in the Schrödinger representation may be considered to be worth while only in those cases when the process under investigation is not connected with field divergences and does not require the regularization of the equations. In such cases if we write the equations of motion in the Schrödinger representation we shall obtain expressions of simpler structure which do not require integration over the fourth coordinates. However, in order to investigate field processes in quantum electrodynamics it is more convenient to write the equation of motion in the Tomonaga-Schwinger form. This guarantees a manifestly covariant form for the Hamiltonian of each order, and allows the equations of motion to be regularized directly.

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Some Sum Rules for the Cross Sections of Electric Quadrupole Transitions in the Nuclear Photoeffect

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Two parameters which characterize the cross sections for quadropole transitions in the nuclear photoeffect are estimated [formulas (22), (25)]. Other (known) parameters which characterize the dipole transition cross section are used for this purpose. The estimates indicate that, in intermediate and heavy nuclei, the "center of gravity" of the quadropole transition cross section is situated at energies exceeding 10-20 mev.

THE previous theoretical estimates[1-3] of the parameters which characterize the total cross sections for electric quadropole transitions in the nuclear photoeffect are based on the liquid drop model of the nucleus (let us denote this total cross section by $\sigma_{E2}(\nu)$, where ν is the photon energy).

These estimations allow us to assume that the cross section σ_{E2} has at least two maxima. The first maximum is in the range of energies of the order of 1 mev, which correspond to the eigenfrequency of nuclear surface vibration. The second maximum takes place on the right of the dipole resonance energy, at energies of the order of 20-40 mev — which correspond to the lowest eigenfrequency of the nuclear matter polarization quadropole vibrations. The cross section area under the second maximum is appreciably larger than the cross section area under the first one.

In order to obtain a model-independent confirmation of the conclusion of Danos and Steinwedel[2-3] on the existence and on the role of the second maximum, we will consider two sum rules which characterize the cross section for quadropole transitions. These sum rules [see formulas (19), (20) and also (22), (25)] relate the cross sections $\sigma_{E2}(\nu)$ with some constants (with respect to ν) which depend on the nuclear structure. For the calculation of one of these constants (the calculation of the other one is trivial), we make use of that phenomenological expression for the coordinate distribution of two protons in a nucleus which is experimentally confirmed in the case of dipole transitions. For this purpose, the first step of this work consists in reconsidering two known sum rules which correspond to the cross section for dipole transitions.

1. DIPOLE TRANSITIONS

In the present section, we are interested in the two following sum rules which have been considered previously[4-8]

$$\int_0^\infty \sigma_{E1}(\nu)\,d\nu = 2\pi^2 \frac{\hbar}{c}\frac{1}{i\hbar}\langle D\dot{D} - \dot{D}D\rangle, \tag{1}$$

$$\int_0^\infty \sigma_{E1}(\nu)\frac{d\nu}{\nu} = \frac{4\pi^2}{\hbar c}\langle D^2\rangle. \tag{2}$$

$\sigma_{E1}(\nu)$ is the total cross section for dipole transitions and the symbol $\langle\,\rangle$ means averaging over the ground state of the nucleus; D is any of the components of the dipole moment operator in the long wave length approximation. To be specific, let $D \equiv D_z = \sum_\alpha e_\alpha z_\alpha$, where e_α and z_α are the charge and the z-component of the coordinate $\mathbf{q}_\alpha$ of the α-th nucleon. The dot above the operator stands for $i/\hbar$ times the commutator with the Hamiltonian of the nucleus.

The coordinates $\mathbf{q}_\alpha$ are related by the condition

$$\sum_\alpha \mathbf{q}_\alpha = 0, \tag{3}$$

which expresses the fact that we are dealing with the relative coordinates subspace.

Using Eq. (3) and after some transformations, we get

$$\int_0^\infty \sigma_{E1}(\nu)\,d\nu = 2\pi^2 \frac{\hbar e^2}{mc}\frac{NZ}{A}(1+\Delta). \tag{4}$$

where N and Z are the neutron and proton numbers; m is the nucleonic mass; Δ is a positive quantity for the calculation of which it is necessary to know 1) the ground state wave function of the nucleus, and 2) the exchange part of the nuclear potential; in the absence of exchange, $\Delta = 0$.

