Some Remarks on the Validity of the Hydrodynamic Description of Quantum Systems

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It is shown that the hydrodynamic description of quantum systems imposes some serious restrictions on the dimensions of the investigated system. A result of this is that the description is not valid for atomic nuclei or multiple creation of mesons at the initial stages of expansion of the meson liquid.

THE hydrodynamic description of motion assumes that it is possible to assign to each element of the considered medium an energy density ϵ, a momentum density g, a medium density ρ and a pressure p, which are functions of coordinates and time. The whole medium has to be broken up into physically infinitely small volumes Δx^3, and the time into physically infinitely small intervals Δt.

It is only formally that one can consider the space-time lattice thus formed, as having infinitely small interpoint distances. Actually, these distances are subject to lower, as well as upper bounds. The upper bounds are trivial: $\Delta x \ll L$, $\Delta t \ll T$, where L is the dimension of the system and T is a time, characteristic of the considered process. As far as the lower bounds are concerned, the classical and the quantum theories give different restrictions. We are going to consider the restrictions characteristic of a quantum system.

Let Δx^3 be the volume element, with $\Delta x = L/n$, $n \gg 1$. The momentum of the element will be $g\Delta x^3$. On the other hand, the momentum dispersion Δp related to the localization of matter in the interval Δx, will be $> \hbar/\Delta x$. In order that it be possible to describe the motion with a momentum density g, it is required that the mean value of the momentum $g\Delta x^3$ be larger than the possible dispersion, i.e., that $g\Delta x^3 \gg \hbar/\Delta x$ or

$$g \gg \hbar/\Delta x^4. \quad (1)$$

In the nonrelativistic case, the energy density is $\epsilon = g^2/2\rho$; hence

$$\varepsilon \gg n^8\hbar^2/2L^8\rho. \quad (2)$$

In the relativistic case $\epsilon \approx gc$, i.e.,

$$\varepsilon \gg n^4\hbar c/L^4. \quad (3)$$

These relationships for the energy can also be obtained from the relation $\Delta E\Delta t > \hbar$ for $\Delta t \approx \Delta x/v$ or $\Delta t \approx \Delta x/c$ respectively, let us note that any more detailed model consideration can only increase the values of the right hand sides of these inequalities.

We now apply these inequalities to two definite problems.

A. HYDRODYNAMIC DESCRIPTION OF THE ATOMIC NUCLEUS

In this case, the characteristic dimension is the nuclear radius $L = R = r_0 A^{1/3}$, $r_0 = 1.3 \times 10^{-13}$ cm, A = atomic weight of the nucleus. The matter density is mA/V, where m is the nucleon mass and $V = 4\pi R^3/3$.

Let us now make use of the inequality (2) and apply it to the total nuclear excitation energy $E \approx \epsilon V$ due to the hydrodynamic motions. It then follows from (2) that:

$$E \gg \frac{n^8\hbar^2}{2R^8}\frac{V^2}{mA} = \frac{1}{2}\left(\frac{4\pi}{3}\right)^2 n^8 \frac{\hbar^2}{2mr_0^2} A^{-5/3}. \quad (4)$$

We have $\hbar^2/2mr_0^2 \approx 10$ mev. Even if the nuclear radius is divided only into three parts ($n = 3$), we still get a tremendous excitation energy — so large that the nucleus cannot exist as a whole (for $A = 200$, $E \gg 80$ mev). Therefore it is not possible to expect, for instance, that the moment of inertia of a nucleus computed from the motion of an ideal liquid in an ellipsoidal container have any relation to the actual situation.

The quantization of the hydrodynamic rotational motion of the nucleus will not help, because the most important parameter of the problem — the moment of inertia — is calculated from the classical theory. This does not mean that one cannot at all speak of a rotation of the nucleus. In the quantum theory, the reciprocal of the moment of inertia, $a = 1/I$, is an operator. In order to be able to speak of a rotation of the nucleus it is necessary that the dispersion $\overline{\Delta a^2}$ be much smaller than $\bar{a}^2$, i.e., there must be a certain "rigidity" of the nuclear shape.

B. HYDRODYNAMIC DESCRIPTION OF MULTIPLE MESON CREATION

Recently, attempts have been made[1-3] to consider the multiple meson creation in the collision of two relativistic nucleons as the process of expansion of the excited meson liquid. When a large number of mesons is present, it can be expected that the quantization will have no sizable effect, because the mesons are subject to Bose statistics. The restrictions for such a classical description come, however, from the fact that one has to apply this concept not to the system as a whole, but to small space-time regions.

In the case considered here, we are talking about a meson liquid occupying the volume of an oblate ellipsoid: $V = (4\pi/3)(\hbar/\mu c)^3\, 2mc^2/E$ (here μ is the meson mass, m – the nucleon mass and E – the nucleons' energy in the center-of-mass system). The minor half axis of the ellipsoid is, taking into account the Lorentz contraction, equal to $L = (\hbar/\mu c)\, mc^2/E$. Because of the large magnitude of this contraction, we can restrict ourselves to a one-dimensional problem. Instead of (3), we get for the energy density per unit length:

$$\varepsilon \gg n^2\hbar c/L^2 \quad \text{for} \quad E \gg n^2\hbar c/L. \tag{5}$$

If we substitute the value of L, we find that

$$n^2 \ll m/\mu. \tag{6}$$

This inequality shows that in the initial stage of the process immediately following the nucleon collision, the hydrodynamic description is absolutely inapplicable: in this stage the meson liquid which occupies the ellipsoid undergoes quantum fluctuations of momentum and energy.

The further behavior of this liquid becomes quite indefinite from the hydrodynamic point of view, and the ellipsoid can break into "drops". One can therefore think that the process of multiple meson creation is actually a purely quantum effect.

If one still assumes that, in this quantum phase, the motion is a more or less regular one-dimensional expansion of the ellipsoid, it is easy to show that, in order that the hydrodynamic description be valid, it is necessary that the energy of the primary nucleons be greater than $10^{14} - 10^{15}$ ev (number of layers $n \approx 10$, $\Delta p/p \approx 10\%$). The same numerical conclusion is reached when one considers the final stages of the expansion, when L becomes comparable to $\hbar/\mu c$ and the motion becomes three-dimensional.

1 E. Fermi, Progr. Theoret. Phys. 5, 570 (1950).
2 E. Fermi, Phys. Rev. 81, 683 (1955).
3 C. Z. Belen'kii and L. D. Landau, Usp. Fiz. Nauk 56, 309 (1955).

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