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Effect of the Diffuseness of the Nuclear Boundary on Neutron Scattering

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It is shown that the scattering amplitude for a potential with a diffuse boundary whose width is small compared with the wavelength of the incident particle can be expressed in terms of the phase shifts for scattering by a rectangular potential of a smaller radius and three numerical parameters which determine the shape of the potential. The parameters have been computed for potential (1) and the scattering amplitude has been determined. The problem of passage through a barrier whose width is small compared with the wavelength of the incident particle is also considered.

IN THE OPTICAL MODEL of neutron scattering at low energies (up to 3 Mev) a rectangular-well potential has been assumed[1]. In actuality the potential clearly becomes zero in a smooth fashion. The thickness of the diffuse boundary should be of the order of the range of the nuclear forces. In the considered range of energies the wavelength of the incident neutron is large compared with the surface thickness; its wavelength inside the nucleus is of the same order as the surface thickness. Under these circumstances the diffuse boundary can have an appreciable effect on the magnitude of the total cross section and on its dependence on the atomic weight. We shall use in the present work the smallness of the surface thickness compared with the wavelength of incident neutron. It will turn out that under these circumstances it is possible to neglect near the boundary the total and the "centrifugal" energies in the Schrödinger equation. This simplifies the problem considerably and permits the scattering amplitude of a neutron of arbitrary momentum to be expressed in terms of three numerical parameters which are determined solely by the shape of the potential in the boundary region. Furthermore, since the Schrödinger equation in the region of the boundary is then the same as for an s-wave, it is possible to give exact solutions for a number of potentials, as for example

$$V(r) = -K_0^2 [1 + e^{\varkappa(r-R_0)}]^{-1} \tag{1}$$

In this paper these numerical parameters will be calculated for a potential of the form (1).

It is possible to treat in a completely analogous manner the passage of a particle through a barrier whose thickness is small compared to the wavelength of the incoming particle. Section 3 of this paper is devoted to that problem.

1. INFLUENCE OF THE DIFFUSENESS OF THE NUCLEAR BOUNDARY ON NEUTRON SCATTERING

We shall write the Schrödinger equation for the wave-function, multiplied by r, for a neutron of given momentum in the form

$$d^2 u_l(r)/dr^2 + [k^2 - V_0(r) - V_1(r) - l(l+1)/r^2]\, u_l(r) = 0. \tag{2}$$

Here the complex potential $V(r)$ (multiplied by $2m/\hbar^2$) has been split into two parts:

$$V_0(r) = 0 \quad \text{for} \quad r > R_1, \qquad V_1(r) = 0 \quad \text{for} \quad r < R_1, \quad r > R_2. \tag{3}$$

We now consider (2) under the condition

$$(k\Delta R)^2 \ll 1, \quad l \sim kR \quad (\Delta R = R_2 - R_1), \tag{4}$$

which expresses the smallness of the boundary thickness compared with the wavelength of the incoming neutron. We now show that for $R_1 < r < R_2$ it is possible to neglect in (2) the terms k^2 and $l(l+1)/r^2$. The precision of this approximation corresponds to keeping all terms of order up to and including $k\Delta R$. We write (2) in the form

$$U_l(r) = j_l(kr) - \frac{h_l(kr)}{k}\int_0^r j_l(kr')\, V_1(r')\, u_l(r')\, dr' - \frac{j_l(kr)}{k}\int_r^\infty h_l(kr')\, V_1(r')\, u_l(r')\, dr'. \tag{5}$$

for $r > R_1$

$$j_l(x) = \frac{1}{2i}[h_l(x)\, e^{2i\delta_l} - h_l^*(x)], \qquad h_l(x) = \sqrt{\frac{\pi x}{2}}\, H^{(1)}_{l+1/2}(x), \tag{6}$$

where δ_l is the complex scattering amplitude for the potential $V_0(r)$.