The theory[4,5] gives the following estimate for Δ:

$$\Delta \approx 0.1A^2/NZ \approx 0{,}4. \tag{5}$$

This estimate is in excellent agreement with a more exact investigation which can be carried out in the case of photofission of the deuteron.

The right hand side of the sum rule (2) is computed in Ref. 7. (See also Refs. 6 and 8). In the present Section, we are going to consider the sum rule (2) from a somewhat different, phenomenological point of view.

Let us introduce the functions $n_p(\mathbf{r})$ and $n_{2p}(\mathbf{r}, \mathbf{r}')$ describing the coordinate distribution of two protons in a nucleus

$$n_{2p}(\mathbf{r}, \mathbf{r}') = [Z(Z-1)]^{-1} \times \sum\sum_{p_1 \neq p_2} \langle \delta(\mathbf{r}-\mathbf{q}_{p_1})\,\delta(\mathbf{r}'-\mathbf{q}_{p_2})\rangle, \tag{6}$$

$$n_p(\mathbf{r}) = \int n_{2p}(\mathbf{r}, \mathbf{r}')\,d\mathbf{r}'. \tag{7}$$

Using the distributions n_p and n_{2p} the expression $\langle D^2\rangle$ can be rewritten in the form

$$\langle D^2\rangle = \frac{e^2}{3}[Z\langle r^2\rangle_p + Z(Z-1)\langle \mathbf{r}\mathbf{r}'\rangle_{2p}], \tag{8}$$

$$\langle r^2\rangle_p = \int r^2 n_p(\mathbf{r})\,d\mathbf{r}; \tag{9}$$

$$\langle \mathbf{r}\mathbf{r}'\rangle_{2p} = \int \mathbf{r}\mathbf{r}' n_{2p}(\mathbf{r}, \mathbf{r}')\,d\mathbf{r}\,d\mathbf{r}'.$$

Let us call "zeroth" the nucleon coordinate distribution for which all the distributions

$$n_{\alpha\beta}(\mathbf{r},\mathbf{r}') = \langle \delta(\mathbf{r}-\mathbf{q}_\alpha)\,\delta(\mathbf{r}'-\mathbf{q}_\beta)\rangle$$

do not depend on α and β; in particular, they do not depend on whether the α-th and β-th nucleons are both protons, both neutrons, or are different.

The functions n_{2n} and n_{pn} which describe the coordinate distribution of two neutrons and of two different nucleons are, in the case of a zeroth distribution, equal among themselves and equal to n_{2p}. Taking this into account, and averaging the square of the Eq. (3) over the nuclear ground state, we find that the following equality is true in the case of a zeroth distribution:

$$\langle r^2\rangle_p^{(0)} + (A-1)\langle \mathbf{r}\mathbf{r}'\rangle_{2p}^{(0)} = 0. \tag{10}$$

Substituting (10) into (8), we get

$$\langle D^2\rangle^{(0)} = \frac{e^2}{3}\frac{NZ}{A-1}\langle r^2\rangle_p^{(0)} \approx \frac{e^2}{5}\frac{NZ}{A-1}R^2. \tag{11}$$

where R is the nuclear radius.

It follows that in the general case the sum rule (2) can be transformed into the form:

$$\int_0^\infty \sigma_{E1}(\nu)\frac{d\nu}{\nu} = \frac{4\pi^2}{5}e^2\frac{NZ}{A-1}R^2(1-\Lambda), \tag{12}$$

$$\Lambda = -[(Z-1)(A-1)/NZ\langle r^2\rangle_p] \times \int \mathbf{r}\mathbf{r}' n_{2p}^{(1)}(\mathbf{r}, \mathbf{r}')\, d\mathbf{r}\, d\mathbf{r}', \quad (13)$$

$$n_{2p}^{(1)}(\mathbf{r}, \mathbf{r}') = n_{2p}(\mathbf{r}, \mathbf{r}') - n_{2p}^{(0)}(\mathbf{r}, \mathbf{r}'). \quad (14)$$

