# On Radiation in Anisotropic Media

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A general solution is given for the problem of the field produced by a given distribution of external currents in an infinite homogeneous medium possessing arbitrary anisotropy (including gyrotropic behavior). In the particular case of a magnetoactive medium described by a tensor of the form (3.1), the multipole expansion of the radiation field has been obtained. In conclusion, the radiation field of a dipole in a magnetoactive medium is considered in greater detail.

## 1. INTRODUCTION

THE classical electrodynamical problem of the radiation of a given distribution of external currents in a homogeneous anisotropic medium* has not, so far as is known to us, received a general solution up to the present time. In papers by Ginsburg[1] and Kolomenskii[2] the special case is considered of the radiation of a point charge moving in a transparent anisotropic medium (Ref. 1 deals with the case of an inactive crystal, and Ref. 2 with that of a gyrotropic crystal). These writers, moreover, employ Hamilton's method, expanding the field inside a "box" in spatial Fourier series of the harmonic functions $\exp(ik_\lambda r)$. The time-dependent coefficients of the series are determined in the general case by a system of inhomogeneous linear differential equations. In the particular case of a monochromatic radiation field, which is the only one in which we shall hereafter be interested, the system of differential equations reduces to a system of linear algebraic equations, and the problem becomes considerably easier. In principle it can then be solved for a medium with arbitrary anisotropy and an arbitrary distribution of sources of the field. But with such a general statement of the problem it is more convenient to use a different approach not involving resolution of the field in terms of spatial harmonics $\exp(i\mathbf{k}_\lambda \mathbf{r})$, which are in reality not in any way singled out in the problem in question.

Namely, we at once seek an expression for the field produced by the given distribution of external currents j in the unbounded homogeneous medium with arbitrary anisotropy in a form that corresponds in the case of an isotropic medium to the well-known expression of the field in terms of retarded potentials, for example, the Hertz vector

$$\Pi = \frac{1}{i\omega\varepsilon}\int_V j e^{-ik\sqrt{\varepsilon}\rho}\, dV/\rho.$$

The general expression so obtained, including also quasistationary fields, indeed turns out to be somewhat complicated, but for radiation problems one is interested only in the asymptotic representation of the indicated expression for the field, corresponding to the wave zone of the sources. The method we use for the solution of this last problem has been illustrated in this paper in the special case of an anisotropic medium — an ionized gas in a constant magnetic field (magnetoactive medium).

## 2. GENERAL EXPRESSION FOR THE FIELD IN AN INFINITE HOMOGENEOUS ARBITRARILY ANISOTROPIC MEDIUM WITH SOURCES

We start with the field equations for an anisotropic medium with a given distribution of external currents (the medium is assumed nonmagnetic):

$$\mathrm{curl}\,\mathbf{E} = -ik\mathbf{H},\quad \mathrm{rot}\,\mathbf{H} = ik\hat{\varepsilon}\mathbf{E} + \frac{4\pi}{c}\mathbf{j}_{ex} \qquad (2.1)$$

where $\hat{\epsilon}$ is the dielectric permeability tensor (in the general case complex)* and $\mathbf{j}_{ex}$ is the external current density, assumed to be a continuous function of position. It is required to find the solution of the equations (2.1) satisfying the condition of finiteness at all points of space and the radiation condition. This latter condition means, as for the isotropic case, that the field from a confined source must consist at infinity of diverging waves.

It is expedient to start from the equations for the field vectors (for example, for E) without at

* Under the name of anisotropic medium we include both optically inactive and also active (gyrotropic) crystals.

* The quantity $\hat{\epsilon}\mathbf{E}$ represents a vector with the components $\epsilon_{ik}E_k$ (summation over repeated indices is assumed).

once going over to auxiliary potentials. Eliminating the vector **H** from Eq. (2.1), we find for **E** the equation

$$\text{curl curl } \mathbf{E} - k^2\hat{\varepsilon}\mathbf{E} \tag{2.2}$$

$$= \text{grad div } \mathbf{E} - \nabla^2\mathbf{E} - k^2\hat{\varepsilon}\mathbf{E} = \frac{4\pi k}{ic}\mathbf{j},$$

where the second form of the left-hand member has meaning only in a rectangular system of coordinates, which we shall use for the time being. For simplicity in writing, the subscript "ex" has been dropped.

