corrections contained in (13), were computed with a simple Hylleraas function instead of the eight–term function; thus there is no guarantee that the value of some of these terms would not change if they were evaluated with an eight–term function. This is especially true of the orbit–orbital part of the 2nd order relativistic correction,[8] where not only the magnitude, but even the sign depends on the choice of wave function. Furthermore, one should evaluate more correctly the Lamb shift in the electric field of the nucleus; this has been done so far using screened wave functions.

As for the relativistic corrections whose sign does not depend on the choice of wave function, for example the spin–spin part of the two and three–body interactions, it may be expected that the use of an eight–term Hylleraas function instead of the simpler one will not produce a noticeable change in the numerical values of these small quantities; this is indicated, for example, by the fact that in computing E_n^0 with various Hylleraas functions, the results were found to differ only in the fourth place.

On the basis of this analysis it may be expected that including the effects discussed above will lead to agreement between theoretical and experimental value for I_0 within the limits of experimental accuracy.

In conclusion the author wishes to express his gratitude to Iu. M. Shirokov for his continued interest in this analysis and to V. N. Ts'itovich for checking the formula.

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SOVIET PHYSICS JETP VOLUME 5, NUMBER 2 SEPTEMBER, 1957

Non-Linear Theory of Betatron Oscillations in a Strong Focusing Synchrotron

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(Submitted to JETP editor November 15, 1955)
J. Exptl. Theoret. Phys. (U.S.S.R.) 32, 316–322 (February, 1957)

A new method has been developed for investigating betatron resonances. Parametric resonances are investigated.

1. EQUATIONS OF MOTION

THE equations describing betatron oscillations about some plane periodic orbit in a strong focussing synchrotron[1] have the following form:

$$\frac{d^2r}{d\theta^2}+\frac{1}{P}\left(\frac{l}{2\pi}\right)^2\frac{\partial H}{\partial r}r = \frac{1}{P}\left(\frac{l}{2\pi}\right)^2\left\{\frac{1}{\rho}(P-\rho H_0)-\delta\left(\frac{\partial H}{\partial r}\right)r - \frac{\partial H}{\partial z}z - \sum_{n\geqslant 2}\frac{\partial^n H}{\partial r^n}\left[\frac{r^n}{n!}-\frac{r^{n-2}z^2}{2!\cdot(n-2)!}+\frac{r^{n-4}z^4}{4!\,(n-4)!}-\cdots\right]\right\}; \quad (1)$$

$$\frac{d^2z}{d\theta^2}-\frac{1}{P}\left(\frac{l}{2\pi}\right)^2\frac{\partial H}{\partial r}z=\frac{1}{P}\left(\frac{l}{2\pi}\right)^2\left\{H_{\rho_0}+\delta\left(\frac{\partial H}{\partial r}\right)z - \frac{\partial H}{\partial z}r+\sum_{n\geqslant 2}\frac{\partial^n H}{\partial r^n}\left[\frac{r^{n-1}z}{(n-1)!}-\frac{r^{n-3}z^3}{3!\,(n-3)!}+\cdots\right]\right\}, \quad (2)$$

where r denotes the radial and z the vertical deviation of the particles from their periodic orbit; $\rho(\theta)$ is the radius of the orbit. The "angle" θ changes by 2π over a period length l; $\partial H/\partial r$ is the gradient of the magnetic field, $\delta(\partial H/\partial r)$ is the error in the gradient; $1/P = e/cp$, where p is the momentum of the particles. The series of unessential terms in (1) and (2) are discarded; we neglect H_θ in the frequency.

Let us set

$$r = x(\theta)\Phi_r^*(\theta)\exp(i\nu_r\theta) + x^*(\theta)\Phi_r(\theta)\exp(-i\nu_r\theta), \quad (3)$$

$$r'_\theta = x(\theta)\frac{d\Phi_r^*}{d\theta}\exp(i\nu_r\theta) + x^*(\theta)\frac{d\Phi_r}{d\theta}\exp(-i\nu_r\theta);$$

$$z = y(\theta)\Phi_z^*(\theta)\exp(j\nu_z\theta) + y^*(\theta)\Phi_z(\theta)\exp(-i\nu_z\theta), \quad (4)$$

