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Charge Invariance in Nuclear Reactions

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It is shown that there exists a relation between the cross sections of the reactions He^3 $(pp)\,He^3$, $He^3\,(nn)\,He^3$, $He^3\,(np)\,T$; $T\,(pn)\,He^3$; $T\,(pp)\,T$; $T(pp)\,T$. This relation is based on the hypothesis of charge invariance of nuclear forces; because of it we can reduce the number of independent parameters (phases) necessary for a complete description of these reactions. The cross sections of these reactions are all expressed by the same parameters $\delta(T)$ and $\epsilon(T)$ [the phases and mixture parameters in a state with isotopic spin T] and thus it is possible to carry out a phase analysis of all these reactions simultaneously. The method we have applied is a general one and is applicable, with only slight changes, to all reactions involving light nuclei.

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1. INTRODUCTION

IN the present work we have investigated the relation between the cross sections of certain nuclear reactions that follows from the assumption of charge invariance of nuclear reactions. We apply our considerations to reactions that are possible for a system of four nucleons [2 protons and 2 neutrons ($2p$, $2n$); 3 protons and 1 neutron ($3p$, n); 1 proton and 3 neutrons (p, $3n$)] at small energies:

$$T\,(pp)\,T;\ \ He^3\,(pp)\,He^3;\ \ T\,(nn)\,T;$$

$$He^3\,(nn)\,He^3;\ \ T\,(pn)\,He^3;\ \ He^3\,(np)\,T.$$

It will be evident from what follows, however, that our method is a general one and is applicable for reactions involving any type of light nuclei.

We divide the space in the center-of-mass system of the colliding particles into two parts: the internal, which is bounded by some radius r_0, and the external, which fills all the remaining space. The radius r_0 is so chosen that at distances $r > r_0$ there is no nuclear interaction between the colliding and outgoing particles. We know the exact wave functions of the particles in the external region for such a choice of r_0. In the case of uncharged particles, these are the ordinary wave functions of free motion, and in the case of charged particles, they are Coulomb functions.

In reactions involving light nuclei, the wave function in the internal region ought to satisfy the requirements of charge invariance. In order to understand to what they reduce, let us consider as a concrete case the reactions $T(pp)\,T$; $T\,(pn)\,He^3$; He^3 $(pp)\,He^3$ for small energies, when other conceivable reactions, such as $T\,(pd)\,d$, are not possible. In a collision of a proton with a nucleus of tritium in the internal region ($r \leqslant r_0$), there is formed some sort of state of the system consisting of two protons and two neutrons ($2p$, $2n$). This state decays either to $T+p$ or to He^3+n, because the others are not possible for small energies.

For $r = r_0$, the wave function in a state with isotopic spin T is a combination of two terms $\varphi^{(1)}(T)$ and $\varphi^{(2)}(T)$:

$$\varphi^{(1)}(T) = \psi_T\psi_p\Psi'^{(1)}(T) \equiv \psi(T+p)\,\Psi'^{(1)}(T),$$
$$\varphi^{(2)}(T) = \psi_{He}\cdot\psi_n\Psi'^{(2)}(T) \equiv \psi(He^3+n)\,\Psi'^{(2)}(T), \tag{1.1}$$

where Ψ_T, Ψ_p, Ψ_{He^3} and Ψ_n are the wave functions of the internal state T, p, He^3 and n, while $\Psi^{(1)}(T)$ and $\Psi^{(2)}(T)$ are functions of the relative motion of the systems $T+p$ and $He+n$, respectively. It follows from charge invariance that $\varphi^{(1)}$ (T) and $\varphi^{(2)}$ (T) ought to change into one another for an exchange in the places of the neutron and the proton; consequently,

$$\Psi'^{(1)}(T) = \Psi'^{(2)}(T) \equiv \Psi'(T).$$

The total wave function for $r \leqslant r_0$ is a superposition of states with isotopic spin $T=0$ and 1: $\varphi_{T=0}$ and $\varphi_{T=1}$. The wave function $\varphi_{T=1}$ ought to be symmetric for the substitution $\Psi_T \rightleftarrows \Psi_{He^3}$, Ψ_p Ψ_n and therefore,

$$\varphi_{T=1} = \varphi^{(1)}(1) + \varphi^{(2)}(1). \tag{1.2}$$

The function $\varphi_{T=0}$ ought to change sign in such an exchange, i. e.,

$$\varphi_{T=0} = \varphi^{(1)}(0) - \varphi^{(2)}(0). \tag{1.3}$$

For the scattering $He^3\,(pp)\,He^3$ [the system ($3p$, n)], only the state with $T=1$ is possible, and for $r = r_0$, the wave function has the form

$$\Phi_{T=1} = \psi(He^3+p)\,\Psi'^{(3)}(1), \tag{1.4}$$

where, as a consequence of charge invariance, $\Psi^{(3)}(1) = \Psi^{(2)}(1) = \Psi^{(1)}(1) \equiv \Psi(1)$.

It remains for us to determine the concrete form of the relative motion $\Psi(T)$ for $r = r_0$. First, let us consider the state with total momentum J and parity $P = (-)^l$ (l=orbital momentum). Since each particle in the system (T +p) and (He^3 +n) has spin ½, for any l (except $l = 0$), there exist two states with given $J = l$ and $P = (-)^l$: states with total spin $s = 0$ and $s = 1$. If we denote the angular part of the wave function for given J, l, s by $Y^J_{l,s}$, and the radial part by $R^J_{l,s}$, then the total function $\Psi(T)$ in the states J, P has the form

$$\Psi^J_P(T) = R^J_{l1}(T)\,Y^J_{l1} + R^J_{l0}(T)\,Y^J_{l0}. \qquad (1.5)$$

The angular parts are easily constructed with the help of the usual rules of momentum addition, while the radial functions for $r \sim r_0$ have the form

$$R^J_{ls}(T) = a^J_{ls}(T)\,R^{(-)}_{kl}(r) - b^J_{ls}(T)\,R^{(+)}_{kl}(r). \qquad (1.6)$$

Here $a^J_{ls}(T)$ and $b^J_{ls}(T)$ are certain constants for each J, l, s and T, while $R^{(\pm)}_{kl}(r)$ are radial functions of the free motion

$$R^{(\pm)}_{kl}(r) = i\sqrt{\frac{\pi k}{2r}}\{\pm J_{l+1/2}(kr) \qquad (1.7)$$
$$+ iN_{l+1/2}(kr)\} \to \frac{1}{r}e^{\pm i\left(kr - \frac{\pi l}{2}\right)}.$$

Here $J_{l+1/2}$ and $N_{l+1/2}$ are Bessel functions of the first and second kind, the signs (±) referring to outgoing and ingoing waves, respectively, while k is the wave vector, in magnitude equal to the wave vector of the relative motion in the system He^3 + n, which is produced after the decay of the "intermediate" system.

Similarly, in states with total momentum J and $P = (-)^{l+1}$ ($l = J + 1$), the function Ψ (T) has the form

$$\Psi^{J=l+1}_P(T) = R^{l+1}_{l1}(T)\,Y^{l+1}_{l1} + R^{l+1}_{l+2,1}(T)\,Y^{l+1}_{l+2,1}, \qquad (1.8)$$

where the $R^J_{ls}(T)$ are defined by Eq. (1.6).