Exploiting the linearity of the field equations, we shall seek the solution of Eq. (2.2) in the form

$$E_i = \frac{4\pi k}{ic}\int\limits_V T_{ik}(\mathbf{r}, \mathbf{r}_1)\, j_k(\mathbf{r}_1)\, dV_1, \tag{2.3}$$

where $T_{ik}(\mathbf{r}, \mathbf{r}_1)$ is a tensor of the second rank depending on the coordinates of the point of observation and the source point ($\mathbf{r}$ and $\mathbf{r}_1$) and the integration is taken over the entire volume of the sources. In the isotropic case, with $\epsilon_{ik} = \epsilon\delta_{ik}$, where $\delta_{ik}$ is the unit tensor, we have

$$T_{ik} = \frac{1}{4\pi q^2}\left(q^2\delta_{ik} + \frac{\partial^2}{\partial x_i\,\partial x_k}\right)e^{-iq\rho}/\rho, \tag{2.4}$$

where $\rho = \mathbf{r} - \mathbf{r}_1$ and $q^2 = k^2\epsilon$. From the point of view of the theory of linear differential equations $T_{ik}(\mathbf{r}, \mathbf{r}_1)$ is the Green's tensor for the equations (2.2).

Substituting Eq. (2.3) into Eq. (2.2), we obtain the following equation for $T_{ik}$:

$$D_{i\alpha}T_{\alpha k} = \delta_{ik}\,\delta(\rho), \tag{2.5}$$

$$D_{ik} = \frac{\partial^2}{\partial x_i\,\partial x_k} - \nabla^2\delta_{ik} - \alpha_{ik}.$$

Here $\alpha_{ik} = k^2\epsilon_{ik}$ and $\nabla^2$ is the Laplacian operator. It can be verified without difficulty that for $\epsilon_{ik} = \epsilon\delta_{ik}$ the tensor (2.4) indeed satisfies the equations (2.5).

In analogy with the case of the isotropic medium [Eq. (2.4)], we shall seek the solution of Eq. (2.5) in the form

$$T_{\alpha k} = D'_{\alpha k}I_0(\mathbf{r}, \mathbf{r}_1), \tag{2.6}$$

where $D'_{\alpha k}$ is also some tensor differential operator and $I_0$ is a scalar function of the coordinates of the points $\mathbf{r}$ and $\mathbf{r}_1$. In order for Eq. (2.6) to be a solution of Eq. (2.5), it is necessary and sufficient that the following relations hold

$$D_{i\alpha}D'_{\alpha k} = D_0\delta_{ik}, \qquad D_0 I_0 = \delta(\rho). \tag{2.7}$$

Such a separation of the problem (into an algebraic and an analytical part) decidedly simplifies its solution.

The first of the equations (2.7) indicates that the operator $D_0$ is equal to the determinant of the tensor $D_{ik}$ and the tensor $D'_{ik}$ is the algebraic complement of the tensor $D_{ki}$. Using the invariant representation of algebraic complements and determinants (cf. for example Ref. 3) we obtain:

$$D'_{ik} = {}^1/_2\, e_{k\alpha\beta}\, e_{imn}\, D_{\alpha m}\, D_{\beta n}, \tag{2.8}$$

$$D_0 = {}^1/_6\, e_{\alpha\beta\gamma}\, e_{ijk}\, D_{\alpha i}\, D_{\beta j}\, D_{\gamma k}, \tag{2.9}$$

where $e_{ijk}$ is the completely antisymmetric unit pseudotensor of the third rank ($e_{123} = 1$). Substituting Eq. (2.9) into Eq. (2.7), we obtain an equation for $I_0$, which determines this quantity apart from a nonsingular contribution $I_0'$ which satisfies the equation $D_0 I_0' = 0$. The lack of uniqueness of the function $I_0$ is obviously entirely analogous to that of the Green's function. As for the Green's function, the nonsingular term $I_0'$ is determined by the supplementary conditions (in our case, the radiation condition).