In what follows, we are going to assume that the coordinate distribution of a single nucleon, given by the function $n_p(\mathbf{r})$, is the same in the zeroth case and in a non-zeroth case; it follows then that $n_{2p}^{(1)}(\mathbf{r}, \mathbf{r}')$ satisfies the condition

$$\int n_{2p}^{(1)}(\mathbf{r}, \mathbf{r}')\, d\mathbf{r}' = 0. \quad (15)$$

The previous calculations[6-8] of the quantity $\langle D^2 \rangle$ show that the main reason for the deviation of the true coordinate distribution of identical nucleons from the zeroth distribution is the Pauli principle; according to it, the approach to one another of two identical nucleons is appreciably less probable than that of two different nucleons. Therefore, to get the qualitative character of the function $n_{2p}^{(1)}$, it will be sufficient to consider the simplest gas model of the nucleus – so far as this model takes the Pauli principle into account. The result of such an investigation amounts to the following: the function $n_{2p}^{(1)}$ is approximately proportional to $n_{2p}^{(0)}$ everywhere except in a region where the distance between the points $\mathbf{r}$ and $\mathbf{r}'$ is less or of the order of a, a being comparable to the mean internucleonic distance. Considering this region as being small with respect to the nuclear volume (i.e. neglecting the edge effect), we can approximate the function $n_{2p}^{(1)}(\mathbf{r}, \mathbf{r}')$ with the help of a δ-function of $\mathbf{r}-\mathbf{r}'$:

$$n_{2p}^{(1)}(\mathbf{r}, \mathbf{r}') = \Omega_p\Big[n_{2p}^{(0)}(\mathbf{r}, \mathbf{r}') - \delta(\mathbf{r}-\mathbf{r}')\, n_p\Big(\frac{\mathbf{r}+\mathbf{r}'}{2}\Big)\Big]. \quad (16)$$

where Ω_p means (a^3/R^3).

Substituting (16) into (13), we get

$$\Lambda = \Omega_p (Z-1)\, A/N. \quad (17)$$

2. QUADROPOLE TRANSITIONS

The expression of the cross section for quadrupole transitions in the long wave length approximation has the form

$$\sigma_{E2}(\nu) = 4\pi^2 \sum_f{}' \left(\frac{\nu}{\hbar c}\right)^3 |Q_{f0}|^2\, \delta(E_f - E_0 - \nu). \quad (18)$$

Q is any of the non-diagonal components of the quadrupole moment operator. To be specific, let

$$Q \equiv Q_{13} = {}^1/_2 \sum_\alpha e_\alpha x_\alpha z_\alpha.$$

Using the matrix multiplication rule, we readily get

$$\int_0^\infty \sigma_{E2}(\nu)\frac{d\nu}{\nu^2} = \frac{4\pi^2}{(\hbar c)^3}\frac{-i\hbar}{2}\langle Q\dot{Q} - \dot{Q}Q\rangle; \quad (19)$$

$$\int_0^\infty \sigma_{E2}(\nu)\frac{d\nu}{\nu^3} = \frac{4\pi^2}{(\hbar c)^3}\langle Q^2\rangle. \quad (20)$$

In the derivation of Eq. (20), we have neglected the mean square of the nuclear eigen-quadrupole moment with respect to $\overline{(Q_{00})^2}$, i.e., we have assumed $\langle Q^2\rangle \gg \overline{Q_{00})^2}$. Q_{00} is the mean value of the operator Q in the nuclear ground state. The bar means averaging over degeneracy (if any).

It is easy to note that Eq. (19) is analogous to Eq. (1), and Eq. (20) to Eq. (2). This analogy is very useful. Let us first consider the right hand side of Eq. (19). After elementary calculations, we get

$$-(i\hbar/2)\langle Q\dot{Q} - \dot{Q}Q\rangle = (\hbar^2/4m)\, e^2 Z \langle x^2\rangle_p (1+\xi). \quad (21)$$