$$z'_\theta = x(\theta)\frac{d\Phi_z^*}{d\theta}\exp(i\nu_z\theta) + y^*(\theta)\frac{d\Phi_z}{d\theta}\exp(-i\nu_z\theta),$$

where $\Phi_{r,z}$ are the Floquet functions with well-known properties

$$\Phi_{r,z}(\theta + 2\pi) = \Phi_{r,z}(\theta)\exp(2\pi i\nu_{r,z}),$$

$\nu_{r,z}$ is equal to the number of betatron oscillations in the length l. This yields

$$\frac{dx}{d\theta} + i\nu_r x = i\left(\frac{l}{2\pi}\right)^2\frac{1}{P\omega_r}f_r(\theta) \quad (5)$$

$$\times\left\{\frac{1}{\rho}(P - \rho H_0) - \delta\left(\frac{\partial H}{\partial r}\right)(xf_r^* + x^*f_r)\right.$$

$$\left. - \frac{\partial H}{\partial z}(yf_z^* + y^*f_z) - \sum_{n\geqslant 2}\frac{\partial^n H}{\partial r^n}\left[\frac{(xf_r^* + x^*f_r)^n}{n!}\cdots\right]\right\};$$

$$\frac{dy}{d\theta} + i\nu_z y \quad (6)$$

$$= i\left(\frac{l}{2\pi}\right)^2\frac{1}{P\omega_z}f_z(\theta)\left\{H_{\rho_0} + \delta\left(\frac{\partial H}{\partial r}\right)(yf_z^* + y^*f_z)\right.$$

$$\times\frac{\partial H}{\partial z}(xf_r^* + x^*f_r)$$

$$\left. + \sum_{n\geqslant 2}\frac{\partial^n H}{\partial r^n}\left[\frac{(xf_r^* + x^*f_r)^{n-1}(yf_z^* + y^*f_z)}{(n-1)!} - \cdots\right]\right\},$$

$$-i\omega_{r,z} = \Phi_{r,z}\frac{d\Phi_{r,z}^*}{d\theta} - \Phi_{r,z}^*\frac{d\Phi_{r,z}}{d\theta}, \quad (7)$$

$$\Phi_{r,z}(\theta) = f_{r,z}(\theta)\exp(i\nu_{r,z}\theta); \quad (8)$$

$$f_{r,z}(\theta) = f_{r,z}(\theta + 2\pi).$$

It is important to note that Eqs. (5) and (6) have periodic coefficients with maximum period $2\pi M$, where M is the number of periodic elements.

2. FIRST RESONANCE APPROXIMATION

The distance between linear resonances is equal to $1/2M$ in the $\nu_r\,\nu_z$ plane. One of the main problems of the theory is to locate the boundaries of the so-called safe region between these resonances, i.e., a region where the amplitude does not exceed a certain given value. It is therefore of the greatest interest to know the values of ν_r, ν_z which lie on the boundaries of the safe region, i.e., sufficiently near to the exact resonance values. This allows us to single out the resonance harmonics in a disturbance and to neglect non-resonance ones in the first approximation. Since the disturbance is generally small, this operation gives a good approximation. We shall show in Sec. 4 how the non-resonance harmonics of a disturbance must be treated. The resonance equations obtained in this simple fashion in the first approximation coincide with the so-called "abbreviated" equations obtained by the method of Kryslov and Bogoliubov.[2]

An important advance can be made in the method. Specifically, it may be noted that with the proper choice of variables the resonance equation may be written in the form of Hamilton's equation. In all cases of practical interest, these variables turn out to be the squares of the amplitudes, A_r^2, A_z^2, and some phase variables φ_r, φ_z which usually appear in the "abbreviated" equations. This is especially useful for finding an integral of the motion in the difficult case of many simultaneous resonances. A remarkable property of the Hamiltonian is that it is independent of the angular variable θ; $\mathcal{H} = \mathcal{H}(A_r^2, A_z^2, \varphi_r, \varphi_z)$. This opens up wide possibilities for analysis.