Because of (4) one can expand the functions j_l and h_l in the region $R_1 < r < R_2$ in powers of $k(r - R_1)$ and keep only the first two terms. Utilizing the identity

$$j_l' h_l - j_l h_l' = 1,$$

we obtain for (5)

$$u_l(r) = F_l^{(1)} + (r - R_1) F_l^{(2)} + \int_0^r (r - r') V_1(r') u_l(r') dr', \tag{7}$$

$$F_l^{(1)} = j_l - \frac{1}{k} j_l h_l [\xi_l + k\Phi_l \eta_l]; \quad F_l^{(2)} = k\chi_l F_l^{(1)}, \tag{8}$$

$$j_l = j_l(kR_1), \quad h_l = h_l(kR_1), \quad \Phi_l = h_l'/h_l; \quad \chi_l = j_l'/j_l,$$
$$\xi_l = \int V_1(r) u_l(r) dr; \quad \eta_l = \int (r - R_1) V_1(r) u_l(r) dr. \tag{9}$$

Because of the matching conditions $j_l(kR_1)$ is of the order k/k' (k' is the wave vector in the region $r < R_1$), an anomalously small quantity. At the same time j'_l is of order unity. Therefore in the expansion of the products $j_l(kr)h_l(kr')$ and $j_l(kr')h_l(kr)$ terms of order $j_l' h_l' (k\Delta R)^2$ have been kept.

Differentiating (7) twice with respect to r one can convince oneself that with the employed precision

$$d^2 u_l / dr^2 = V_1(r) u_l(r). \tag{10}$$

It then follows that for $R_1 < r < R_2$

$$u_l(r) = A\varphi_1(r) + B\varphi_2(r), \tag{11}$$

where $\varphi_1(r)$ and $\varphi_2(r)$ are two linearly independent solutions of (10), satisfying the boundary conditions

$$\varphi_1(R_1) = 1, \quad \varphi_1'(R_1) = 0; \quad \varphi_2(R_1) = 0, \quad \varphi_2'(R_1) = 1. \tag{12}$$

Assuming $\varphi_1(r)$ and $\varphi_2(r)$ to be known one could obtain the scattering amplitude by matching (11) to the solutions in the regions $r < R_1$ and $r > R_2$. In the usual matching procedure one needs $u_l'(r)$. However the function $u_l(r)$ has been obtained only with a precision to $k\Delta R$, and in the process of differentiation the precision decreases by one order. We therefore use the following procedure. It follows from (7) that

$$A = F_l^{(1)}, \quad B = F_l^{(2)}. \tag{13}$$

It turns out therefore that $u_l(r)$ is expressed in the region $R_1 < r < R_2$ the standard functions and by the quantities ξ_l and η_l which enter into $F_l^{(1)}$ and $F_l^{(2)}$. On the other hand, because of (9) ξ_l and η_l are given by $u_l(r)$ in the region $R_1 < r < R_2$. Equations for ξ_l and η_l are therefore obtained by inserting (11) into (9); taking (13) and (8) into account one has

$$\xi_l = (\alpha_1 + \alpha_2 k\chi_l)\left\{j_l - \frac{1}{k} j_l h_l [\xi_l + k\Phi_l \eta_l]\right\},$$
$$\eta_l = (\beta_1 + \beta_2 k\chi_l)\left\{j_l' - \frac{1}{k} j_l' h_l [\xi_l + k\Phi_l \eta_l]\right\}, \tag{14}$$

$$\alpha_{1,2} = \int V_1(r) \varphi_{1,2}(r) dr;$$
$$\beta_{1,2} = \int (r - R_1) V_1(r) \varphi_{1,2}(r) dr. \tag{15}$$

The scattering amplitude $(\beta_l - 1)/2ik$ can be given in terms of ξ_l and η_l by

$$\beta_l = e^{2i\delta_l} - \frac{2i}{k} \int j_l(kr) V_1(r) u_l(r) dr. \tag{16}$$