According to Eqs. (2.5) and (2.9)

$$D_0\exp\{\pm i\mathbf{p}\rho\} = \Delta(\mathbf{p})\exp\{\pm i\mathbf{p}\rho\}, \tag{2.10}$$

where $\Delta(p)$ is the determinant of the matrix $p^2\delta_{ik} - p_i p_k - \alpha_{ik}$. Consequently, we shall satisfy the second of the equations (2.7) if we set:

$$I_0(\mathbf{r}, \mathbf{r}_1) = \frac{1}{(2\pi)^3}\int\limits_{-\infty}^{\infty}\frac{\exp\{\pm i\mathbf{p}\rho\}}{\Delta(\mathbf{p})}d\mathbf{p}. \tag{2.11}$$

The validity of the expression (2.11) for $I_0$ from the point of view of the radiation condition (i.e., the legitimacy of setting $I'_0 = 0$) is confirmed by the asymptotic behavior of the integral (2.11) at infinity. We shall not carry out the analysis of this integral for the general anisotropic case. In the following Section the investigation is carried through for several special cases, and shows that the radiation condition is satisfied.

Thus all of the quantities of interest have been found, and for $T_{ik}$ we have finally

$$T_{ik}(\mathbf{r}, \mathbf{r}_1) = {}^1/_2\, e_{k\alpha\beta}\, e_{imn} D_{\alpha m} D_{\beta n} I_0(\mathbf{r}, \mathbf{r}_1), \quad (2.12)$$

with $I_0(r, r_1)$ given by the expression (2.11).

The solution (2.3) can be given a different and more familiar form. We introduce the vector

$$\Pi = \frac{4\pi k}{ic} \int_V I_0(\mathbf{r}, \mathbf{r}_1)\, \mathbf{j}(\mathbf{r}_1)\, dV_1. \quad (2.13)$$

Then from Eqs. (2.3) and (2.12) we obtain:

$$E_i = {}^1/_2\, e_{k\alpha\beta}\, e_{imn} D_{\alpha m} D_{\beta n} \Pi_k. \quad (2.14)$$

Thus we can regard the vector $\Pi$ as the Hertz vector. But in this connection it must be kept in mind that the definition of $\Pi$ by Eq. (2.13) will not reduce on passage to the isotropic case to the well-known definition of the Hertz vector for an isotropic medium, since the expression (2.11) does not* go over into $e^{-iq\rho}/\rho$.

The set of formulas (2.11), (2.13) and (2.14) [or (2.3), (2.11) and (2.12)] completely solves the problem proposed. The difficulty encountered in the practical application of these formulas is that of calculating the integral (2.11). But for the determination of the field in the wave zone ($k\rho \gg 1$) it is necessary to know only the asymptotic value of this integral with accuracy to terms of the order $1/k\rho$. For this purpose it is clear that in all concrete cases the method of steepest descent can be successfully applied. In the following Section we carry out such a calculation for one special case of an anisotropic medium.

## 3. THE WAVE FIELD OF AN ARBITRARY DISTRIBUTION OF CURRENTS IN A MAGNETOACTIVE MEDIUM

We shall apply the results obtained above to the special case of a magnetoactive medium such as an ionized gas in a constant magnetic field. If the axis of symmetry (i.e., the magnetic field) is directed along the $z$ axis of the coordinates, then the tensor $\epsilon_{ik}$ has the form (cf., for example, Ref. 4, p. 326):

$$\varepsilon_{ik} = \begin{Vmatrix} \varepsilon & -ig & 0 \\ ig & \varepsilon & 0 \\ 0 & 0 & \eta \end{Vmatrix} \quad (3.1)$$

* Apart from a constant factor, the quantity $(\nabla^2 + q^2) I_0$ goes over into $e^{-iq\rho}/\rho$ in the case of an isotropic medium.