3. PARAMETRIC RESONANCE

On the left and the lower boundaries of the safe region (Fig. 1), one can neglect the effect of all resonances except the parametric one. Let us consider, for instance, the left boundary. In that case the resonace equations in the first "resonance approximation" take the following form (leaving out everywhere the index r):

$$dA^2/d\theta = 2gA^2\cos\varphi = -\partial\mathcal{H}/\partial\varphi, \quad (9)$$

$$d\varphi/d\theta = 2(\varepsilon + \alpha A^2 - g\sin\varphi) = \partial\mathcal{H}/\partial A^2, \quad (10)$$

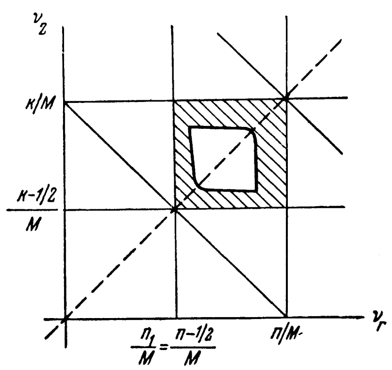

FIG. 1.

where, by definition,

$$x = \frac{A}{2|\Phi|_{\max}} \exp\left[-i\left(\frac{\varphi}{2} - \frac{n\theta}{2M} + \frac{\gamma}{2}\right)\right], \quad (11)$$

$$A = 2|\Phi|_{\max}|x|,$$

$$\varepsilon = \nu^{(1)} - \frac{n}{2M}, \quad (12)$$

$$\nu^{(1)} = \nu + \frac{l^2}{8\pi^3 M w} \int_0^{2\pi M} |\Phi|^2 \delta\left(\frac{\partial H/\partial r}{H_0\rho_0}\right) d\theta,$$

$$ge^{i\gamma} = -\frac{il^2}{8\pi^3 M w} \int_0^{2\pi M} \delta\left(\frac{\partial H/\partial r}{H_0\rho_0}\right) f^2 e^{in\theta/M} d\theta \quad (13)$$

$$\approx -\frac{il^2}{8\pi^3 M w} \int_0^{2\pi M} \delta\left(\frac{\partial H/\partial r}{H_0\rho_0}\right) \Phi^2 d\theta,$$

$$\alpha = \frac{l^2}{16\pi^3 w} \frac{1}{4|\Phi|^2_{\max}} \frac{1}{P} \int_0^{2\pi} \frac{\partial^3 H}{\partial r^3} |\Phi|^4 d\theta. \quad (14)$$

It should be stated that we have made use here of the general non-linear property of a strong focussing accelerator

$$\partial^3 H(\theta)/\partial r^3 \approx -\partial^3 H(\theta + \pi)\partial r^3. \quad (15)$$

This is related to the fact that the frequencies ν_r and ν_z are generally chosen near to each other

$$|\nu_r - \nu_z| \ll \nu_{r,z}. \quad (16)$$

Therefore,

$$\int_0^{\pi} |\Phi_r\Phi_z|^2 d\theta \approx \int_{\pi}^{2\pi} |\Phi_r\Phi_z|^2 d\theta, \quad (17)$$

$$\int_0^{2\pi} |\Phi_r\Phi_z|^2 \frac{\partial^3 H}{\partial r^3} d\theta \approx 0. \quad (18)$$

For this reason, in case of parametric resonance (and also for the case of the simultaneous action of parametric and external resonance for the same degrees of freedom) the variables may be separated in spite of the non-linearity of the equations. If ν_r differs considerably from ν_z, then we must use

$$\varepsilon + \alpha_1 A_r^2 + \alpha_2 A_z^2$$

instead of $\epsilon + \alpha A^2$ in Eq. (10), where

$$\alpha_2 = -\frac{l^2}{8\pi^3 w} \frac{1}{4|\Phi|^2_{\max}} \times \frac{1}{P} \int_0^{2\pi} \frac{\partial^3 H}{\partial r^3} |\Phi_r\Phi_z|^2 d\theta, \quad (19)$$

and the canonical variables are now $\alpha_1 A_r^2$, $\alpha_2 A_z^2$. A is chosen such that it coincides with the amplitude of betatron oscillation. We have discarded in (9) and (10) non-linear terms which arise from derivatives $\partial^n H/\partial r^n$ of higher order than the third. Computations show that such an approximation is permissible as a rule.