$\langle X^2\rangle \approx R^2/5$; ξ is a positive quantity which vanishes in the absence of exchange forces. Let us try to obtain the relationship between ξ and the quantity Δ introduced above. For this purpose let us assume some simple expression for the nuclear exchange potential energy U; for instance,

$$U = {}^1/_2 \sum_{\alpha,\beta}{}' U_{\alpha\beta}(\mathbf{q}_\alpha - \mathbf{q}_\beta)\, P_{\alpha\beta},$$

where $P_{\alpha\beta}$ is an operator representing the coordinates $\mathbf{q}_\alpha$ and $\mathbf{q}_\beta$ (calculations with such a U are carried out in Ref. 4). Substituting this expression in the sum rule (1) and (19), we get after some calculations

$$\Delta = -\sum_{\alpha\beta} \langle \frac{1}{2\hbar^2} (e_\alpha - e_\beta)^2 (z_\alpha - z_\beta)^2 U_{\alpha\beta} P_{\alpha\beta} \rangle \Big/ \sum_\alpha \langle \frac{e_\alpha}{m} \rangle,$$

$$\xi = -\sum_{\alpha\beta} \langle \frac{1}{2\hbar^2} (e_\alpha - e_\beta)^2 (x_\alpha z_\alpha - x_\beta z_\beta)^2 U_{\alpha\beta} P_{\alpha\beta} \rangle \Big/ \sum_\alpha \langle \frac{e_\alpha}{m} (x_\alpha^2 + z_\alpha^2) \rangle.$$

Comparing these expressions, one sees that ξ has a tendency to be smaller than Δ by a factor of about $(r_0/R)^2$. This estimate is of course very inaccurate; however, it does not matter for what follows whether we put $\xi = 0$, in the other limiting case, $\xi = \Delta$. To be specific, let $\xi = 0$. Finally

$$\int_0^\infty \sigma_{E2}(\nu) \frac{d\nu}{\nu^2} = \frac{\pi^2}{5} \left(\frac{e^2}{\hbar c}\right) \left(\frac{1}{mc^2}\right) ZR^2. \tag{22}$$

For the evaluation of the right hand side of Eq. (20), let us first recall that in the evaluation of $< D^2 >$ we had to take into account the relationship (3) between the coordinates q_α and Eq. (10), which follows from (3). In particular, we were not able to write the zeroth distribution $n_{2p}^{(0)}(\mathbf{r}, \mathbf{r}')$ in the form of a product of distributions $n_p(\mathbf{r})$ and $n_p(\mathbf{r}')$ because it would have led to a qualitatively incorrect result: the right hand side of the sum rule (4) would have been proportional to Z rather than to ZN/A. In the present case, however, where we have to compute $< Q^2 >$, we can consider the coordinates q_α as independent, not taking Eqs. (3) and (10) into account. It is easy to show that the error due to this approximation is of the order of $1/A$. In particular, we will assume that the zeroth distribution $n_{2p}^{(0)}(\mathbf{r}, \mathbf{r}')$ is simply a product of the distribution $n_p(\mathbf{r})$ and $n_p(\mathbf{r}')$. Further the correction for the zeroth distribution is given by formula (16) according to the condition (17) and with the additional assumption on the multiplicativity of

$$n_{2p}^{(0)}(\mathbf{r}, \mathbf{r}').$$

With these assumptions, the expression

$$\langle Q^2 \rangle = {}^1/_4 e^2 [Z \langle x^2 z^2 \rangle_p + Z(Z-1) \langle xx'zz' \rangle_{2p}] \tag{23}$$

becomes equal to

$$\langle Q^2 \rangle = {}^1/_4 e^2 Z \langle x^2 z^2 \rangle_p (1 - N\Lambda/A). \tag{24}$$

Letting $\langle x^2 z^2 \rangle_p \approx R^4/35$, , we finally get

$$\int_0^\infty \sigma_{E2}(\nu) \frac{d\nu}{\nu^3} = \frac{\pi^2}{35} \left(\frac{e^2}{\hbar c}\right) (1 - N\Lambda/A). \tag{25}$$

We see that in this case the correlation has an effect twice as small ($N/A \approx ½$) as in the case of the sum rule (12) for dipole transitions.

As already mentioned, the function $\Lambda = \Lambda(A)$ can be obtained directly from experiment by substituting the observed value of σ_{E1} into the left hand side of the sum rule (12). However, the errors on the measurements of σ_{E1} are now such that the theoretical values of Λ, calculated in Ref. 7, are probably closer to truth than the experimental ones. The theoretical values of Λ can be approximated with sufficient accuracy by the following formula

$$\Lambda = 0.84 (1 + 22/A). \tag{26}$$

For $A = 50$, $\Lambda \approx 0.6$; for $A = 240$, $\Lambda \approx 0.76$. These values confirm the initial assumption on the smallness of the correlation radius a with respect to the nuclear radius R.