In the absence of absorption $\epsilon$, $\eta$ and $g$ are real; when there is absorption, these quantities are, generally speaking, complex. The explicit expressions for the components of $\epsilon_{ik}$ in terms of the parameters of the plasma are of no interest to us here. We note that if we formally set $g = 0$ we obtain the case of a uniaxial crystal with the axis of symmetry along the $z$ axis*. This circumstance can be used later on to obtain various results for a uniaxial crystal.

Our problem is to calculate the asymptotic expression (for $k\rho \to \infty$) for the integral (2.11) in the case of a medium described by a tensor of the form (3.1). The determinant $\Delta(\mathrm{p})$ is biquadratic in $p_3$ and can be written in the form (cf., for example, Ref. 5, p. 134):

$$\Delta(\mathrm{p}) = \alpha_3 (p_3^2 - s_3^2)(p_3^2 - t_3^2), \quad (3.2)$$

where $\alpha_3 = k^2\eta$, and $s_3(p_1, p_2)$ and $t_3(p_1, p_2)$ are the $z$ components of the wave vectors $\mathbf{s}$ and $\mathbf{t}$ corresponding to the ordinary and extraordinary plane waves, expressed in terms of the other two components $p_1$ and $p_2$. The components $s_3$ and $t_3$ are determined as the roots of the equation $(z_1 = s_3^2,\ z_2 = t_3^2)$:

$$\alpha_3 z^2 - [2\alpha_1\alpha_3 - (\alpha_1 + \alpha_3) p^2] z + (p^2 - \alpha_3)(\alpha_1 p^2 - \alpha_1^2 - \beta^2) = 0, \quad (3.3)$$

$$\alpha_1 = k^2\varepsilon, \quad \beta = k^2 g, \quad p^2 = p_1^2 + p_2^2.$$

Substituting Eq. (3.2) into Eq. (2.11), we obtain:

$$I_0(\mathbf{r}, \mathbf{r}_1) = -\frac{1}{(2\pi)^3 \alpha_3} \int_{-\infty}^{\infty} \frac{\exp\{\pm i\mathrm{p}\rho\}\, dp_1 dp_2 dp_3}{(p_3^2 - s_3^2)(p_3^2 - t_3^2)} \quad (3.4)$$

This can easily be reduced to a one-dimensional integral. In fact, let us first consider the region $z > 0$ and take the minus sign in the exponent of the expression (3.4). Supposing further that, in the extraction of roots, $s_3$ and $t_3$ are defined by

* For an absorbing crystal it is here necessary to postulate that the principal axes of the dielectric permeability tensor and of the conductivity tensor coincide, i.e., to require a sufficiently high degree of symmetry of the crystal.

those branches for which $\mathrm{Im}(s_3, t_3) < 0$, we perform the integration over $p_3$, taking the residues at the poles $p_3 = s_3, t_3$. If we go over to polar coordinates $\rho, \vartheta, \varphi$ for the result obtained and introduce $p_1 = p\cos\eta$, $p_2 = p\sin\eta$, then, after integration over $\eta$, we obtain

$$I_0 = \frac{i}{8\pi\alpha_3}\int_{-\infty}^{\infty}\frac{pH_0^{(2)}(p\rho\sin\vartheta)}{s_3^2 - t_3^2}\left[e^{-is_3\rho\cos\vartheta}/s_3 - e^{-it_3\rho\cos\vartheta}/t_3\right]dp. \tag{3.5}$$

Here $H_0^{(2)}(z)$ is the Hankel function of second kind and zeroth order.

This expression for $I_0$ is valid for arbitrary values of the parameters $\epsilon$, $\eta$ and $g$. But in what follows, for simplicity of exposition, we confine ourselves to the case of no absorption, i.e., the case of real $\epsilon$, $\eta$ and $g$.

We apply the method of steepest descents to the integral (3.5). We first transform to a new variable of integration $\xi = \xi' + i\xi''$ by means of the equation $kn_i(\xi)\sin\xi = p$, where $n_i(\xi)$ is the index of refraction of the $i$th plane wave ($i = 1$ for ordinary, $i = 2$ for extra-ordinary) with its normal making angle $\xi$ with the $z$ axis. In the first term of Eq. (3.5) one must put $i = 1$, and in the second $i = 2$. The possibility of complex values of the angle $\xi$ means that our consideration includes so-called inhomogeneous plane waves.