According to (9) and (10), amplitude A sustains beats of a much longer period than the period of free oscillation $2\pi/\nu$. Thus the derivative $1/2\, d\varphi/d\theta$ which may be considered constant during the course of a few periods of free oscillations) may be considered as the difference between some effective "instantaneous" frequency

$$\nu_\theta = \nu^{(1)} + \alpha A^2 - g\sin\varphi$$

and its resonance value $n/2M$.

Equations (9) and (10) yield an integral of the motion (20)

$$\mathcal{H} = 2(\varepsilon - g\sin\varphi)A^2 + \alpha A^4 = \text{const}.$$

Note that the equations $dA^2/d\theta = 0$ and $d\varphi/d\theta = 0$ determine a periodic solution for x since according to (11) x is a periodic function of θ if A and φ are constant.

Substituting these periodic solutions into Eq. (20), we obtain formulas relating $\mathcal{H}$ and ϵ:

$$\text{a)} \quad -\frac{\alpha\mathcal{H}}{g^2} = \left(1 + \frac{\varepsilon}{g}\right)^2 \quad (21)$$

$$\text{b)} \quad -\frac{\alpha\mathcal{H}}{g^2} = \left(1 - \frac{\varepsilon}{g}\right)^2,$$

$$\text{c)} \quad \mathcal{H} \equiv 0.$$

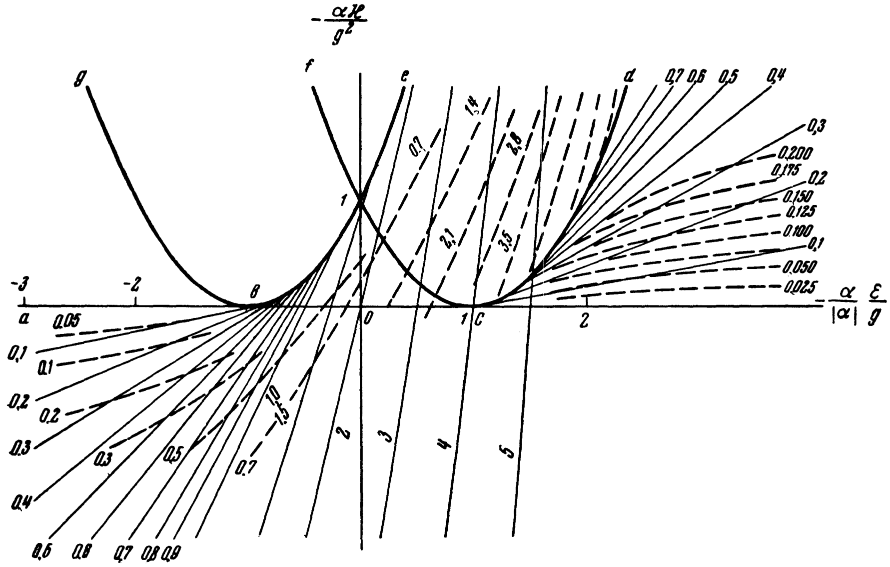

FIG. 2

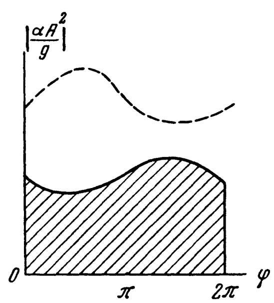

FIG. 3

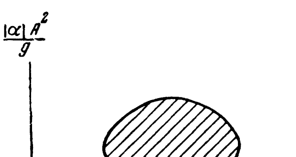

FIG. 4

Equations (21) obviously define curves on the plane $(-\alpha\mathcal{H}/g^2,\ \epsilon/g)$ which divide it into regions which yield different types of phase diagrams, $A^2 = A^2(\varphi, \mathcal{H}, \epsilon)$. A simple analysis which we shall not carry out here, shows that in the region below *abcd* (Fig. 2) the phase diagrams are unbounded (Fig. 3), while in the region inside *ebcd*, the phase diagrams are bounded (Fig. 4). Motion is impossible above *abc*. As for the curves given by (21), and which divide out the region, the situation is as follows: The phase diagrams for the points which lie on *eb* are simply points (which just correspond to periodic solutions); however characteristic asymptotic motions appear on *bc* and *cd*. Specifically

$$A \to 0 \quad \text{for} \quad \mathcal{H} \equiv 0, \ -1 < \varepsilon/g < 1; \tag{22}$$

