Multiple Interaction Hamiltonians in Quantum Electrodynamics

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A method is described for constructing electromagnetic interaction Hamiltonians of a many-particle system, based on the usual Hamiltonian formalism. For illustration the Hamiltonian for the pair interaction of spinor particles is derived.

INTRODUCTION

THE exact solution of problems connected with the many body problem in quantum electrodynamics is confronted by considerable difficulties both of principle and of technique. An approximate treatment of this problem is possible by means of representing complicated motions of a system of N charged particles as a superposition of simpler processes, in each of which $N'(N' = 2, 3. . ., N)$ real and a certain number of virtual particles take part, and by separately investigating each of these processes. Due to the work of Schwinger it is now known how to obtain the Green's function, and also how to obtain and to solve (approximately) the corresponding equation for each of these simp-. ler systems.

In this article we propose a different method of describing the elementary interactions of which the complex motion of the system of N particles is composed. In place of a set of Green's functions we shall make the system of N particles, corresponding to a set of interaction Hamiltonians of different orders. Each of these corresponds to a process with a strictly defined number of real and virtual particles. As will be seen below, it will be possible, by using this method, to calculate in a simpler way the relativistic and field corrections to the optical spectra of atoms which have their origin in the multiple interactions of the particles.

An explicit expression for the Hamiltonian of the N-th order which also contains the term representing N-fold interactions (N real particles participate simultaneously) can be obtained from the usual equation of quantum electrodynamics

$$(\hat{E} - \hat{\mathscr{H}})\psi(x) = 0;$$

$$\hat{\mathscr{H}} = \sum_{j=0}^{\infty} \hat{H}_j, \tag{1}$$

written, for example, in the Schrödinger or the Tomonaga-Schwinger representation. In order to do this, it is necessary to assume all the processes of lower multiplicity ($1, 2, \ldots, N - $ 1th orders),

to be virtual, and to exclude the terms corresponding to them from the equations of motion (1) by means of unitary transformations. A summation over the intermediate states of the $1, 2, \ldots,$ $(N - 1)$-th orders is carried out in Eq. (1). The result then describes correctly electromagnetic processes in systems with $N'(N' \geq N)$ real and virtual particles, and its Hamiltonian will contain explicit expressions of $N' = N, N + 1, \ldots,$ $2N - 1$ orders which can be directly used to describe processes in which N' real and virtual particles participate.

The representation of the Hamiltonian of the N-th order by summing the processes with all possible intermediate states possesses a number of advantages which make this method convenient for the solution of specific problems.

First of all, in each approximation (in each H_N) the relativistic invariance of the method is guaranteed, since the terms are classified only according to the number of particles which they describe. This allows one to obtain correct non-relativistic approximations for the interaction Hamiltonians of all orders, even in the Schrödinger representation, which is quite useful in view of the fact that until now a number of authors[1,2] have attempted unsuccessfully to obtain them from Breit's equation.

In the second place, a "preliminary" summation over the intermediate states of the interaction Hamiltonian makes it unnecessary to calculate higher approximations of the interaction energy by means of perturbation theory, and consequently a knowledge of the complete sets of eigenfunctions of the system under consideration in the zeroth approximation is not required. The energy corrections of the required order to the energy terms of the atom will be simply determined by the matrix elements of the corresponding Hamiltonians.

In the third place, in this case, just as in the case of the expansion of the S-matrix in a power series, a "term by term" regularization is possible

since the removal of the field divergences from each Hamiltonian of the N-th order means their removal from the corresponding approximations of the theory.

1. SETTING UP THE N-TH ORDER HAMILTONIANS

The basic equation of electrodynamics (1) has the following form in the Schrödinger representation:

$$[(i\partial / \partial t) - \mathscr{H}]\psi(\mathbf{r}, t) = 0;$$
$$\mathscr{H} = H_0 + H_1^{tr} + H_2^l. \tag{1.1}$$

Here $\psi(\mathbf{r}, t)$ is the wave function of a certain system of particles in configuration space, H_0, H_1^{tr}, H_2^l — are respectively the Hamiltonians of the free fields, of the interaction of transverse photons with particles, and of the Coulomb interaction between particles. We assume all fields to be quantized and we do not introduce a potential of an external field*.