We can now write down the boundary values of the for wave function for $r = r_0$ as

$$\psi(r_0) = (\varphi_{T=1} + \varphi_{T=0})_{r=r_0}, \qquad (1.9)$$

where the φ_T are determined from Eqs. (1.1)-(1.6).

All further calculations are similar, both those for the case $J = l$, $P = (-)^l$, and those for the case $J = l + 1$, $P = (-)^l$. Therefore, we need carry out only the first of these.

2. BOUNDARY CONDITIONS

In this Section, we establish the form of the wave function of a system of two neutrons and two protons ($2p$, $2n$) for such energies for which, as $r \to \infty$, only those states corresponding to the pairs (T+p) and (He^3 +n) are energetically allowed. The same general wave function of the system for $r \geqslant r_0$ has the form (we are at present interested only in a state with total angular momentum l and $l = J + 1$):

$$\begin{aligned}\psi(r > r_0) = \psi(\mathrm{T} + p)\{[C_l\,\mathscr{R}^{(-)}_{k_1,l}(r) - D_l\,\mathscr{R}^{(+)}_{k_1,l}(r)]\,Y^{l+1}_{l,1} + \\ + [C_{l+2}\,\mathscr{R}^{(-)}_{k_1,l+2}(r) - D_{l+2}\,\mathscr{R}^{(+)}_{k_1,l+2}(r)]\,Y^{l+1}_{l+2,1}\} + \\ + \psi(\mathrm{He}^3 + n)\{[A_l\,R^{(-)}_{k,l}(r) - B_l\,R^{(+)}_{k,l}(r)]\,Y^{l+1}_{l,1} + \\ + [A_{l+2}\,R^{(-)}_{k,l+2}(r) - B_{l+2}\,R^{(+)}_{k,l+2}(r)]\,Y^{l+1}_{l+2,1}\}.\end{aligned} \qquad (2.1)$$

Here, as before, Ψ(T +p) and Ψ(He^3 +n) denote the internal wave functions of the pairs T +p and He^3 + n, while the expressions in the curly brackets are functions of the relative motion of T and p, and He^3 +n, respectively. $\mathscr{R}^{(\pm)}_{k_1 l}(r)$ and $R^{(\pm)}_{kl}(r)$ are the Coulomb and ordinary free solutions; A, B, C, D are constants, about which we shall speak later, and k_1 and k are the wave vectors of the relative motion of the systems T + p and He^3 + n, respectively, and differ from each other only by the mass differences of T + p and He^3 + n. The function (2.1) is divided in the usual way into two parts

$$\psi(r > r_0) = \psi_{r=1} + \psi_{r=0},$$

which at $r = r_0$ transform into the wave function $\delta_{T=1}$ and $\delta_{T=0}$, respectively. The functions $\Psi_{T=1}$ and $\Psi_{T=0}$ differ from each other by the different values of the constants A, B, C, and D, which we shall distinguish by the arguments 1 and 0: $A = A(1) + A(0)$; $B = B(1) + B(0)$, etc.

At $r = r_0$, the wave function (2.1) ought to join together with the internal wave function (1.9). We shall fist concern ourselves with states with isotopic spin $T = 1$, and shall connect the function $\delta_{\mathrm{T}=1}$ with the function $\delta_{T=1}$ which is defined by the values of the constants $A(1)$, $B(1)$, $C(1)$, $C(1)$. The boundary conditions are written in the following fashion:

$$\psi_{T=1}(r_0) = \varphi_{T=1}(r_0), \quad \psi'_{T=1}(r_0) = \varphi'_{T=1}(r_0). \tag{2.2}$$

Now, substituting the functions from (1.2, (1.6), (1.7), and equating terms with corresponding l, we get

$$\begin{aligned} a^{l+1}_{l1}\,(1) &= A_l(1);\; a^{l+1}_{l+2,1}(1) = A_{l+2}(1); \\ b^{l+1}_{l\,1}\,(1) &= B_l(1);\; b^{l+1}_{l+2,\,1}(1) = B_{l+1}(1), \end{aligned} \tag{2.2'}$$

$$\begin{aligned} &A_l(1)\,R^{(-)}_{kl}(r_0) - B_l(1)\,R^{(+)}_{kl}(r_0) \\ &\quad = C_l(1)\,\mathcal{R}^{(-)}_{k_1 l}(r_0) - D_l(1)\,\mathcal{R}^{(+)}_{k_1 l}(r_0); \\ &A_l(1)\,R^{(-)\prime}_{kl}(r_0) - B_l(1)\,R^{(+)\prime}_{kl}(r_0) \\ &\quad = C_l(1)\,\mathcal{R}^{(-)\prime}_{k_1 l}(r_0) - D_l(1)\,\mathcal{R}^{(+)\prime}_{k_1 l}(r_0), \\ &A_{l+2}(1)\,R^{(-)}_{k,\,l+2}(r_0) - B_{l+2}(1)\,R^{(+)}_{k,\,l+2}(r_0) \\ &\quad = C_{l+2}(1)\,\mathcal{R}^{(-)}_{k_1,\,l+2}(r_0) - D_{l+2}(1)\,\mathcal{R}^{(+)}_{k_1,\,l+2}(r_0), \\ &A_{l+2}(1)\,R^{(-)\prime}_{k,l+2}(r_0) - B_{l+2}(1)\,R^{(+)\prime}_{k,\,l+2}(r_0) \\ &\quad = C_{l+2}(1)\,\mathcal{R}^{(-)\prime}_{k_1,\,l+2}(r_0) - D_{l+2}(1)\,\mathcal{R}^{(+)\prime}_{k_1,\,l+2}(r_0) \end{aligned} \tag{2.2''}$$

(the primes denote differentiation with respect to r). We get directly from (2.2″):

$$\begin{aligned} C_l(1) &= \gamma^{(l)}_{11}\,A_l(1) + \gamma^{(l)}_{12}\,B_l(1), \\ D_l(1) &= \gamma^{(l)}_{21}\,A_l(1) + \gamma^{(l)}_{22}\,B_l(1) \end{aligned} \tag{2.3}$$

and corresponding equations for $l+2$. Here,

$$\begin{aligned} \gamma^{(l)}_{11} &= \frac{1}{G}\,[R^{(-)\prime}_{kl}(r_0)\,\mathcal{R}^{(+)}_{k_1,l}(r_0) - R^{(-)}_{kl}(r_0)\,\mathcal{R}^{(+)\prime}_{k_1,\,l}(r_0)], \\ \gamma^{(l)}_{12} &= -\frac{1}{G}\,[R^{(+)\prime}_{kl}(r_0)\,\mathcal{R}^{(+)}_{k_1,\,l}(r_0) - R^{(+)}_{kl}(r_0)\,\mathcal{R}^{(+)\prime}_{k_1,\,l}(r_0)], \\ \gamma^{(l)}_{21} &= \frac{1}{G}\,[R^{(-)\prime}_{kl}(r_0)\,\mathcal{R}^{(-)}_{k_1,\,l}(r_0) - R^{(-)}_{kl}(r_0)\,\mathcal{R}^{(-)\prime}_{k_1,\,l}(r_0)], \\ \gamma^{(l)}_{22} &= -\frac{1}{G}\,[R^{(+)\prime}_{kl}(r_0)\,\mathcal{R}^{(-)}_{k_1,\,l}(r_0) - R^{(+)}_{kl}(r_0)\,\mathcal{R}^{(-)\prime}_{k_1,\,l}(r_0)], \end{aligned}$$