Expanding $j_l(kr)$ in powers of $k(r - R_1)$ we obtain

$$\beta_l = e^{2i\delta_l} - \frac{2i}{k} j_l \xi_l - 2i j_l' \eta_l. \tag{17}$$

Inserting (14) in (17) yields

$$\beta_l = \frac{h_l^*}{h_l} \frac{k\chi_l(1 + \alpha_2) + \alpha_1 - k\Phi_l^*[1 - \beta_1 - \beta_2 k\chi_l]}{k\chi_l(1 + \alpha_2) + \alpha_1 - k\Phi_l[1 - \beta_1 - \beta_2 k\chi_l]}. \tag{18}$$

This result can be written in the form

$$\beta_l = \frac{h_l^*}{h_l} \frac{(k\chi_l)_{\text{эф}} - k\Phi_l^*}{(k\chi_l)_{\text{эф}} - k\Phi_l},$$
$$(k\chi_l)_{\text{эф}} = \frac{k\chi_l(1 + \alpha_2) + \alpha_1}{1 - \beta_1 - \beta_2 k\chi_l}. \tag{19}$$

Using the fact that $\varphi_1' \varphi_2 - \varphi_2' \varphi_1$ is a constant it is easy to show that the parameters $\alpha_{1,2}$ and $\beta_{1,2}$

satisfy the relationship

$$\alpha_1\beta_2 - \alpha_2\beta_1 + \alpha_2 - \beta_1 = 0. \qquad (20)$$

The scattering amplitude is thus given by the scattering phase shifts due to $V_0(r)$ [see (8) and (6)] and three parameters which do not depend on k or l.

2. SCATTERING FROM A POTENTIAL OF THE FORM (1)*

Choosing R_1 and R_2 such that $e^{\alpha(R_1 - R_0)}$ and $e^{\alpha(R_0 - R_2)}$ are negligible, $V_0(r)$ can be considered to be a square well, and therefore

$$k\chi_l = K' f_l(K'R_1),$$
$$f_l(x) = [\sqrt{x}J_{l+1/2}(x)]' / \sqrt{x}J_{l+1/2}(x), \qquad (21)$$
$$K' = \sqrt{K_0^2 + k^2}.$$

We now have to obtain $\alpha_{1,2}$ and $\beta_{1,2}$. From (15), (10) and (12) we find

$$\alpha_1 = \varphi_1'(R_2); \quad \alpha_2 = \varphi_2'(R_2) - 1,$$
$$\beta_1 = \varphi_1'(R_2)\,\Delta R - \varphi_1(R_2) + 1; \qquad (22)$$
$$\beta_2 = \varphi_2'(R_2)\,\Delta R - \varphi_2(R_2).$$

With the considered potential $V(r)$ one can solve Eq. (10) exactly[2]. In the actual cases the imaginary part of the potential is small[1]. One can therefore neglect the absorption in the diffuse boundary. This means that one can consider the quantities $K_0\Delta R$ and K_0/α to be real. We shall assume this from now on.

Neglecting terms of the order $e^{\alpha(R_1 - R_0)}$ one obtains easily

$$\varphi_1(r) = \frac{1}{2}\left\{e^{iK_0(r-R_1)} F\left(i\frac{K_0}{\alpha},\ i\frac{K_0}{\alpha},\ 1+2i\frac{K_0}{\alpha},\ -e^{\alpha(r-R_0)}\right) + \text{c.c.}\right\},$$
$$\varphi_2(r) = \frac{1}{2i}\left\{e^{iK_0(r-R_1)} F\left(i\frac{K_0}{\alpha},\ i\frac{K_0}{\alpha},\ 1+2i\frac{K_0}{\alpha},\ -e^{\alpha(r-R_1)}\right) - \text{c.c.}\right\} \qquad (23)$$