$$-\frac{\alpha A^2}{g} \to \frac{\varepsilon}{g} - 1, \tag{23}$$

$$\sin\varphi \to +1 \quad \text{for} \quad -\frac{\alpha\mathcal{H}}{g^2} = \left(\frac{\varepsilon}{g} - 1\right)^2,$$

$$\frac{\varepsilon}{g} > 1, \ \alpha < 0;$$

$$\frac{\alpha A^2}{g} \to -\left(1 + \frac{\varepsilon}{g}\right), \tag{24}$$

$$\sin\varphi \to -1 \quad \text{for} \quad -\frac{\alpha\mathcal{H}}{g^2} = \left(\frac{\varepsilon}{g} + 1\right)^2,$$

$$\frac{\varepsilon}{g} < -1, \quad \alpha > 0.$$

If the form of the phase diagram is known, it is easy to compute A_{max} Besides the parabolas (21), Fig. 2 shows the lines $|\alpha A^2/g|_{max} = \text{const}$ (solid lines). These lines may be obtained in the regions of practical importance from the formulas

$$\left.\begin{array}{ll} -\frac{\alpha\mathcal{H}}{g^2} = 2\left|\frac{\alpha A^2}{g}\right|_{max}\left(-\frac{\alpha\varepsilon}{|\alpha|g}-1\right)-\left(\frac{\alpha A^2}{g^2}\right)^2_{max} & 0<-\frac{\alpha\mathcal{H}}{g^2}<\left(1-\frac{\varepsilon}{g}\right)^2, \\ -\frac{\alpha\mathcal{H}}{g^2} = 2\left|\frac{\alpha A^2}{g}\right|_{min}\left(-\frac{\alpha\varepsilon}{|\alpha|g}+1\right)-\left(\frac{\alpha A^2}{g}\right)^2_{min} & -\alpha\varepsilon>0; \end{array}\right. \tag{25}$$

$$\left.\begin{array}{ll} -\frac{\alpha\mathcal{H}}{g^2} = 2\left|\frac{\alpha A^2}{g}\right|_{max}\left(-\frac{\alpha\varepsilon}{|\alpha|g}+1\right)-\left(\frac{\alpha A^2}{g}\right)^2_{max} & -\frac{\alpha\mathcal{H}}{g^2}<0, \\ -\frac{\alpha\mathcal{H}}{g^2} = 2\left|\frac{\alpha A^2}{g}\right|_{min}\left(-\frac{\alpha\varepsilon}{|\alpha|g}-1\right)-\left(\frac{\alpha A^2}{g}\right)^2_{min} & -\alpha\varepsilon<0. \end{array}\right. \tag{26}$$

It should be noted at this point that in addition to the main solution given by formula (25) in the region $-\alpha\epsilon > 0$, there exists a second solution which has considerably larger amplitude (dotted lines in Fig. 3). If (25) corresponds to $d\varphi/d\theta \gtrless 0$, then this second solution corresponds to $d\varphi/d\theta \lessgtr 0$. For this solution the lines $|\alpha A^2/g|_{min} = \text{const}$ have the form:

$$-\frac{\alpha\mathcal{H}}{g^2} = 2\left|\frac{\alpha A^2}{g}\right|_{min}\left(-\frac{\alpha\varepsilon}{|\alpha|g}-1\right) - \left(\frac{\alpha A^2}{g}\right)^2_{min}, \quad -\alpha\varepsilon>0,$$
$$0<-\frac{\alpha\mathcal{H}}{g^2}<\left(1-\left|\frac{\varepsilon}{g}\right|\right)^2, \tag{27}$$

i.e., they are just like (25) for A_{max} in the fundamental solution. This A_{min}, however, does not coincide with A_{max} of the main solution. It may be easily verified that the line $A_{max} = \text{const}$ of the main solution must stop at the point of tangency with the parabola. Its downward continuation turns out to be the line $A_{min} = \text{const.}$ of the second solution. This fact becomes more evident if one considers for example the curve (Fig. 5)

$$-\frac{\alpha\mathcal{H}}{g^2} = 2\left(\frac{\varepsilon}{g}-1\right)\left|\frac{\alpha A^2}{g}\right| - \left(\frac{\alpha A^2}{g}\right)^2, \tag{28}$$