$$G = -\frac{2ik_1}{r_0^2}, \tag{2.4}$$

The $R^{(\pm)}_{rl}$ were determined earlier by (1.7), while the $\mathcal{R}^{(\pm)}_{r1,l}$ are functions of the motion in the Coulomb field

$$\mathcal{R}^{(\pm)}_{k_1,\,l}(r) = \frac{1}{r}\,[G_l(k_1 r) \pm iF_l(k_1 r)],$$

where F_l and G_l are the regular and nonregular Coulomb functions.* Carrying out the calculations for states with $T=0$, we get

$$\begin{aligned} C_l(0) &= -\,[\gamma^{(l)}_{11}A_l(0) + \gamma^{(l)}_{12}B_l(0)], \\ D_l(0) &= -\,[\gamma^{(l)}_{21}A_l(0) + \gamma^{(l)}_{22}B_l(0)] \end{aligned} \tag{2.5}$$

and corresponding equations for $l+2$.

We now consider the boundary conditions at infinity. Let the case of interest to us be the reactions T(pn) He^{3}and T(pp) T. In both these processes, the ingoing waves of the nuclei He^3 and n ought to be absent. Therefore, the condition below ought to be satisfied [see Eq. (2.1)]:

$$\begin{aligned} A_l &\equiv A_l(1) + A_l(0) = 0, \\ A_{l+2} &\equiv A_{l+2}(1) + A_{l+2}(0) = 0. \end{aligned} \tag{2.6}$$

The second pair of boundary conditions is connected with normalization of the ingoing waves of T and p. Actually, in the general problem, when we consider collisions of protons with the nucleus T, the amplitude of the ingoing (T+p)-wave is found by a simple expansion of plane waves (with consideration of the spins of the particles) in a series in the functions Y^J_{ls}. The corresponding formulas are ordered repeatedly[1], and the second pair of boundary conditions has the form:

$$\begin{aligned} &C_l \equiv C_l(1) + C_l(0) = \gamma^{(l)}_{11}(A_l(1) - A_l(0)) \\ &\quad + \gamma^{(l)}_{12}(B_l(1) - B_l(0)) = C^{(0)}_l, \\ &C_{l+2} \equiv C_{l+2}(1) + C_{l+2}(0) = \gamma^{(l+2)}_{11}(A_{l+2}(1) \\ &\quad - A_{l+2}(0)) + \gamma^{(l+2)}_{12}(B_{l+2}(1) - B_{l+2}(0)) = C^{(0)}_{l+2}. \end{aligned} \tag{2.7}$$

Here we have made use of Eq. (2.3), while the quantities on the right-hand side, $C^{(0)}_l$ and $C^{(0)}_{l+2}$, are the amplitudes (unknown from the statement of the problem) of the outgoing waves of T +p. The boundary conditions for the reactions He^3 (np)T and He^3 (nn) He^3 are formulated in a similar way. In these cases, the ingoing waves of T + p are absent.

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*The coefficients $\gamma^{(l)}_{in}$ depend on the energy and on r_0. The value of r_0 is bounded above and below. Below, it is bounded by the condition that for $r > r_0$, nuclear interaction between the colliding and outgoing particles is absent. At the same time, r_0 cannot be too large, since the Coulomb interaction can be neglected only in the zone of action of the nuclear forces.

3. MATRIX OF THE REACTION FOR STATES WITH GIVEN *T*

In this Section, we shall study in detail the physical meaning and the structure of the amplitudes $A_l(T)$, $B_l(T)$, $C_l(T)$ and $D_l(T)$. It is evident from Eq. (2.1) that $A_l(T)$ and $C_l(T)$ are the amplitudes of the ingoing waves of ($He^3 + n$), respectively, while $B_l(T)$ and $D_l(T)$ are the amplitudes of the outgoing waves. The amplitudes *B* and *D* are expressed by the amplitudes of the ingoing waves *A* and *C*:

$$\begin{aligned}
B_l(T) &= S^{(l+1)}_{l1,\,l1}(T)\,A_l(T) + S^{l+1}_{l1,\,l+21}(T)\,A_{l+2}(T) + \\
&\quad + \sqrt{\frac{k_1}{k}}\,[M^{l+1}_{l1,\,l1}(T)\,C_l(T) + M^{l+1}_{l1,\,l+21}(T)\,C_{l+2}(T)],\\
B_{l+2}(T) &= S^{l+1}_{l+21,\,l1}(T)\,A_l(T) + S^{l+1}_{l+21,\,l+21}(T)\,A_{l+2}(T) + \\
&\quad + \sqrt{\frac{k_1}{k}}\,[M^{l+1}_{l+21,\,l1}(T)\,C_l(T) + M^{l+1}_{l+21,\,l+21}(T)\,C_{l+2}(T)],\\
D_l(T) &= \sqrt{\frac{k}{k_1}}\,[{}'M^{l+1}_{l1,\,l1}(T)\,A_l(T) + {}'M^{l+1}_{l1,\,l+21}(T)\,A_{l+2}(T)] + \\
&\quad + \Omega^{l+1}_{l1,\,l1}(T)\,C_l(T) + \Omega^{l+1}_{l1,\,l+21}(T)\,C_{l+2}(T),\\
D_{l+2}(T) &= \sqrt{\frac{k}{k_1}}\,[{}'M^{l+1}_{l+21,\,l1}(T)\,A_l(T) + {}'M^{l+1}_{l+21,\,l1}(T)\,A_{l+2}(T)] + \\
&\quad + \Omega^{l+1}_{l+21,\,l1}(T)\,C_l(T) + \Omega^{l+1}_{l+21,\,l+21}(T)\,C_{l+2}(T).
\end{aligned} \tag{3.1}$$

The matrix of the coefficients *S*, *M*, $'M$, Ω is known as the reaction matrix. The element with indices $S^J_{ls,\,l's'}(T)$ describes the transition between the state ($Jl's'T$) and the state ($JlsT$). The indices for the other elements can be interpreted similarly. Since we assume the validity of the hypothesis of charge invariance, there are no transitions between states with different isotopic spins *T* and the amplitudes of the outgoing waves with given *T* are expressed in terms of the amplitudes of the ingoing waves with the same *T*. Inasmuch as the nuclear interaction depends on the isotopic spin, the reaction matrices for $T = 1$ and $T = 0$ will be different.

The matrix elements $S^J_{ls,\,l's'}(T)$ connect the amplitudes of the incident and outgoing waves of $He^3 + n$. Therefore, this part of the matrix describes the elastic process of scattering He^3 (*nn*) He^3. The elements $M(T)$ describe the reaction T (*pn*) He^3, the elements $'M(T)$, the reaction He^3 (*np*) T, and the elements $\Omega(T)$, the elastic scattering *T* (*pp*) T (all these refer to states with given *T*).