Let us carry out the unitary transformation

$$\psi(\mathbf{r}, t) = e^{iS_1}\psi'(\mathbf{r}, t), \tag{1.2}$$

by means of which (1.1) is brought into the form

$$[(i\partial / \partial t) - \mathscr{H}_1]\psi(\mathbf{r}, t) = 0; \tag{1.2a}$$

$$\mathscr{H}_1 = e^{-iS_1}\mathscr{H}e^{iS_1} = \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \tag{1.3}$$

$$\times [S_1, [S_1, \ldots [S_1, H_0 + H_1^{tr} + H_2^l]\ldots]].$$

If one demands that

$$H_1^{tr} - i[S_1, H_0] = 0, \tag{1.4}$$

then the Hamiltonian of Eq. (1.3) will be free of the interaction term of the first order (see also Ref. 3). But, as has been noted already above, it will contain explicit expressions for the Hamiltonians of the 2-nd and the 3-rd orders

$$H_2 = H_2^l - \tfrac{1}{2}i[S_1, H_1^{tr}]; \tag{1.5}$$

$$H_3 = -i[S_1, H_2^l - \tfrac{1}{3}i[S_1, H_1^{tr}]]. \tag{1.6}$$

Carrying out in turn the 2, 3, ..., k, ..., $(N-1)$th unitary transformations of the type (1.2) and imposing on the operators S_k the conditions

$$H_k - i[S_k, H_0] = 0, \tag{1.7}$$

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* The introduction into (1.1) of the interaction of the particles with an external field does not change the following calculations.

analogous to (1.4) we shall obtain as a result the equation

$$[(i\partial / \partial t) - \mathscr{H}_{N-1}]\psi(\mathbf{r}, t) = 0;$$
$$\mathscr{H}_{N-1} = \exp\{-iS_{N-1}\}\mathscr{H}_{N-2}\exp\{iS_{N-1}\}, \tag{1.8}$$

which correctly describes processes in which N' ($N' \geq N$) real and virtual particles take part. In its Hamiltonian explicit expressions occur for the interaction Hamiltonians of the N, $N+1$, ..., $(2N-1)$-th orders.

The Hermitian nature of the operators $\mathscr{H}_N$ occurring in (1.8) and of the individual terms H_k which they contain is a direct consequence of the Hermitian nature of the original Hamiltonian $\mathscr{H}$ and of its terms (1.3), and may be easily proved by the method of induction with the aid of relations (1.2) - (1.8). We shall not go into this, but shall directly state a rule according to which the operators S_k should be constructed to satisfy condition (1.7). In order to do this we shall represent the operator H_k in the form

$$H_k = \int d\mathbf{p}\{h_k(\mathbf{p}) + h_k^*(\mathbf{p})\}. \tag{1.9}$$

Then S_k turns out to be equal to

$$S_k = -i\int d\mathbf{p} \tag{1.10}$$

$$\times \left(h_k(\mathbf{p}) - h_k^*(\mathbf{p})\right)\Big/\left(\sum_n E_i(\mathbf{p}_j) - \sum_p E_i(\mathbf{p}_j)\right),$$

where $\sum_n E_i \left(\sum_p E_i\right)$ is the sum of the free energies of particles of the i-th kind with momenta $\mathbf{p}_j$, absorbed (created) by the absorption (creation) operators occurring in $h_k(\mathbf{p})$; p is the set of all the three dimensional momenta occurring in the integrand of (1.9). $E_i(\mathbf{p}_j)$ stands for the diagonalized operator for the free energy which in the case of Bose-particles is defined by equation

$$E_i^B(\mathbf{p}_j) = e_{ij} = +\sqrt{\mathbf{p}_j^2 + m_i^2} \tag{1.11a}$$

and in the case of Fermi-particles is defined by equation

$$E_i^F(\mathbf{p}_j) = \varepsilon_{ij} = \pm\sqrt{\mathbf{p}_j^2 + m_i^2}. \tag{1.11b}$$

It follows from (1.10) that S_k is a self-conjugate quantity. The verification of this formula presents no difficulties and may be carried out by substituting into (1.7) formula (1.10) and the expressions H_0 and $h_k(\mathbf{p})$ which have the form of a product of certain coefficient functions by the product of field operators.