$$\alpha<0, \quad \varepsilon>0.$$

The intersection of this parabola with the line $-\alpha\mathcal{H}/g^2 = \text{const}$ yields two solutions. The larger one on the right gives $A = A_{min}$ for the second solution, while the smaller one on the left gives $A = A_{max}$ for the main solution. They only coincide at the point $|\alpha A^2/g| = (\epsilon/g) - 1$, i.e. on the line cd. It follows that if one draws the line $A_{max} = \text{const}$ for the largest possible amplitude and continues it downward until it intersects the abscissa, then the region to the left of this line (for $-\alpha\epsilon > 0$) must be forbidden. Indeed a large number of particles will be lost in the shaded region of Fig. 6 as they reach the second solution during the start of the acceleration (and also during scattering).

The range of the beat amplitude may be characterized by the quantity $\xi = A^2_{max}/A^2_{min}$. The following relation between ϵ/g and ξ can be obtained from Eqs. (25) and (26):

$$\left|\frac{\varepsilon}{g}\right| = \frac{\xi+1}{\xi-1} + \frac{\xi+1}{2\xi}\left|\frac{\alpha A^2}{g}\right|_{max}; \tag{29}$$
$$-\alpha\varepsilon>0, \quad 0<-\frac{\alpha\mathcal{H}}{g^2}<\left(1-\left|\frac{\varepsilon}{g}\right|\right)^2,$$

$$\left|\frac{\varepsilon}{g}\right| = \frac{\xi+1}{\xi-1} - \frac{\xi+1}{2\xi}\left|\frac{\alpha A^2}{g}\right|_{max}; \tag{30}$$
$$-\alpha\varepsilon<0; \quad -\frac{\alpha\mathcal{H}}{g^2}<0.$$

Another criterion for determining the boundaries of the safe region arises from the fact that the quantity $\langle \mathcal{H}^2 \rangle^{1/2}$ must not correspond to an amplitude which exceeds some given limiting value. In computing $\langle \mathcal{H}^2 \rangle$, one must take account of scattering by the residual gas.

The Hamiltonian form of Eqs. (9) and (10) permits one to calculate the change in amplitude resulting from a slow (adiabatic) change in the parameters. The dotted lines in Fig. 2 are adiabatic invariants; more exactly, they are defined by

$$I = \frac{|\alpha|}{g}\int A^2 d\varphi = \text{const.} \tag{31}$$

The integral appearing in (31) is numerically equal to the shaded area of Fig. 3 or 4. Figure 2 clearly shows the behavior of the amplitude for slowly varying parameters. In particular, the oscillations due to synchrotron oscillations in the momentum may be assumed adiabatic even at the start of the acceleration.

Evidently passage through resonance can only occur if $\alpha\ (d\epsilon/d\theta) > 0$.

4. THE RESONANCE THEORY OF DISTURBANCES

The effects of non-resonance harmonics can be analyzed by the so-called resonance theory of disturbance. This theory is based on a substitution suggested by Liuponov [3,4] Assume for example that the equation

$$(dx/d\theta) + i\nu x = F(\theta,\ x,\ x^*) \qquad (32)$$

has on its right hand side free, linear, quadratic, etc., terms with periodic coefficients. We shall seek solutions of the form

$$x = \Phi(\theta) + s + a_1(\theta)\, s + a_2(\theta)\, s^* \qquad (33)$$

$$+ b_1(\theta)\, s^2 + b_2(\theta)\, ss^* + b_3(\theta)\, s^{*2} + \cdots,$$

where Φ, a_i, b_i are periodic functions of θ to be determined. As for s, this new variable must satisfy a first approximation resonance equation accurate to terms which are quadratic in the disturbance on the right side of (32). This requirement, together with the periodicity condition, uniquely defines the substitution (33).

The procedure in the next approximation is clear. The practical second order effects in the resonance equations amount to corrections in the coefficients of the equation in the first approximation. This is essentially the effect of simultaneous oscillation in the momentum $\Delta p/p$ and other disturbances.

Non-resonance harmonics produce new resonances in higher approximation because of the non-linearity of the equation. In practice, however, they generally do not play any role.

The author wishes to express his profound gratitude to professor V. B. Berestetskii, V. V. Vladimirski and L.L. Gol'din for most helpful advice and discussions.

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