Here $F(\alpha, \beta, \gamma; z)$ is the hypergeometric series. Neglecting terms of the order $e^{\alpha(R_0 - R_2)}$ and utilizing the formulae

$$F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} F\left(\alpha, \alpha+1-\gamma, \alpha+1-\beta; \frac{1}{z}\right)(-z)^{-\alpha}$$
$$+ \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)} F\left(\beta, \beta+1-\gamma, \beta+1-\alpha; \frac{1}{z}\right)(-z)^{-\beta},$$
$$F(i\gamma, i\gamma, 1+2i\gamma; z) \sim \frac{\Gamma(1+2i\gamma)}{\Gamma(i\gamma)\Gamma(1+i\gamma)}(-z)^{-i\gamma}\ln(-z)$$
$$+ \frac{\Gamma(1+2i\gamma)}{\Gamma(i\gamma)\Gamma(1+i\gamma)}[2\psi(1) - \psi(i\gamma) - \psi(1+i\gamma)](-z)^{-i\gamma} \quad \text{at} \quad z \to \infty,$$
$$\psi(x) = \Gamma'(x)/\Gamma(x),$$

we obtain from (23) for $r \approx R_2$

$$\varphi_1(r) = \sqrt{\frac{\tanh \pi\gamma}{\pi\gamma}}\left\{-[\gamma\Delta_1 + K_0(r-R_0)]\sin[K_0(R_0-R_1)+\Delta]\right.$$
$$\left. + \frac{\gamma\pi}{\tanh\gamma\pi}\cos[K_0(R_0-R_1)+\Delta]\right\}, \qquad (24)$$

* The author is grateful to L. D. Landau for pointing out the importance of investigating potentials of this form.

$$\varphi_2(r)=\frac{1}{K_0}\sqrt{\frac{\tanh\pi\gamma}{\pi\gamma}}\Big\{[\gamma\Delta_1+K_0(r-R_0)]\cos[K_0(R_0-R_1)+\Delta]$$
$$-\frac{\gamma\pi}{\tanh\gamma\pi}\sin[K_0(R_0-R_1)+\Delta]\Big\}, \qquad (25)$$
$$\Delta=\arg\frac{\Gamma(1+2i\gamma)}{\Gamma^2(1+i\gamma)};\quad \Delta_1=2\psi(1)-2\mathrm{Re}\,\psi(1+i\gamma),\ \gamma=K_0/\alpha.$$

The solution to the problem is obtained by inserting (24) into (22) and using (19). Before writing the final formulae we note that we have to neglect k^2 compared to K_0^2 since in the actual case $K_0\Delta R\sim 1$; in other words, we take $K_0\approx K'$ since we have already neglected terms of the order $(k\Delta R)^2$ compared to unity.

The following expressions occur when substituting (24), (22) and (21) into (19):

$$f_l(K'R_1)\cos[K'(R_0-R_1)+\Delta]$$
$$-\sin[K'(R_0-R_1)+\Delta],$$
$$f_l(K'R_1)\sin[K'(R_0-R_1)+\Delta]$$
$$+\cos[K'(R_0-R_1)+\Delta].$$

We consider the first of these expressions. It is proportional to

$$[\sqrt{K'R_1}J_{l+1/2}(K'R_1)]'\cos[K'R_0+\Delta-K'R_1]$$
$$-\sqrt{K'R_1}J_{l+1/2}(K'R_1)\sin[K'R_0+\Delta-K'R_1].$$

The function $\sqrt{x}J_{l+1/2}(x)$ has the form

$$C_l(x)\sin\left(x-\frac{l\pi}{2}\right)+\tilde{C}_l(x)\cos\left(x-\frac{l\pi}{2}\right),$$

where $C_l(x)$ and $\tilde{C}_l$ are of the order 1 for $x>l$. In the present case $x=K'R_1\gg kR_1\sim l$. Further,

$$[\sqrt{x}J_{l+1/2}(x)]'=C_l(x)\cos(x-l\pi/2)-\tilde{C}_l(x)\sin(x-l\pi/2)$$
$$+C_l'(x)\sin(x-l\pi/2)+\tilde{C}_l'(x)\cos(x-l\pi/2).$$