In the Tomonaga-Schwinger representation the Hamiltonians H_N are obtained in the same way as in the Schrödinger representation and have the same appearance. For example, the terms H_2, H_3 in the Tomonaga-Schwinger representation are identical with (1.5) and (1.6). The difference arises only in the explicit expression for the operators H_k and S_k, which in the Tomonaga-Schwinger representation are defined on an arbitrary space-like hypersurface $\sigma = \sigma(x_\mu)$, which passes through a fixed point in four-dimensional space (x_μ)

$$H_k = H_k(x_\mu), \qquad S_k = S_k[\sigma(x_\mu)], \quad \text{(B)}$$

where S_k is a functional. At the same time the expressions which define $S_k[\sigma]$ also take on a different form. Thus, to Eq. (1.7) in the Tomonaga-Schwinger representation there corresponds (in complete agreement with Ref. 4, where the expression $S_1[\sigma(x_\mu)]$ is investigated) the equation

$$-\frac{\delta}{\delta\sigma(x)} S_k[\sigma(x)] = H_k(x), \quad \text{(1.7a)}$$

by integrating which we shall obtain an explicit expression for S_k

$$\text{(1.10a)}$$
$$S_k[\sigma(x)] = -\frac{1}{2}\int_{-\infty}^{\infty} H_k(x')\,\varepsilon[\sigma(x), \sigma(x')]\,d^4x',$$

where

$$\varepsilon[\sigma(x), \sigma(x')] = \pm 1 \quad \text{for} \quad x_0 - x_0' \gtrless 0. \quad \text{(C)}$$

It may easily be seen that with the aid of the transformations (1.10) one may eliminate all the effects from the equation of motion with the exception of the self-energy parts. The reason for this is that S_k commutes here with H_0 and does not lead to the appearance in the equation of a corresponding counter term with a negative sign. Therefore, the self-energy effects should be eliminated from the equations of motion (1.8) with the aid of subtraction devices used in the regularization procedure.

2. EXAMPLE. PAIR INTERACTION OF SPINOR PARTICLES

The term corresponding to pair interaction of particles (for example, of electrons) in the first order in $e^2(H_{2ee})$ is contained in the Hamiltonian H_2 [see Eq. (1.5)]. The expressions for this term, and also for S_1 were first obtained by Snyder[3]

who investigated in detail the general covariance properties of H_{2ee}, and who also attempted to obtain an expression for the electromagnetic mass of the electron in this approximation. The formula (1.5) was also utilized in the investigation of the problem of positronium[5] and the results obtained in that case agreed with those of other authors[6,7]. Therefore, we shall not investigate expression (1.5) in detail, but shall show how it can be written, and shall compare it with certain well-known formulas.

In the simplest case of electromagnetic interaction of spinor particles of the same mass, the Hamiltonian of Eq. (1.1) consists of the following terms:

$$\mathcal{H} = H_{0e} + H_{0\gamma} + H_{1e}^{tr} + H_{2ee}^{l}, \quad \text{(2.1)}$$

where
$$H_{0e} = \int d\mathbf{p}\,\psi^*(\mathbf{p})\cdot(\alpha\mathbf{p} + \beta m)\,\psi(\mathbf{p}), \quad \text{(2.2)}$$

$$H_{0\gamma} = \int d\mathbf{k}\,k\,a_j^*(\mathbf{k})\,a_j(\mathbf{k}), \quad \text{(2.3)}$$

$$H_{1e}^{tr} = \frac{-q}{4\pi^{3/2}}\int d\mathbf{p}_1\,d\mathbf{p}_2\,d\mathbf{k}\,k^{-1/2}\delta(\mathbf{k} - \mathbf{p}_{12})\,[a_j(\mathbf{k})$$
$$+ a_j^*(-\mathbf{k})]\,\psi_1^*\alpha_j\psi_2, \quad \text{(2.4)}$$

$$H_{2ee}^{l} = \frac{q^2}{2(2\pi)^3} \quad \text{(2.5)}$$

$$\times \int d\mathbf{p}_1\,d\mathbf{p}_2\,d\mathbf{p}_3\,d\mathbf{p}_4\,p_{12}^{-2}\cdot\delta(\mathbf{p}_2 + \mathbf{p}_{34})\,\psi_1^*\psi_2\psi_3^*\psi_4$$

are, respectively, the Hamiltonians for the free electrons, for the free protons, for the interaction between electrons and free protons, and for the Coulomb interaction; α_j, are the Dirac matrices; $\psi_\alpha(\mathbf{p}_k) = \psi_{k\alpha}$ is the absorption operator for an electron with momentum $\mathbf{p}_k$; $a_i(\mathbf{k})$ is the absorption operator for a transverse photon with momentum $\mathbf{k}$;

$$\text{(D)}$$
$$\mathbf{p}_{ab} = \mathbf{p}_a - \mathbf{p}_b; \qquad p_\alpha = |\mathbf{p}_\alpha|; \qquad p_\alpha^0 = \mathbf{p}_\alpha/p_\alpha;$$