From the definition of the index of refraction it follows that

$$p^2 + s_3^2 = k^2 n_1^2,\quad p^2 + t_3^2 = k^2 n_2^2. \tag{3.6}$$

From this we obtain

$$s_3 = kn_1\cos\xi,\quad t_3 = kn_2\sin\xi. \tag{3.7}$$

Substitution of Eq. (3.6) into Eq. (3.3) gives an equation for $n_i$, with roots of the form

$$n_{1,2}^2 = \frac{[\varepsilon(\varepsilon-\eta) - g^2]\sin^2\xi + 2\varepsilon\eta \pm \sqrt{[\varepsilon(\eta-\varepsilon) + g^2]^2\sin^4\xi + 4\eta^2 g^2\cos^2\xi}}{2(\varepsilon\sin^2\xi + \eta\cos^2\xi)}. \tag{3.8}$$

Noting further that

$$s_3^2 - t_3^2 = \frac{k^2}{\eta}\sqrt{(\varepsilon-\eta)^2 n_i^4\sin^4\xi - 4\eta g^2(n_i^2\sin^2\xi - \eta)}, \tag{3.9}$$

we obtain for $I_0$ the following expression

$$I_0 = \frac{i}{8\pi k^3}\sum_{i=1}^{2}\int_{C_i}\frac{(n_i'\sin\xi + n_i\cos\xi)H_0^{(2)}(k\rho n_i\sin\vartheta\sin\xi)\,\mathrm{tg}\,\xi}{[(\varepsilon-\eta)^2 n_i^4\sin^4\xi - 4\eta g^2(n_i^2\sin^2\xi - \eta)]^{1/2}}\exp\{-ik\rho n_i\cos\vartheta\cos\xi\}. \tag{3.10}$$

The integration contours $C_i$ lie in the complex $\xi$-plane.

In all of what follows we shall carry out the investigation only for one type of wave, dropping the index $i$. Omitting from consideration for the time being the region of angles around $\vartheta = 0$, we employ the asymptotic representation of the function $H_0^{(2)}(z)$. With the intent of carrying out all calculations to the accuracy of terms of order $1/k\rho$, we can keep only the first term in the asymptotic representation of $H_0^{(2)}(z)$. Substitution of this into Eq. (3.10) and introduction of the polar angles $\theta$ and $\chi$ of the vectors $\mathbf{r}$ and $\mathbf{r}_1$ (Fig. 1) gives the following expression for $I_0$:

$$I_0(\mathbf{r}, \mathbf{r}_1) = \frac{ie^{i\pi/4}}{8\pi k^3}\sqrt{\frac{2}{\pi k\rho\sin\vartheta}}\int_C F[(\xi, \mathbf{r}_1)\, e^{-ikrn\cos(\xi-\theta)}d\xi, \tag{3.11}$$

$$F(\xi, \mathbf{r}_1) = \sqrt{\frac{\sin\xi}{n(\xi)}} \times \frac{(n'\sin\xi + n\cos\xi)\exp\{ikr_1 n\cos(\xi-\chi)\}}{\cos\xi\,[(\varepsilon-\eta)^2 n^4\sin^4\xi - 4\eta g^2(n^2\sin^2\xi - \eta)]^{1/2}}. \tag{3.12}$$

From the condition $k\rho \gg 1$ it necessarily follows that $kr \gg 1$ and $kr_1 \gtrsim 1$. Thus the integral (3.11) satisfies the requirements for applicability of the method of steepest descents[6]. The saddle point $\xi_0(\theta)$ is determined by the equation

$$(d/d\xi)[n(\xi)\cos(\xi-\theta)] = 0,$$

from which we have

$$n'(\xi_0)/n(\xi_0) = \mathrm{tg}(\xi_0 - \theta). \tag{3.13}$$

But $n'(\xi_0)/n(\xi_0) = \operatorname{tg}\alpha(\xi_0)$, where $\alpha(\xi_0)$ is the angle between the wave normal and the energy-flux vector of the plane wave, under the condition that the angle between the normal and the $z$ axis is equal to $\xi_0$ (cf., for example, Ref. 4, p. 463). Thus[4] $\alpha(\xi_0) = \xi_0(\theta) - \theta$, i.e., $\xi_0(\theta)$ is that angle between the wave normal of the plane wave and the direction of the magnetic field for which the energy-flux vector of this wave makes the angle $\theta$ with the magnetic field. From this it is clear that the angle $\xi_0(\theta)$ is always real.