It is not difficult to see that the matrix elements $M(T)$ and $'M(T)$ are identically equal to zero. We now show that this is a consequence of the assumed hypothesis of charge invariance. Actually, absorption of particles ought not to take place in the reaction process; therefore, the total particle flux incident on the sphere $r = r_0$ ought to be equal to the flux of outgoing particles on the sphere $r = r_0$. But the current through the sphere r_0 can be computed by using the wave functions (1.2) or (1.3), in which we must substitute the values for $\Psi^{(1)}$ and $\Psi^{(2)}$ from Eq. (1.8). In this case, use is made of the fact that the total number of particles ($He^3 + n$), as also the total number of particles (T + p) found in states with $J = l \pm 1$ and entering the sphere, is proportional to

$$|a^{l+1}_{l1}(T)|^2 + |a^{l+1}_{l+2,\,1}(T)|^2,$$

while the number of particles leaving the sphere is proportional to

$$|b^{l+1}_{l1}(T)|^2 + |b^{l}_{l+2,1}(T)|^2.$$

From the flux equality, it follows that

$$\begin{aligned}
&|b^{l+1}_{l1}(T)|^2 + |b^{l+1}_{l+21}(T)|^2 \\
&\qquad = |a^{l+1}_{l1}(T)|^2 + |a^{l+1}_{l+21}(T)|^2.
\end{aligned} \tag{3.2}$$

Making use of the conditions (2.2′), we rewrite (3.2′) in the form

$$|A_l(T)|^2 + |A_{l+2}(T)|^2 = |B_l(T)|^2 + |B_{l+2}(T)|^2. \tag{3.3}$$

Substituting $B_l(T)$ and $B_{l+2}(T)$ from (3.1) in (3.3), we get:

$$\begin{aligned}
&|A_l(T)|^2 + |A_{l+2}(T)|^2 \\
&\qquad = |A_l(T)|^2\,[\,|S^{l+1}_{l1,\,l1}(T)|^2 + |S^{l+1}_{l+21,\,l1}(T)|^2] \\
&\qquad + |A_{l+2}(T)|^2\;[\,|S^{l+1}_{l+21,\,l+21}(T)|^2 + |S^{l+1}_{l1,\,l+21}(T)|^2]
\end{aligned}$$

$$+ [A_l(T) A^*_{l+2}(T) (S^{l+1}_{l1,\, l1}(T)\; S^{l+1^*}_{l1,\, l+21}(T) + S^{l+1}_{l+21,\, l1}(T) S^{l+1^*}_{l+21,\, l+21}(T)) + \text{C.C.}] + [\text{terms containing } C_l(T) \text{ and } C_{l+2}(T)]. \qquad (3.4)$$

All the terms on the right-hand side of (3.4), except the first three, depend on the elements of M. Equation (3.4) ought to be satisfied for all values of $A_l(T)$, $A_{l+2}(T)$, $C_l(T)$ and $C_{l+2}(T)$. This is possible only if the matrix M is equal to zero, while the matrix

$$S = \begin{vmatrix} S^{l+1}_{l1,\, l1}(T) & S^{l+1}_{l+21,\, l1}(T) \\ S^{l+1}_{l1,\, l+21}(T) & S^{l+1}_{l+21,\, l+21}(T) \end{vmatrix}$$

is unitary, *i. e.*, if $S^+ = S^{-1}$. Similarly, it can be shown that the matrix $'M$ is equal to zero. This also follows from the fact that, thanks to the invariance of the Hamiltonian relative to a time inversion, the total matrix of the reaction ought to be symmetric relative to the main diagonal[2]

$$S = \tilde{S}; \quad \Omega = \tilde{\Omega}; \quad 'M = \tilde{M}, \qquad (3.5)$$

where $\sim$ denotes the transposed matrix. It is not difficult to show that the most general form of the matrix S which satisfies all the above conditions is

$$S(T) = \begin{vmatrix} \cos \varepsilon^{l+1}_l(T) \exp\{i\delta^{l+1}_l(T)\}; & i \sin \varepsilon^{l+1}_l(T) \exp\left\{\frac{i}{2}(\delta^{l+1}_l(T) + \delta^{l+1}_{l+2}(T))\right\} \\ i \sin \varepsilon^{l+1}_l(T) \exp\left\{\frac{i}{2}(\delta^{l+1}_l(T) + \delta^{l+1}_{l+2}(T))\right\}; & \cos \varepsilon^{l+1}_l(T) \exp\{i\delta^{l+1}_{l+2} T)\} \end{vmatrix} \qquad (3.6)$$

the $\delta^J_{ls}(T)$ have the simple meaning of the scattering phases in states J, l, s, T, while the parameter $\epsilon^J_l(T)$ determines the amount of transitions $l \rightleftarrows (l+2)$. Frequently, $\epsilon^J_l(T)$ is known as the mixing parameter.

We shall now show that the matrices $S(T)$ and $\Omega(T)$ are expressible in terms of each other. For this purpose, we make use of the connection between C and D and A and B, which is given by the boundary conditions (2.3). Since $'M = 0$, the latter two of Eqs. (3.1) connected the amplitudes C and D. Substituting in the right and left sides of these equations for C and D their values from (2.3), and expressing B in terms of A and $S^J_{ls,\, l's'}(T)$ by means of the first two of Eqs. (3.1), we get (upon equating terms with equal powers of A) four equations which connect $S(T)$ and $\Omega(T)$. Solving them for Ω, we find:

$$\begin{aligned} \Omega_0(T) &= \frac{1}{N}\{(\gamma^{(l)}_{21} + \gamma^{(l)}_{22} S_0(T))(\gamma^{(l+2)}_{11} + \gamma^{(l+2)}_{12} S_2(T)) - \gamma^{(l+2)}_{12}\gamma^{(l)}_{22} S^2_1(T)\}, \\ \Omega_1(T) &= \frac{S_1(T)}{N}\{\gamma^{(l)}_{11}\gamma^{(l)}_{22} - \gamma^{(l)}_{12}\gamma^{(l)}_{21}\}, \\ \Omega_2(T) &= \frac{1}{N}\{(\gamma^{(l+2)}_{21} + \gamma^{(l+2)}_{22} S_2(T))(\gamma^{(l)}_{11} + \gamma^{(l)}_{12} S_0(T)) - \gamma^{(l)}_{12}\gamma^{(l+2)}_{22} S^2_1(T)\}, \\ N &= (\gamma^{(l)}_{11} + \gamma^{(l)}_{12} S_0(T))(\gamma^{(l+2)}_{11} + \gamma^{(l+2)}_{12} S_2(T)) - \gamma^{(l)}_{12}\gamma^{(l+2)}_{12} S^2_1(T), \end{aligned} \qquad (3.7)$$

where we have introduced the notation

$$\begin{aligned} S_0(T) &= S^{l+1}_{l1,\, l1}(T); \quad S_1(T) = S^{l+1}_{l1,\, l+21}(T) = S^{l+1}_{l+21,\, l1}(T), \\ S_2(T) &= S^{l+1}_{l+21,\, l+21}(T), \quad \Omega_0(T) = \Omega^{l+1}_{l1,\, l1}(T), \\ \Omega_1(T) &= \Omega^{l+1}_{l1,\, l+21}(T) = \Omega^{l+1}_{l+21,\, l1}(T); \\ \Omega_2(T) &= \Omega^{l+1}_{l+21,\, l+21}(T). \end{aligned}$$

Making use of Eqs. (3.7) and (7.1, it is not difficult to prove that the matrix Ω (T) is unitary and symmetric, *i. e.*, that it satisfies all those conditions which ought to be laid down for the scattering matrix. Thus the total scattering matrix of the system ($2p$, $2n$) which satisfies states with given J, T and P, is expressed by only three real parameters, for which we can choose, for example, $\delta^{l+1}_l(T)$, $\delta^{l+1}_{l+2}(T)$ and $\epsilon^{l+1}_l(T)$. We note here that we here that we could, with the same success, have expressed them all not by δ and ϵ but by three other parameters $\omega^{l+1}_l(T)$, $\omega^{l+1}_{l+2}(T)$ and $x^{l+1}_l(T)$ which would be related to the matrix $\Omega(T)$ in the same way as the first group were related with the matrix $S(T)$.

