in accordance with some chosen convention, and the almost-local operator

$$N_n (\Xi^n) = M_n (\Xi^n) + i \Gamma_n (\Xi^n)$$

(M and Γ are Hermitean). The operator M_n will again be uniquely determined via the unitarity condition in terms of the operator functions S_k of lower index, while its form will of course depend on the way the T-product definition is extended (in particular, if the second method suggested above is used, it will be zero).

Substituting the resolution (17) in (14) and making some combinational transformations, we get an expression for S_n ($\xi_1 \ldots , \xi_n$) as a sum

$$S_n (\xi_1, \ldots , \xi_n) \quad (18)$$

$$= \sum_{\left(\substack{\Xi_1, \ldots , \Xi_m \\ 1 \leqslant m \leqslant n} \right)} T [N_{\nu_1}(\Xi_1^{\nu_1}) \ldots N_{\nu_m} (\Xi_m^{\nu_m})],$$

in which it is assumed that the summation is extended over all possible separation of the set $\{\xi_1 \ldots , \xi_n \}$ into a sum of "points" $\Xi_1^{\nu_1}$, $\ldots , \Xi_m^{\nu_m}$, where the points Ξ, consisting of different points ξ, are assumed to be different, while the order of the points Ξ in the entries $N_{\nu_1} \ldots N_{\nu_m}$ is irrelevant.*

The summation in (18) is carried out first over the number m of points Ξ_i; second, over all possible distributions of the ν_i variables ξ in each point $\Xi_i^{\nu_i}$, and third, over all possible assignments of the variables $\xi_1 \cdots \xi_n$ among the m groups with $\nu_1 \ldots \nu_m$ in each group. We can therefore write the expression in more detail in the form

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*We emphasize once more that the almost-local operators N_{ν_i} are assumed to enter into the T-product as a whole; the T-ordering is done only between different Ξ_i, and not inside them.

$$(19)$$

$$S_n(\xi_1, \ldots, \xi_n) = T(N_1(\xi_1) \ldots N_1(\xi_n))$$

$$+ \sum_{\left(\begin{array}{c} 2 \leqslant m \leqslant n-1 \\ \Sigma \nu_i = n \end{array}\right)} \frac{1}{m!} P(\xi_1, \ldots, \xi_{\nu_1} | \xi_{\nu_1+1}, \ldots, \xi_{\nu_1+\nu_2} | \ldots | \xi_{\nu_1+\cdots+\nu_{m-1}+1}, \ldots, \xi_n)$$

$$\times T[N_{\nu_1}(\xi_1, \ldots, \xi_{\nu_1}) N_{\nu_2}(\xi_{\nu_1+1}, \ldots, \xi_{\nu_1+\nu_2}) \ldots N_{\nu_m}(\xi_{\nu_1+\cdots+\nu_{m-1}+1}, \ldots, \xi_n)]$$

$$+ M_n(\Xi^n) + i\Gamma_n(\Xi^n).$$

The factor $1/m!$ appears here because each specific subdivision occurs $m!$ times in the sum (19), differing only in an irrelevant order of arrangment of the factors under the sign of the T−product.

Formulas (18) or (19) give us the desired representation of the operator functions $S_n(\xi_1 \ldots, \xi_n)$ valid for any values of the arguments. They express S_n in terms of the sequence of almost-local operators

$$N_1(\xi) = i\Lambda(\xi), \qquad (20)$$

$$N_2(\xi_1, \xi_2), \ldots, N_n(\xi_1, \ldots, \xi_n), \ldots,$$

in each of which the Hermitean part M_n is uniquely determined by the functions S_k (i.e., in the last analysis, by the operators N_k)of lower index, while the anti−Hermitean part $i\Gamma_n$ remains arbitrary and must be given in the formulation of the theory. Formula (19) is completely analogous in structure to the expression I–30 of the local theory.

Combinatorial transformations, which are the same as those used in the local theory (cf. B.S. 4.30–4.34) eable us to convert the expansion (3) with the operators functions (19) to the concise formula:

$$S(g) = T\left(\exp\left\{\int N(\xi; g)\, g(\xi)\, d\xi\right\}\right) \quad (21)$$

$$= T\left(\exp\left\{i\int \Lambda(\xi; g)\, g(\xi)\, d\xi + \int M(\xi; g)\, g(\xi)\, d\xi\right\}\right),$$

if we define the almost−local Hermitean functionals $M(\xi; g)$ and $M(\xi; g)$ by the expansions:

$$(22)$$

$$\Lambda(\xi; g) = \Lambda(\xi) + \sum_{\nu=2}^{\infty} \frac{1}{\nu!} \int \Gamma_\nu(\xi, \xi_1, \ldots, \xi_{\nu-1})\, g(\xi_1)$$

$$\ldots g(\xi_{\nu-1})\, d\xi_1 \ldots d\xi_{\nu-1},$$

$$M(\xi; g) = \sum_{\nu=2}^{\infty} \frac{1}{\nu!} \int M_\nu(\xi_1, \xi_1, \ldots, \xi_{\nu-1})\, g(\xi_1) \quad (23)$$

$$\ldots g(\xi_{\nu-1})\, d\xi_1 \ldots d\xi_{\nu-1}.$$

The functional $\Lambda(\xi; 1)$ can now be regarded as the total Lagrangian of the system. Its individual terms, which are determined by the almost−local operators

$$\Lambda(\xi) = \Gamma_1(\Xi^1), \quad \Gamma_2(\Xi^2), \ldots, \Gamma_n(\Xi^n), \ldots,$$

differ from one another only in the manner of turning on the interaction" (cf. the discussion in B. S. at the end of Section 4), which we are free to choose, and differ also in the number of simple points x ... which are contained in a single composite point Ξ. Therefore there is no basic physical distinction between them, so that combining them in the total Lagrangian $\Lambda(\Xi; 1)$, which must be given in order to characterize the physical system, is entirely natural.