The integration path of steepest descent is determined by the equation

$$n(\xi)\cos(\xi - \theta) = \psi(\theta) - i\zeta^2, \qquad \psi(\theta) = n(\xi_0)\cos\alpha(\xi_0), \tag{3.14}$$

where $\zeta$ is a real variable ranging from $-\infty$ to $+\infty$. Integrating over the path of steepest descent and taking the function $F(\xi, \mathbf{r}_1)\,d\xi/d\zeta$ for $\xi = \xi_0$ out from under the sign of integration, we obtain an asymptotic expression for $I_0(\mathbf{r}, \mathbf{r}_1)$ valid to terms of order $1/k\rho$:

$$I_0(\mathbf{r}, \mathbf{r}_1) = A(\theta)\, e^{ikr_1 n(\xi_0)\cos\gamma}\, e^{-ikr\psi(\theta)}/kr. \tag{3.15}$$

Here $\gamma = [\chi - \xi_0(\theta)]$ is the angle between $\mathbf{r}_1$ and the wave normal $\mathbf{N}$ of the plane wave that has its energy-flux vector directed at the angle $\theta$ (see Fig. 1), and the function $A(\theta)$ has the form $[\,(n_0 = n(\xi_0),\ \alpha_0 = \alpha(\xi_0)\,]$:

$$A(\theta) = \frac{i}{4\pi k^3}\,\frac{\cos\theta}{\cos\xi_0\cos\alpha_0} \tag{3.16}$$
$$\times\left[\frac{n_0\sin\xi_0}{\sin\theta\,[(n_0'' - n_0)\cos\alpha_0 - 2n_0'\sin\alpha_0]\,[(\varepsilon - \eta)^2 n_0^4\sin^4\xi_0 - 4\eta g^2(n_0^2\sin^2\xi_0 - \eta)]}\right]^{1/2}.$$

Substitution of Eq. (3.15) into Eq. (2.11) gives the expression for the wave field:

$$\mathbf{\Pi} = 4\pi k^2 A(\theta)\, e^{-ikr\psi(\theta)}\,\mathbf{Z}(\theta)/kr, \tag{3.17}$$

$$\mathbf{Z}(\theta) = \frac{1}{i\omega}\int_V \mathbf{j}(\mathbf{r}_1)\, e^{ikr_1 n(\xi_0)\cos\gamma}\, dV_1. \tag{3.18}$$

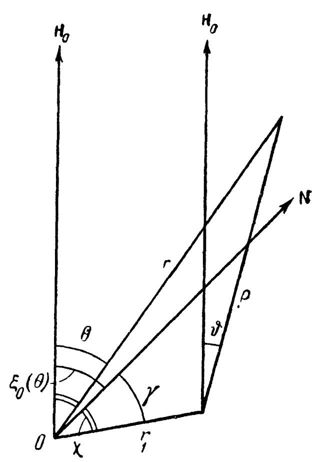


FIG. 1

## 4. EXPANSION OF THE WAVE FIELD IN TERMS OF MULTIPOLES

Expansion of the function $\mathbf{Z}(\theta)$ into a series in powers of $a/\lambda$ ($a$ is the order of magnitude of the linear dimensions of the system of currents and $\lambda$ is the wavelength) must give the expansion of the wave field (3.17) in terms of multipoles:

$$\mathbf{\Pi} = \sum_{s=0}^{\infty}\mathbf{\Pi}_s \tag{4.1}$$
$$= 4\pi k^2 A(\theta)\,(kr)^{-1} e^{-ikr\psi(\theta)}\sum_{s=0}^{\infty}\mathbf{Z}_s(\theta).$$