All the derivations of this Section apply with equal force to states with $T=1$ and with $T=0$. Since we deal experimentally with states which are superpositions of states with equal T, the total matrix of the reaction in states with momentum J are some combination of the scattering matrices which correspond to $T=1$ and $T=0$, and will therefore be determined by the same six real parameters. In the general case, where the hypothesis of charge invariance is invalid, the number of parameters in the reaction matrix is equal to 10^2.

4. THE REACTIONS T (*pp*) T, T (*pn*) He^3, He^3 (*nn*) He^3 AND He^3 (*np*) T

We begin by investigating the case of the collision of a proton with a tritium nucleus. The boundary conditions at infinity reduce in this case to the absence of ingoing waves of He^3+*n*, *i. e.*, the conditions reduce to the four equations (2.6) and (2.7). Substituting in Eq. (2.7) the expressions for $C_l(T)$ and $C_{l+2}(T)$ [(2.3) and (2.5)], expressing the $B_l(T)$ and $B_{l+2}(T)$ therein in terms of $A_l(T)$ and $A_{l+2}(T)$, and taking Eq. (2.6) into account, we get the following equations for A_l (1) and A_{l+2}(1):

$$A_l(1)\,[2\gamma_{11}^{(l)}+\gamma_{12}^{(l)}S_0^{(+)}]+A_{l+2}(1)\,\gamma_{12}^{(l)}S_1^{(+)}=C_l^{(0)},$$

$$\begin{aligned}A_l(1)\,\gamma_{12}^{(l+2)}S_1^{(+)}+A_{l+2}(1)\,[2\gamma_{11}^{(l+2)}\\+\gamma_{12}^{(l+2)}S_2^{(+)}]=C_{l+2}^{(0)},\end{aligned}\qquad(4.1)$$

$$S_i^{(+)}=S_i(1)+S_i(0)\qquad(i=0,\,1,\,2).$$

Solving (4.1) for the various A, we get:

$$\begin{aligned}A_l(1)=-A_l(0)&=\frac{1}{N_1}\{C^{(0)}[2\gamma_{11}^{(l+2)}\\&+\gamma_{12}^{(l+2)}S_2^{(+)}]-C_{l+2}^{(0)}\gamma_{12}^{(l)}S_1^{(+)}\},\\A_{l+2}(1)=-A_{l+2}(0)&=\frac{1}{N_1}\{-C_l^{(0)}\gamma_{12}^{(l+2)}S_1^{(+)}\\&+C_{l+2}^{(0)}[2\gamma_{11}^{(l)}+\gamma_{12}^{(l)}S_0^{(+)}]\},\\N_1=[2\gamma_{11}^{(l)}&+\gamma_{12}^{(l)}S_0^{(+)}].[2\gamma_{11}^{(l+2)}\\&+\gamma_{12}^{(l+2)}S_2^{(+)}]-\gamma_{12}^{(l)}\gamma_{12}^{(l+2)}S_1^{(+)2}.\end{aligned}\qquad(4.2)$$

We can now compute the matrix elements of the total reaction matrix, which corresponds to the reactions T (*pp*) T and T (*pn*) He^3. The amplitude of the outgoing (T+ *p*) waves is equal to: in a state with orbital momentum l: $D_l\equiv D_l(1)+D_l(0)$; in states with orbital momentum $l+2$: $D_{l+2}\equiv D_{l+2}(1)+D_{l+2}(0)$. To begin with, we consider the first sum.

$$\begin{aligned}D_l&=\gamma_{21}^{(l)}(A_l(1)-A_l(0))+\gamma_{22}^{(l)}(B_l(1)-B_l(0))\\&=A_l(1)\,[2\gamma_{21}^{(l)}+\gamma_{22}^{(l)}S_0^{(+)}]+A_{l+2}(1)\,\gamma_{22}^{(l)}S_1^{(+)}\end{aligned}$$

The first equation was written on the basis of Eqs. (2.3) and (2.5), and the second on the basis of Eq. (3.1). Replacing $A_l(T)$ and $A_{l+2}(T)$ by their values in (4.2), we get an expression of the following form for D_l:

$$D_l=\Omega_{l1,l1}^{l+1}C_l^{(0)}+\Omega_{l1,l+21}^{l+1}C_{l+2}^{(0)};\qquad(4.3)$$

$$\left.\begin{aligned}\Omega_{l1,l1}^{l+1}&=\frac{1}{N_1}\{(2\gamma_{21}^{(l)}+\gamma_{22}^{(l)}S_0^{(+)})\,(2\gamma_{11}^{(l+2)}\\&\quad+\gamma_{12}^{(l+2)}S_2^{(+)})-\gamma_{22}^{(l)}\gamma_{12}^{(l+2)}S_1^{(+)2}\},\\\Omega_{l1,l+21}^{l+1}&=\frac{1}{N_1}S_1^{(+)}2(\gamma_{11}^{(l)}\gamma_{22}^{(l)}-\gamma_{12}^{(l)}\gamma_{21}^{(l)}).\end{aligned}\right\}\qquad(4.4)$$

According to the same interpretation of Eq. (4.3), which expresses the amplitude of the outgoing wave by the amplitudes of ingoing waves, the coefficients $\Omega_{l1,l1}^{l+1}$ and $\Omega_{l1,l+21}^{l+1}$ are the elements of the scattering matrix of the reaction T (*pp*) T.

Continuing similar calculations for D_{l+2}, we get

$$D_{l+2}=\Omega_{l+21,l1}^{l+1}C_l^{(0)}+\Omega_{l+21,l+21}^{l+1}C_{l+2}^{(0)},\qquad(4.5)$$

where

$$\Omega_{l+21,l1}^{l+1}=\Omega_{l1,l+21}^{l+1},$$

$$\begin{aligned}\Omega_{l+21,l+21}^{l+1}=\frac{1}{N_1}\{(2\gamma_{21}^{(l+2)}+\gamma_{22}^{(l+2)}S_2^{(+)})\,(2\gamma_{11}^{(l)}\\+\gamma_{12}^{(l)}S_0^{(+)})-\gamma_{22}^{(l+2)}\gamma_{12}^{(l)}S_1^{(+)2}\},\end{aligned}\qquad(4.6)$$

The matrix elements Ω completely describe the scattering T (*pp*) T in states with given J and P.