The functional $M(\xi; g)$, on the other hand, has no direct physical relation to the system under consideration. Its appearance is due to the arbitrariness discussed above in the definition of the continuation of T−products into the domain of coupled arguments: if we use a definition which does not maintain unitarity, for coupled arguments then we must correct things by adding to the Lagrangian the "anti−Hermitean added term"− $iM(\xi; g)$.

It should be emphasized that in order to construct the S−matrix according to formula (21), we must first of all give some definite method for extending the definition of the T−product into the domain of coupled arguments. As soon as such a definition is made, we can assign a definite total Lagrangian $\Lambda(\xi; 1)$. The functional $M(\xi; 1)$ is then uniquely determined from the condition of unitarity,* so that (21) gives us the value of the S-matrix. If we now shift from this definition of the T−product to some other, then to get the same S-matrix we must change the form not only of the functional $M(\xi; 1)$ but also of the functional $\Lambda(\xi; 1)$. Thus the total Lagrangian $\Lambda(\xi; 1)$ determines the S−matrix uniquely only with respect to a fixed method of defining the T−product in the region of coupled arguments.

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*In practise, the functional $M(\xi; 1)$ can only be determined by successive use of the unitarity requirement in order to find the almost−local operators $M_2 \ldots M_{n,} \ldots$

Such a situation is completely analogous to that which occurs in the local theory; there, too, the form of the total Lagrangian L $(x; 1)$ determines the S—matrix only with respect to some given fixed • method of regularization; in order to get the same S—matrix with different methods of regularization requires the use of different total Lagrangians L $(x; 1)$.**

5. DISCUSSION

The fundamental result obtained above is the proof that, by generalizing the method of Bogoliubov and Stueckelberg to the non—local theory, one can construct for any Lagrangian an S—matrix satisfying the physically obvious conditions A—D. We have succeeded in overcoming the difficulty with the unitarity of the S—matrix which arises in constructing the theory in the Heisenberg representation.

Physically, this progress proved possible, apparently, because of the more consistent application of the non—local point of view. In fact, the introduction of a form factor means in descriptive terms the assignment to the elementary particles of a certain internal structure. Since we avoid defining it, and regard the form factor as something put into the theory from outside, this means that we avoid.—.at least at this stage of the theory, the study of the laws which govern the internal structure of elementary particles, and regard it as given; all our equations refer essentially only to processes in which the internal structure of the particles does not change.

But from this point of view it appears unnatural to require of the equations of the theory that they answer the question: what happens if, roughly speaking, the particles "penetrate on another"; to this there corresponds mathematically not only the attachment of several points x' . . . to one point ξ, but also the case of coupling of arguments. This information must be put into the theory from outside, just as is done for the form factor and the "fundamental Lagrangian $\Lambda(\xi)$, in such a way, of course,

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**We note that the entire procedure developed for constructing the non—local S—matrix suggests the idea that the coupling of arguments in the non—local theory can be regarded as an unusual method for evaluating the indeterminate forms which occur for coupled (coin - cident)arguments in the local theory, which is usually accomplished by means of regularization. Obviously, therefore, one could arrive at the non—local theory from a "formalistic" point of view, not attributing any physical meaning to the non—local character, but regarding it merely as a method of regularization before making the limiting transition to the local theory. It is not out of the question that such a method of regularization might be of some interest, in view of its close connection with the space—time description .

that it does not come into conflict with the latter.

The method of solution of the equation in Heisenberg representation[7,8] taking the form factor and the quantity Λ (ξ) as given, assigned a completely definite meaning to the operator functions of the S—matrix for coupled arguments, while starting coupled "points" ξ. Thus, from our point of view, this method gave a more detailed description than is admissible for a non—local system, and it was just this which gave rise to the difficulty with unitarity.

In our method for constructing the S—matrix, this defect does not occur: to define the theory completely we require the assignment of the values of all the operator functions of the S—matrix (15) for completely coupled sets of arguments; then the requirement of mutual compatibility of these values (the unitarity condition) determines their Hermitean parts , while their anti—Hermitean parts (16) must be assigned in formulating the theory, and together determine the assumed internal structure of the particles. Such a procedure for constructing the theory, in which in a certain sense we separate the "domains of essential non—locality" and the domains of the more or less customary space—time description (cf. Reference 17), naturally follows from our condition of almost- causlity.

The "total Lagrangian" (22) is the operator which in our case defines the theory. This may serve as an argument in support of the position that, in a consistent non—local theory, we must not limit ourselves to the consideration, of non—local interactions of just one specific kind, such as (1), (2), but must consider general non—local interactions, expressed in terms of Lagrangians which are general functionals of the field operators, like the Lagrangian (22). Since, in addition, one may hope that the introduction of a form factor will make it possible to avoid divergences, in such a non—local theory the demarcation between renormalizable and unrenormalizable theories is erased.

In conclusion, we should like to point out that the theory we have constructed should not necessarily be regarded as complete. Here we have in mind that the possibility is not excluded of establishing some additional limitations on the choice of the set of almost—local operators Γ_n (Ξ^n), starting from some additional physical requirements. Here we are thinking of gauge invariance conditions, which in the light of recently obtained results[18] requires further serious investigation, and also of conditions of the type of the "reality condition."[19] We shall not enter into a discussion of these questions here.

attention to this work and for many suggestions and valuable advice, and also to D. V. Shirkov and V. L. Bonch—Bruevich for fruitful discussion.

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