The expression for the multipole of $s$th order is found by expanding the function $\exp\{ikr_1 n(\xi_0) \times \cos\gamma\}$ in a series of powers of $kr_1$. Thus we obtain:

$$\mathbf{Z}_s = \frac{[in(\xi_0)]^s}{i\omega s!}\int_V (kr_1\cos\gamma)^s\,\mathbf{j}(\mathbf{r}_1)\,dV_1. \tag{4.2}$$

In the case of an isotropic medium, the zeroth term of the expansion ($s = 0$) corresponds, as is well known[7], to dipole radiation, while the following approximation ($s = 1$) corresponds to magnetic dipole and electric quadrupole radiations. The same holds true, naturally, also in the case now considered. In fact, introducing the electric and magnetic dipole moments

$$\mathbf{p} = \frac{1}{i\omega}\int_V \mathbf{j}\,dV_1, \quad \mathbf{m} = \frac{1}{2c}\int_V [\mathbf{r}_1\mathbf{j}]\,dV_1, \tag{4.3}$$

and the tensor $\hat{\pi}$ of the electric quadrupole moment of the given current distribution

$$\pi_{ik} = \frac{1}{2c}\int\limits_V (\xi_i j_k + \xi_k j_i)\, dV_1 \qquad (4.4)$$

($\xi_i$ are the coordinates of the point $\mathbf{r}_1$), we find*:

$$\mathbf{Z}_0 = \mathbf{p},\quad \mathbf{Z}_1(\theta) = n(\xi_0)([\mathbf{m}\mathbf{N}] + \hat{\pi}\mathbf{N}). \qquad (4.5)$$

$\mathbf{N} = \mathbf{N}(\theta)$ is as before the unit vector of the wave normal (Fig. 1). The difference between the result (4.5) and the corresponding result in the case of an isotropic medium consists only in the fact that the unit vector of the direction of observation, grad **r**, (isotropic medium) is replaced by the vector **N** making the angle $\alpha(\xi_0)$ with grad **r**.

## 5. THE WAVE FIELD OF AN ELECTRIC DIPOLE

We introduce the explicit expression for the field of a dipole **p**. To do this, we must substitute

$$\Pi_0 = 4\pi k^2 A(\theta)\, e^{-ikr\psi(\theta)}\, \mathbf{p}/kr \qquad (5.1)$$

into Eq. (2.14) and carry out the indicated operations to the accuracy of terms of the order $1/kr$.

Working out the operator (2.8) in Cartesian coordinates for the present case of a magnetoactive medium, we obtain for the components of the electric field the following expressions:

$$E_i(\mathbf{r}) = 4\pi k^6 A(\theta)\, e^{-ikr\psi(\theta)}\, a_{ik}(\theta, \varphi)\, p_k / kr, \qquad (5.2)$$

where we have introduced the notations

$$a_{11} = (\sigma_+^2 \cos^2\varphi - \varepsilon)\gamma_3 + (\varepsilon - \eta)\sigma_-^2, \qquad (5.3)$$

$$a_{22} = (\sigma_+^2 \sin^2\varphi - \varepsilon)\gamma_3 + (\varepsilon - \eta)\sigma_-^2,$$

$$a_{33} = (\sigma_-^2 - \varepsilon)\gamma_1 - g^2,$$

$$a_{12,21} = (\sigma_+^2 \sin\varphi\cos\varphi \mp ig)\gamma_3 \pm ig\sigma_-^2,$$

$$a_{13,31} = \sigma_+\sigma_-(\gamma_1\cos\varphi \mp ig\sin\varphi),$$

$$a_{23,32} = \sigma_+\sigma_-(\gamma_1\sin\varphi \pm ig\cos\varphi);$$

$$\gamma_1 = n^2(\xi_0) - \varepsilon,\quad \gamma_3 = n^2(\xi_0) - \eta, \qquad (5.4)$$

$$\sigma_+ = n(\xi_0)\sin\xi_0,\quad \sigma_- = n(\xi_0)\cos\xi_0.$$

From the relations that have been obtained it is seen that this result is correct for arbitrary angles $\theta$, and thus the assumptions made above ($z > 0$, angle $\theta$ not too close to zero) are not essential. As regards the first assumption ($z > 0$), this follows from the fact that the right-hand side of Eq. (5.2) is a function of $\theta$ symmetric with respect to $\theta = \pi/2$. The validity of the expression (5.2) at $\theta = 0$ follows from the continuity (absence of singularity) of the right-hand member of Eq. (5.2) at this point.