We now proceed to the reaction T (*pn*) He^3. In order to find the elements of the reaction matrix which describes this process, we proceed exactly as in the previous case, *i. e.*, we compute the amplitudes of the outgoing waves

$$B_l=B_l(1)+B_l(0);\qquad B_{l+2}=B_{l+2}(1)+B_{l+2}(0).$$

For brevity, we do not repeat the derivation, but merely write down the result:

$$B_l = \sqrt{k_1/k}\,\{M^{l+1}_{l1,l1}C^{(0)}_l + M^{l+1}_{l1,l+21}C^{(0)}_{l+2}\}, \quad (4.7)$$
$$B_{l+2} = \sqrt{k_1/k}\,\{M^{l+1}_{l+21,l1}C^{(0)}_l + M^{l+1}_{l+21,l+21}C^{(0)}_{l+2}\},$$

where the elements of the reaction matrix describing the process T (pn) He^3 are

$$M^{l+1}_{l1,l1} = \sqrt{\frac{k}{k_1}}\frac{1}{N_1}\{2\gamma^{(l+2)}_{11}S^{(-)}_0 + \gamma^{(l+2)}_{12}[S^{(+)}_2S^{(-)}_0 - S^{(+)}_1S^{(-)}_1]\},$$

$$M^{l+1}_{l1,l+21} = \sqrt{\frac{k}{k_1}}\frac{1}{N_1}\{2\gamma^{(l)}_{11}S^{(-)}_1 + \gamma^{(l)}_{12}[S^{(+)}_0S^{(-)}_1 - S^{(-)}_0S^{(+)}_1]\},$$

$$M^{l+1}_{l+21,l1} = \sqrt{\frac{k}{k_1}}\frac{1}{N_1}\{2\gamma^{(l+2)}_{11}S^{(-)}_1 + \gamma^{(l+2}_{12}[S^{(+}_2 S_1 - S^{(-)}_2S^{(+)}_1]\}, \quad (4.8)$$

$$M^{l+1}_{l+21,l+21} = \sqrt{\frac{k}{k_1}}\frac{1}{N_1}\{2\gamma^{(l)}_{11}S^{(-)}_2 + \gamma^{(l)}_{12}[S^{(+)}_0S^{(-)}_2 - S^{(+)}_1S^{(-)}_1]\},$$

$$S^{(-)}_i = S_i(1) - S\ (0) \quad (i = 0, 1, 2).$$

We can analyze the reactions He^3 (nn) He^3 and He^3 (np) T with the help of exactly the same considerations. In this case, only the waves of He$^3 + n$ with amplitudes $A^{(0)}_l$ and $A^{(0)}_{l+2}$ appear, while the waves of T$+p$ with amplitudes

$$D_l = \sqrt{k/k_1}\,\{'M^{l+1}_{l1,l1}A^{(0)}_l + 'M^{l+1}_{l1,l+21}A^{(0)}_{l+2}\},$$
$$D_{l+2} = \sqrt{k/k_1}\,\{'M^{l+1}_{l+21,l1}A^{(0)}_l + 'M^{l+1}_{l+21,l+21}A^{(0)}_{l+2}\} \quad (4.9)$$

and the waves of He$^3 + n$ with amplitudes

$$B_l = S^{l+1}_{l1,l1}A^{(0)}_l + S^{l+1}_{l1,l+21}A^{(0)}_{l+2},$$
$$B_{l+2} = S^{l+1}_{l+21,l1}A^{(0)}_l + S^{l+1}_{l+21,l+21}A^{(0)}_{l+2}. \quad (4.10)$$

both diverge.

The corresponding elements of the reaction matrix are:

$$'M^{l+1}_{l1,l'1} = M^{l+1}_{l'1,l1}; \quad (4.11)$$

$$S^{l+1}_{l1,l1} = \frac{1}{2}S^{(+)}_0 + \frac{1}{2N_1}\{2\gamma^{(l)}_{12}\gamma^{(l+2)}_{12}S^{(+)}_1S^{(-)}_1S^{(-)}_0 - \gamma^{(l)}_{12}S^{(-)2}_0(2\gamma^{(l+2)}_{11} + \gamma^{(l+2)}_{12}S^{(+)}_2) - \gamma^{(l+2)}_{12}S^{(-)2}_1(2\gamma^{(l)}_{11} + \gamma^{(l)}_{11}S^{(+)}_0)\},$$

$$S^{l+1}_{l1,l+21} = S^{l+1}_{l+21,l1} = \frac{1}{2}S^{(+)}_1 + \frac{1}{2N_1}\{\gamma^{(l)}_{12}\gamma^{(l+2)}_{12}S^{(+)}_1(S^{(-)}_0S^{(-)}_2 + S^{(-)2}_1) - \gamma^{(l)}_{12}S^{(-)}_0S^{(-)}_1(2\gamma^{(l+2)}_{11} + \gamma^{(l+2)}_{12}S^{(+)}_2) - \gamma^{(l+2)}_{12}S^{(-)}_2S^{(-)}_1(2\gamma^{(l)}_{11} + \gamma^{(l)}_{12}S^{(+)}_0)\}, \quad (4.12)$$

$$S^{l+1}_{l+21,l+21} = \frac{1}{2}S^{(+)}_2 + \frac{1}{2N_1}\{2\gamma^{(l)}_{12}\gamma^{(l+2)}_{12}S^{(+)}_1S^{(-)}_1S^{(-)}_2 - \gamma^{(l+2)}_{12}S^{(-)2}_2(2\gamma^{(l)}_{11}+\gamma^{(l)}_{12}S^{(+)}_0) - \gamma^{(l)}_{12}S^{(-)2}_1(2\gamma^{(l+2)}_{11} + \gamma^{(l+2)}_{12}S^{(+)}_2)\}.$$

We have thus computed the total reaction matrix of the system $(2p, 2n)$ in states with $s = 1$ and $J = l + 1$, $P = (-)^l$.

			Ingoing particles		
		He$^3 + n$		T $+ p$	
		l	$l+2$	l	$l+2$
Outgoing particles	He$^3 + n$ l	$S^{l+1}_{l1,l1}$	$S^{l+1}_{l1,l+21}$	$M^{l+1}_{l1,l1}$	$M^{l+1}_{l1,l+2}$
	$l+2$	$S^{l+1}_{l+21,l1}$	$S^{l+1}_{l+21,l+21}$	$M^{l+1}_{l+21,l1}$	$M^{l+1}_{l+21,l+21}$
	T $+ p$ l	$M^{l+1}_{l1,l1}$	$M^{l+1}_{l+21,l1}$	$\Omega^{l+1}_{l1,l1}$	$\Omega^{l+1}_{l1,l+21}$
	$l+2$	$M^{l+1}_{l1,l+21}$	$M^{l+1}_{l+21,l+21}$	$\Omega^{l+1}_{l+21,l1}$	$\Omega^{l+1}_{l+21,l+21}$,

(4.13)

where matrix elements are given by Eqs. (4.12), (4.8), (4.6) and (4.4). Direct verification shows that this matrix is unitary and symmetric, as was assumed for the reaction matrix. All its elements are expressed by $S_0(T)$, $S_1(T)$, $S_2(T)$, *i. e.*, by the six real parameters $\delta^{l+1}_l(T)$, $\delta^{l+1}_{l+2}(T)$, $\epsilon^{l+1}_l(T)$.