Here

$$C_l'(x)\sim\tilde{C}_l'(x)\sim l^2/x^2\sim l^2(kR)^{-2}(k/K')^2\sim k^2/K'^2\ll 1,$$

and they therefore can be neglected. Our expression therefore equals

$$[\sqrt{K'R_0+\Delta}J_{l+1/2}(K'R_0+\Delta)]'/\sqrt{K'R_1}J_{l+1/2}(K'R_1).$$

The second of the two expressions can be similarly evaluated. In this manner we obtain

$$\frac{1}{(k\chi_l)_{\text{эф}}}=-\frac{\Delta_1}{\alpha}+R_0-R_1+\frac{\gamma\pi}{\tanh\gamma\pi}\frac{1}{f_l(K'R_0+\Delta)}. \qquad (26)$$

Taking into account

$$k^2\Delta R/K'\sim(k\Delta R)^2/K'\Delta R\sim(k\Delta R)^2\ll 1,$$

we obtain by inserting (26) into (19)

$$\beta_l=\frac{h_l^*(x_1)}{h_l(x_1)}\left[\frac{\tanh\pi\gamma}{\pi\gamma}K'f_l(x)-k\Phi_l^*(x_1)\right]\Big/\left[\frac{\tanh\pi\gamma}{\pi\gamma}K'f_l(x)-k\Phi_l(x_1)\right], \qquad (27)$$
$$x=K'R_0+\Delta,\quad x_1=kR_0-k\Delta/\alpha.$$

It has been shown in Ref. 1 that the optical model not only gives the scattering amplitude but also gives the ratio of the average value of the widths of the levels of the compound nucleus to the average level spacing. For our diffuse boundary potential this ratio is

$$\frac{\Gamma}{D} = \frac{2}{\pi} \operatorname{Im} \left\{ k \Big/ \left[\frac{\tanh \pi\gamma}{\pi\gamma} \cot (K'R_0 + \Delta) - ik \right] \right\}. \quad (28)$$

We note that if one assumes that γ does not depend on A and is of order 1, then $\tanh \pi\gamma / \pi\gamma \sim 1/3$ and Γ/D as a function of A has higher and steeper resonances than in the case of a rectangular well potential.

3. TRANSMISSION COEFFICIENT FOR THE PENETRATION OF POTENTIAL BARRIERS AT LOW ENERGIES.

The above problem of the influence of the diffuseness of the nuclear boundary on neutron scattering is formally very closely related to the passage of particles through potential barriers whose thickness is considerably smaller than the wavelength of the incoming particle. We arrive at this problem by putting $V_0(r) = 0$, $l = 0$, and by dropping the boundary condition $u_l(0) = 0$. This has the effect that one has to replace $j_l(kr)$ and $h_l(kr)$ in (5) by e^{-ikr} and $(i/2)e^{ikr}$ respectively if the particle approaches from the direction of positive r.

To obtain the transmission coefficients we consider (5) with the above substitutions. For $r < R_1$ we obtain

$$u(r) = e^{-ikr} \left[1 + \frac{1}{2ik} \int e^{ikr} V_1(r)\, dr \right]. \quad (29)$$

By definition, the transmission coefficient is given by

$$D = \left| 1 + \frac{1}{2ik} \int e^{ikr} V_1(r)\, dr \right|^2. \quad (30)$$

Expanding e^{ikr} in powers of $k(r - R_1)$ we have

$$D = \left| 1 + \frac{e^{ikR_1}}{2ik} \xi + \frac{e^{ikR_1}}{2i} \eta \right|^2 \quad (31)$$

where ξ and η are quantities analogous to the quantities ξ_l and η_l.

We now obtain ξ and η up to the same precision and in a similar manner like in Section 1. Inserting these into (31) we have

$$D = k^2 / (k^2 a^2 + b^2), \quad a = 1 + \alpha_2 - \beta_1, \quad b = \alpha_1 / 2.$$

The parameters $\alpha_{1,2}$ and $\beta_{1,2}$ are obtained according to (15).

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[1]Feshbach, Porter and Weisskopf, Phys. Rev. 96, 448 (1954)

[2]L. Landau and E. Lifshitz, *Quantum Mechanics*, OGIZ, Moscow (1948)

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