Let us examine the surfaces of constant phase

$$\Psi(r, \theta) = kr\psi(\theta) = krn(\xi_0)\cos\alpha(\xi_0) = \text{const}$$

(Fig. 2). Calculation of $\nabla\Psi$ gives

$$(\nabla\Psi)^2 = k^2 n^2(\xi_0), \qquad (5.5)$$

$$\cos(\widehat{\mathbf{N},\rho}) = (\nabla\Psi)_r / kn(\xi_0) = \cos\alpha(\xi_0). \qquad (5.6)$$

Equation (5.5) is, obviously, the special form of the eikonal equation corresponding to the case of a homogeneous medium with axially symmetrical anisotropy.

Equation (5.6) shows that the angle between the normal **N** to the front of the wave produced by a

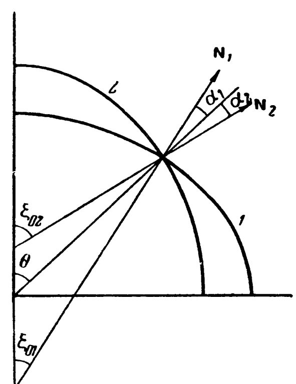

FIG. 2

$1 - \Psi_1(\rho, \theta) = \text{const}$,
$2 - \Psi_2(\rho, \theta) = \text{const}$

point source and the direction of observation **r** (the angle $\theta$) is always equal to the angle between the

* Corresponding calculations can be found in Ref. 7, p. 382.

wave-normal and the energy-flux vector of the plane wave propagated in the direction $\xi_0(\theta)$ (Fig. 2). This result is seen to be quite natural if we take into account the fact that in the wave zone the field of an arbitrary distribution of sources can always be represented in a sufficiently small region of space as the field of a plane wave.

It is of interest to consider the radiation of a dipole for several special cases of the orientation of the dipole itself and of the choice of the direction of observation. Without presenting the corresponding calculations from Eqs. (5.2), (5.3) and (5.4), we give only the final results.

1. *Dipole orientated along the magnetic field* ($\theta = 0$). Just as in the case of an isotropic medium, the dipole does not radiate along its own axis. In a direction perpendicular to the axis of the dipole there is emitted only the ordinary linearly polarized wave, with the amplitude

$$\frac{k^2}{r} p \sqrt{\left|\frac{\varepsilon(\eta-\varepsilon)+g^2}{\eta(\eta-\varepsilon)}\right|}. \qquad (5.7)$$

2. *Dipole orientated perpendicular to the field* ($p_1 = p$, $p_2 = p_3 = 0$). Along its own axis the dipole emits the extraordinary wave elliptically polarized in the plane $(x, y)$ perpendicular to the magnetic field. The amplitudes along the $x$ and $y$ axes are, respectively,

$$\frac{k^2}{r} p \sqrt{\left|\frac{\varepsilon[\varepsilon(\eta-\varepsilon)+g^2]}{\varepsilon[\varepsilon(\eta-\varepsilon)+g^2]+\eta g^2}\right|} \begin{cases} g^2/\varepsilon^2 \\ g/\varepsilon \end{cases} \qquad (5.8)$$

Radiation along the axis of the dipole is characteristic of a magnetoactive medium. When we go over to the uniaxial crystal ($g = 0$), or even further to the isotropic medium ($g = 0$, $\epsilon = \eta$), the amplitudes (5.8) go to zero. Along the magnetic field both waves are emitted, and now with circular polarizations. The amplitudes of the waves are

$$(k^2/r)\, p\, |\eta/(\varepsilon+\eta\pm g)| \qquad (5.9)$$

(the upper sign refers to the ordinary wave).

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