The reaction matrix for the states with $J = l$ is found by precisely the same method. Here two values of total spin are possible, $s = 0$ and $s = 1$. In this case the following four states are possible initially and finally:

$$\mathrm{T} + p: s = 0,\ s = 1; \qquad \mathrm{He}^3 + n: s = 0,\ s = 1.$$

We omit the details and write down the result immediately. The complete reaction matrix has the form:

$$\begin{array}{ll|cc|cc}
 & & \multicolumn{4}{c}{\text{Ingoing particles}} \\
 & & \multicolumn{2}{c|}{\mathrm{He}^3+n} & \multicolumn{2}{c}{\mathrm{T}+p} \\
 & & l,s=1 & l,s=0 & l,s=1 & l,s=0 \\
\hline
\text{Outgoing particles} \quad \mathrm{He}^3+n & l,s=1 & S^l_{l1,l1} & S^l_{l1,l0} & M^l_{l1,l1} & M^l_{l1,l0} \\
 & l,s=0 & S^l_{l0,l1} & S^l_{l0,l0} & M^l_{l0,l1} & M^l_{l0,l0} \\
\hline
\mathrm{T}+p & l,s=1 & M^l_{l1,l1} & M^l_{l0,l1} & \Omega^l_{l1,l1} & \Omega^l_{l1,l0} \\
 & l,s=0 & M^l_{l1,l0} & M^l_{l0,l0} & \Omega^l_{l0,l1} & \Omega^l_{l0,l0}.
\end{array} \tag{4.14}$$

All the elements of this matrix are easily obtained from the corresponding (*i.e.*, located in the same box) elements of the matrix (4.13) with the help of the substitution

$$\gamma_{ik}^{(l+2)} \to \gamma_{ik}^{(l)}. \tag{4.15}$$

By $S_i(T)$ we now mean

$$S_0(T) = S^l_{l1,l1}(T); \tag{4.16}$$

$$S_1(T) = S^l_{l1,l0}(T)\,(=S^l_{l0,l1}(T)); \quad S_2(T) = S^l_{l0,l0}(T).$$

The matrix elements $S^l_{ls,l's'}(T)$ are expressed in terms of the corresponding real parameters $\delta^l_s(T)$ and $\epsilon^s_l(T)$ with the help of formulas similar to (3.6).

At present we are interested only in the wave functions of the system in states with given total momentum. The total scattering amplitude ought to be obtained by a summation over the various l and J. The corresponding formulas have already been derived[1].

5. THE REACTIONS He^3 (pp) He^3 AND T (nn) T

In the foregoing Section, we investigated all reactions which are possible for the system of two protons and two neutrons at low energy. In this case, the system can be found both in states with $T=1$ and in states with $T=0$. In the present Section, we consider reactions in systems ($3p$, n) and ($3n$, p), where only states with $T=1$ are possible. States with isotopic spin $T=2$ will again not be taken into consideration, since they cannot be formed in reactions were there are (in the initial and final states) only particles with isotopic spin ½.

Let us look at the system ($3p$, n). For low energies, such a system ought to have the form He^3+n at infinity and, consequently, only elastic scattering He^3 (pp) He^3 is possible. In as much as only one value of T is possible ($T=1$), the scattering matrix He^3 (pp) He^3 should coincide with the matrix $\Omega(T)$ computed in Sec. 3 for $T=1$ [Eqs. (3.7)]. However, we must take it into account that the charge of He^3 is twice that of T. This leads to a change of the function $R^{(+)}_{k1,\,l}(r)$ which describes the relative motion of He^3 and p for $r>r_0$, in comparison with the case T$+p$. The quantities $\gamma^{(l)}_{ik}$ are changed correspondingly. Equations (3.7) are introduced for the case $J=l+1$. The expressions for the scattering matrix in states with $J=l$ are obtained from (3.7) with the help of the substitution of (4.15) and (4.16) [here the element $\Omega^{l+1}_{l1,l1}(1)$ corresponds to $\Omega^l_{l1,l1}(1)$, the element Ω $\Omega^{l+1}_{l1,l+21}(1)$ to the element $\Omega^l_{l1,l0}(1)=\Omega^l_{l0,l1}(1)$ and the element $\Omega^{l+1}_{l+21,l+21}(1)$ to the element $\Omega^l_{l0,l0}(1)$]. Similarly, in the scattering case T (nn) T [the system ($3n$, p)], where there is no Coulomb interaction between the colliding particles, the elements of the scattering matrix coincide with the matrices $S_{ls,l's'}(T)$ for $T=1$ [the form of this matrix for $J=l+1$ was given above in Eq. (3.6). The total scattering amplitudes for the reactions He^3 (pp) He^3 and T (nn) T, which were obtained by summing over all J, can be obtained by using the equations of Ref. 1.

Thus, we could express the elements of the matrices of all the reactions which are possible for the systems ($2p$, $2n$), ($3p$, n) and (p, $3n$) for small energies, by the parameters $\delta^J_{ls}(T)$ and $\epsilon^J_l(T)$. This greatly simplifies the problem of the experimental determination of the parameters δ and ϵ, since we can use (in finding them) the experimental data on all six reactions written down in the introduction, and not just any one, as usually was the case. It should be remembered that the parameters δ and ϵ depend on the form of the wave function inside the sphere $r\leqslant r_0$. For each of the systems ($2p$, $2n$), ($3p$, n) and ($3n$, p), the form of the wave function for $r=r_0$ is determined by that part of the energy of the system which is connected to the charge invariant nuclear interaction. Therefore, in the common analysis, we must select such energies for the incident particles in the systems ($2p$, $2n$), ($3p$, n) and ($3n$, p), that the energy of the nuclear interaction in all these systems were equal, and therefore the wave functions for $r\leqslant r_0$ likewise. This can be done approximately in the following way. The total energy of the system ($2p$, $2n$) is composed of the mass of the two neutrons and two protons $c^2(2m_n+2m_p)$, the Coulomb interaction of the two protons E_k and the energy of nuclear interaction E_n:

$$E\,(2p,\ 2n) = E_n + E_{\kappa} + 2\,(m_n + m_p)\,c^2.$$

Similarly, the total energy of the system (3*p*, *n*) is equal to

$$E\,(3p,\ n) = E_n + 3E_\kappa + (3m_p + m_n)\,c^2. \qquad (5b)$$

In the system (3*n*, *p*), there is no Coulonb interaction, and

$$E\,(3n,\ p) = E_n + (3m_n + m_p)\,c^2. \qquad (5c)$$

In order to carry out the common analysis, we must so choose the total energy of all systems that E_n will be the same in each. This is satisfied approximately if

$$E\,(2n,\ 2p) - E\,(3p,\ n) \approx 0, \qquad (5\,d)$$

$$E\,(2n,\ 2p) - E\,(3n,\ p) \approx -0{,}4\ \text{MeV}.$$

We recall that here it is not a question of the total energy in the center-of-mass system.