Consider the ratio

$$\bar{\nu}_3 = \int_0^\infty \sigma_{E2}(\nu) \frac{d\nu}{\nu^2} \Big/ \int_0^\infty \sigma_{E2}(\nu) \frac{d\nu}{\nu^3} \tag{27}$$

$$= 7 \frac{(\hbar c)^2}{mc^2} \frac{1}{R^2} \frac{1}{1 + N\Lambda/A} \approx \frac{2.4 \cdot 10^2}{A^{2/3}(1 - \Lambda N/A)}.$$

We took $R = 1.1 \times 10^{-13} A^{2/3}$ cm. In the case of a nucleus with $A \approx 50$, Eq. (27) gives $\bar{\nu}_3 \approx 25$ mev, in the case of uranium $\bar{\nu}_3 \approx 9$ mev. Therefore,

$$\bar{\nu}_3 \approx 10 - 20 \text{ MeV}. \tag{28}$$

Equation (28) gives a lowest bound for the position of the "center of gravity" of the cross section for quadropole transitions. Indeed, the energy ν_0 corresponding to the "center of gravity" is defined by

$$\bar{\nu}_0 = \int_0^\infty \sigma_{E2}(\nu) \nu d\nu \Big/ \int_0^\infty \sigma_{E2}(\nu) d\nu, \tag{29}$$

whence $\bar{\nu}_0 > \bar{\nu}_3$.

Hence we conclude that the "center of gravity" of the cross section for quadropole transition is on the right of the dipole resonance, at energies exceeding 10-20 mev.

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Other Errata

Page	Column	Line	Reads	Should Read
Volume 4				
38	1	Eq. (3)	$\dots\frac{\pi r^2\rho^2\rho^2{}_n}{\rho_s^2},$	$\dots\frac{\pi r^2\rho^2\rho_n}{\rho_s^2},$
196		Date of submittal	May 7, 1956	May 7, 1955
377	1	Caption for Fig. 1	$\delta_{35}=\eta-21\cdot\eta^5$	$\delta_{35}=-21^\circ\eta^5\cdot$
377	2	Caption for Fig. 2	$\alpha_3=6.3^\circ\ \eta$	$\alpha_3=-6.3^\circ\eta$
516	1	Eq. (29)	$s^2/c^2\dots$	s/c
516	2	Eqs. (31) and (32)	Replace A_1s^2/c^2 by A_1	
497		Date of submittal	July 26, 1956	July 26, 1955
900	1	Eq. (7)	$\dots\frac{i}{4\pi}\sum_{c,\,\alpha}\frac{\partial w_a(t,\,P)}{\partial P^\alpha}\dots$ (This causes a corresponding change in the numerical coefficients in the expressions that result from the calculation of the effects of the plasma particles on each other).	$\dots 2\pi^2 i\sum_{c,\,\alpha}\dots$
804	2	Eq. (1)	$\dots\exp\{-(\overline{T}-V')\}$	$\dots\exp\{-(\overline{T}-V')\tau^{-1}\}$
Volume 5				
59	1	Eq. (6)	$v_l\,(1\partial F_0/\partial x)+\dots$ where E_l is the projection of the electric field E on the direction 1	$\overline{(v\partial F_0/\partial x)}+\dots$ where the bar indicates averaging over the angle θ and E_l is the projection of the electric field E along the direction 1
91	2	Eq. (26)	$\Lambda=0.84\,(1+22/A)$	$\Lambda=0.84/(1+22/A)$
253		First line of summary	$Tl^{204,\ 206}$	$Tl^{203,\ 205}$
318	1	Figure caption	$\dots\ e^2mc^2=2.8\cdot10^{-23}$ cm,	$\dots e^2/mc^2=2.8\cdot10^{-13}$ cm,
398		Figure caption	... to a cubic relation. A series of points etc.	... to a cubic relation, and in the region 10 – 20°K to a quadratic relation. A series of points ●, coinciding with points O, have been omitted in the region above 10°K.