$\mathbf{p}$, $\mathbf{k}$ are three-dimensional momenta, and the system of units is used in which $\hbar = c = 1$, $q = \sqrt{4\pi/137}$

In accordance with expression (1.10)

$$S_1 = \frac{-q}{4i\pi^{3/2}} \quad \text{(2.6)}$$

$$\times \int \frac{d\mathbf{p}_1\,d\mathbf{p}_2\,d\mathbf{k}}{V\,k}\,\delta(\mathbf{k} - \mathbf{p}_{12})\frac{a_i(\mathbf{k})\,\psi_1^*\alpha_i\psi_2 - a_i^*(\mathbf{k})\,\psi_2^*\alpha_i\psi_1}{k - \varepsilon_1 + \varepsilon_2}.$$

Now substituting (2.5), (2.6) into (1.5) and retaining there only terms not containing photon operators, we obtain for H_{2ee}

$$H_{2ee} = \frac{q^2}{2(2\pi)^3}$$

$$\times \int dp_1\, dp_2\, dp_3\, dp_4\, \delta\,(p_{12}+p_{34}) \cdot \psi_{1\alpha}^* \psi_{2\beta} \psi_{3\mu}^* \psi_{4\nu}$$

$$\times \left\{ \frac{\delta_{\alpha\beta}\delta_{\mu\nu}}{p_{12}^2} - \frac{\delta_{ij}-p_{12i}^0 p_{12j}^0}{2p_{12}} \right.$$

$$\times \left. \left(\frac{\alpha_{i\alpha\beta}\alpha_{j\mu\nu}}{p_{12}-\varepsilon_1+\varepsilon_2} + \frac{\alpha_{i\alpha\beta}\alpha_{j\mu\nu}}{p_{12}-\varepsilon_4+\varepsilon_3} \right) \right\} \quad (2.7)$$

— an expression which describes pair forces between two electrons. Its covariant nature finds its expression in particular in the fact that in the case of scattering it reduces to the well-known formula due to Möller**

$$H_{2ee}^{\text{pac}} = \frac{q^2}{2(2\pi)^3} \quad (2.8)$$

$$\times \int dp_1\, dp_2\, dp_3\, dp_4\, \delta\,(p_{12}+p_{34}) :$$

$$\frac{\psi_1^* \alpha_i \psi_2 \psi_3^* \alpha_i \psi_4 - \psi_1^* \psi_2 \psi_3^* \psi_4}{p_{12}^2-(\varepsilon_1-\varepsilon_2)^2} :$$

In the case of bound states the first non-relativistic approximation (2.7) corresponds to the non-exchange part of pair interactions and agrees with Breit's formula

$$H_{2ee} = \frac{q^2}{2(2\pi)^3} \quad (2.9)$$

$$\times \int dx_1\, dx_2\, \frac{2\delta_{\alpha\beta}\delta_{\mu\nu}+\alpha_{\alpha\beta}\alpha_{\mu\nu}+(\alpha_{\alpha\beta}x_{12}^0)(\alpha_{\mu\nu}x_{12}^0)}{2x_{12}}$$

$$\times : \psi_\alpha^*(x_1)\,\psi_\beta(x_1)\,\psi_\mu^*(x_2)\,\psi_\nu(x_2):.$$

It follows from (2.9) [or from (2.7)] that Breit's formula for the interaction energy is equally applicable to the description of non-exchange interaction between electrons, between positrons and between electrons and positrons:

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** According to Wick's theorem

$$H_{2ee} = :H_{2ee}: + :\hat{H}_{2ee}:$$

is equal to the sum of its normal product which describes pair interactions and the normal product with one pairing which describes the effect of the electron self-energy.

$$:H_{2ee}^{\text{non-exch}}: \varphi_{ab}(p_1, p_2) = (L_{1a}^+ L_{2b}^+ + L_{1a}^- L_{2b}^- + 2L_{1a}^+ L_{2b}^-)$$

$$\times \int dp_1'\, dp_2'\, G_{ab}^{\text{sym}}(p_1, p_2; p_1', p_2')\, \varphi_{ab}(p_1', p_2'). \quad (2.10)$$

Here $\varphi_{ab}(p_1 p_2)$ is the wave function in configuration space for a system of two particles; G_{ab}^{sym} is the expression for the interaction energy which occurs inside { } in (2.9) or in (2.7);

$$L_k^\pm = \frac{e_k \pm (\alpha p_k + \beta m)}{2e_k}$$

are projection operators. This result does not agree with the "exact three-dimensional equation" (13) of Ref. 8.