6. BEHAVIOR OF THE ELEMENTS OF THE REACTION MATRIX AT THRESHOLD ENERGIES

In order to conclude the general investigation of the reaction matrix, it remains for us to study the behavior of the matrix elements for small energies of the incident particles and those generated during the reaction.

Let us consider the collision of T with *p*. At low energies, only elastic collisions are possible: T*(pp)* T. Beginning with the proton energy 1.019 Mev, (in the laboratory system), the reaction T*(pp)* He^3 becomes possible. Let us investigate the behavior of the elements of the reaction matrix in the neighborhood of 1.019 Mev. For this, we note that the boundary value of the radial wave function for $r = r_0$ [Eqs. (1.5) and (1.7)], must remain finite for all energies of the system; however, the functions entering into (1.5) and (1.7) of the free motion $R_{kl}^{(+)}(r)$ tend to infinity for $k \to 0$, *i. e.*, for energies which approach the threshold of He$^3 + n$. In order that the wave function remain finite as $k \to 0$, the following rules for approach to zero of the phases (as $kr_0 \to 0$) are necessary:

$$\delta_{ls}^{J}(T) \to \text{const}\,(kr_0)^{2l+1}; \quad \varepsilon_l^{J}(T) \to \text{const}\,(kr_0)^{2l+3}. \qquad (6.1)$$

These laws are easily obtained by use of the expansion of $R_{kl}^{(+)}(r_0)$ in a power series in kr_0, and Eqs. (1.6)

With the help of the limiting law (6.1), we can easily determine the behavior of the elements of the reaction matrix (4.13) and (4.14) as $k \to 0$:

$$(S_{ls,l's'}^{J} - 1) \to \text{const}\,(kr_0)^{2l+1},$$

$$S_{l1,l+21}^{J} \to \text{const}\,(kr_0)^{2l+3}, \qquad (6.2)$$

$$M_{ls,ls'}^{J} \to \mathfrak{M}_l\,(kr_0)^{l+{}^1/_2}; \quad M_{l1,l+21}^{J} \to \text{const}\,(kr_0)^{l+{}^1/_2},$$

$$M_{l+21,l1}^{J} \to \text{const}\,(kr_0)^{l+{}^5/_2}.$$

The constant coefficient $\mathfrak{M}_l$ entering into the formula for $M_{l\,s,ls'}^{J}$ is needed by us below. The elements $\Omega_{ls,ls'}^{J}$ remain finite for $k \to 0$, even though they have a break at $k = 0$. We first consider the case $l = 0$, *i. e.*, the element $\Omega_{0s,0s}^{s}$. It is evident from (4.4) and (4.6) that $\Omega_{0\,s,0s}^{s}$ depends only on $\gamma_{ik}^{(l)}$ and $S_i^{(l)}$. With the help of Eq. (3.7), we can express $S_i^{(+)}$, which, for $k \to 0$, ought to remain constant, since they are defined in the form of a wave function for $r < r_0$, and this function has no singularities for $k = 0$. We can expand the coefficients $\gamma_{ik}^{(0)}$ in power series in k. Substituting these expansions in the formula for $\Omega_{0\,s,0s}^{s}$, we get, after lengthy but simple calculations, for the case $(kr_0) \to 0$,

$$\Omega_{0s,0s}^{s} \to (\Omega_{0s,0s}^{s})_{k=0}\left(1 - kr_0\,\frac{1}{2}\,|\mathfrak{M}_0|^2\right). \qquad (6.3)$$

For energies lower than the threshold, only elastic scattering T (*pp*) T is possible; the modulus of $\Omega_{0s,0s}^{s}$ in this region must therefore be unity and, consequently, $|(\Omega_{0s,0s}^{s})_{k=0}| = 1$ (this is also obtained upon direct computation). The wave vector k is positive above the threshold and becomes a pure imaginary below the threshold. It therefore follows from (6.3) that, in the neighborhood of the threshold (from below), $\Omega_{0s,0s}^{s}$ has a constant modulus and varies linearly in phase (with change in $|k|$), while in the neighborhood of the threshold from above, on the other hand, the phase is constant but the modulus varies linearly. Thus, for $k = 0$, a break occurs in both the modulus and the phase of $\Omega_{0s,0s}^{s}$. Consequently, we also have a break in the cross section of T (*pp*) T, since the component $|\Omega_{0s,0s}^{s} - 1|^2$ enters into the cross section. The cross section on both sides of $k = 0$ must change linearly, while a break must be observed at $k = 0$.

The elements $\Omega_{ls,ls}^{J}$ with $l \neq 0$ must behave in precisely the same way as $\Omega_{0s,0s}^{s}$ in the neighborhood of the threshold. The only difference lies in the fact the modulus and the phase will not depend linearly on kr_0, but on a much higher power of kr_0.

The break vanishes only when $\mathfrak{M}_l = 0$, and this, as is evident from (4.8), takes place only when $S_i\ (T=1$ $=S_i(T=0)$ *i. e.*, when the dependence of the reaction matrix on the isotopic spin vanishes, and when the cross section of the reaction $\mathrm{He}^3(np)$ T is equal to zero.

7. CONCLUDING REMARKS

The results of the present research can be summarized as follows. We have succeeded in representing the elements of the reaction matrix (which correspond to possible reactions in the system of four nucleons) at low energies,

$$\mathrm{T}\,(pp)\,\mathrm{T};\;\; \mathrm{T}\,(pn)\,\mathrm{He}^3;\;\; \mathrm{T}\,(nn)\,\mathrm{T};\;\; \mathrm{He}^3\,(pp)\,\mathrm{He}^3;$$

$$\mathrm{He}^3\,(nn)\,\mathrm{He}^3;\;\; \mathrm{He}^3\,(np)\,\mathrm{T},$$

by the same parameters $\Omega^J_{ls}(T)$ and $\epsilon^J_l(T)$. The latter have the simple physical meaning of the scattering phases in a state with isotopic spin T. Thus, a common analysis of all six reactions is possible and this simplifies the problem of the experimental determination of the phases δ and ϵ.

Up to the present, we have expressed all the elements of the reaction matrix in terms of the phases δ and ϵ. This is not always useful. Sometimes it is more suitable to work with the parameters $\omega(T)$ and $x(T)$ which have the same connections with the matrix Ω as $\delta(T)$ and $\epsilon(T)$ have with the matrix $S(T)$. A corresponding calculation of all the formulas of the present work is easily carried out with the help of Eq. (3.7), which must be solved for $S_0(T)$, $S_1(T)$ and $S(T)$. The calculations here are trivial and we shall not carry them out. We must only consider that, as follows from (2.4),

$$\gamma_{11}^{(l)} = \gamma_{22}^{(l)*};\;\; \gamma_{12}^{(l)} = \gamma_{21}^{(l)}; \qquad (7.1)$$

$$|\gamma_{11}^{(l)}|^2 - |\gamma_{12}^{(l)}|^2 = \frac{k}{k_1} \quad \text{for } k<0,$$

and

$$\gamma_{11}^{(l)} = -\gamma_{21}^{(l)*};\;\; \gamma_{12}^{(l)} = -\gamma_{22}^{(l)*} \quad \text{for imaginary } k.$$

In conclusion I should like to thank Ia. A. Smorodinskii for his constant interest in the work.

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115