$$:H_{2ee}: \varphi_{ab}(p_1, p_2) = (L_{1a}^+ L_{2b}^+ - L_{1a}^- L_{2b}^-) \quad (2.11)$$

$$\times \int dp_1'\, dp_2'\, G_{ab}^{\text{chm}}(p_1, p_2; p_1'; p_2')\, \varphi_{ab}(p_1', p_2'),$$

obtained in the Bethe-Salpeter theory. As may be seen from (2.11) the "exact three-dimensional equation" of Salpeter excludes first of all any possibility of interaction between electrons and positrons, and secondly leads to different signs for the interaction between pairs of electrons and between pairs of positrons. Both these facts contradict reality, and therefore the "exact three-dimensional equation" (13) of reference 8 can be used only to describe processes occurring between electrons.

The covariant nature of (2.7) also manifests itself in the fact that it contains a covariant expression for the electromagnetic mass of the electron. In order to obtain it we must calculate the matrix element $\langle a | : \hat{H}_{2ee} : |a\rangle$, which turns out to be equal to

$$\delta m = \frac{q^2}{4m(2\pi)^4} \quad (2.12)$$

$$\times \int dp_2 \left\{ \frac{1}{p_{12}} - \frac{1}{e_2} + \frac{1}{e_2}\frac{m^4}{(p_1 p_2 - e_1 e_2)^2 - m^4} \right\}$$

and contains only a logarithmic divergence as $p_2 \to \infty$.*

The calculation of δm represents a very laborious and painstaking effort, and therefore the explicit formulation of the Hamiltonians of the N-th orders

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* In (2.12) the matrix element was calculated for only the diagonal part of the operator: $\hat{H}_{2ee}:$, since the matrix element of its non-diagonal part is identically equal to zero due to the violation of the law of conservation of parity.

in the Schrödinger representation may be considered to be worth while only in those cases when the process under investigation is not connected with field divergences and does not require the regularization of the equations. In such cases if we write the equations of motion in the Schrödinger representation we shall obtain expressions of simpler structure which do not require integration over the fourth coordinates. However, in order to investigate field processes in quantum electrodynamics it is more convenient to write the equation of motion in the Tomonaga-Schwinger form. This guarantees a manifestly covariant form for the Hamiltonian of each order, and allows the equations of motion to be regularized directly.

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Some Sum Rules for the Cross Sections of Electric Quadrupole Transitions in the Nuclear Photoeffect

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Two parameters which characterize the cross sections for quadropole transitions in the nuclear photoeffect are estimated [formulas (22), (25)]. Other (known) parameters which characterize the dipole transition cross section are used for this purpose. The estimates indicate that, in intermediate and heavy nuclei, the "center of gravity" of the quadropole transition cross section is situated at energies exceeding 10-20 mev.

THE previous theoretical estimates [1-3] of the parameters which characterize the total cross sections for electric quadrupole transitions in the nuclear photoeffect are based on the liquid drop model of the nucleus (let us denote this total cross section by $\sigma_{E2}(\nu)$, where ν is the photon energy).

These estimations allow us to assume that the cross section σ_{E2} has at least two maxima. The first maximum is in the range of energies of the order of 1 mev, which correspond to the eigenfrequency of nuclear surface vibration. The second maximum takes place on the right of the dipole resonance energy, at energies of the order of 20-40 mev — which correspond to the lowest eigenfrequency of the nuclear matter polarization quadrupole vibrations. The cross section area under the second maximum is appreciably larger than the cross section area under the first one.

In order to obtain a model-independent confirmation of the conclusion of Danos and Steinwedel[2-3] on the existence and on the role of the second maximum, we will consider two sum rules which characterize the cross section for quadrupole transitions. These sum rules [see formulas (19), (20) and also (22), (25)] relate the cross sections $\sigma_{E2}(\nu)$ with some constants (with respect to ν) which depend on the nuclear structure. For the calculation of one of these constants (the calculation of the other one is trivial), we make use of that phenomenological expression for the coordinate distribution of two protons in a nucleus which is experimentally confirmed in the case of dipole transitions. For this purpose, the first step of this work consists in reconsidering two known sum rules which correspond to the cross section for dipole transitions.

1. DIPOLE TRANSITIONS

In the present section, we are interested in the