# Some Problems Related to the Statistical Theory of Multiple Production of Particles

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The exact values of the statistical weights of a system consisting of $N$ particles of arbitrary masses are computed by taking into account the laws of conservation of energy and momentum. The solution is expressed as a series expansion in terms of the ratio $\nu_i = \frac{M_i}{E_0}$ ($M$ is the mass of the particle and $E_0$ is the total energy of the system). Since the obtained series converges slowly, another expansion in the form of a power series of the total kinetic energy of the system to the total mass of the heavy particles has been obtained for values of $\nu_i$ which are close to unity.

THE TRANSITION PROBABILITY of a system from one quantum state into another is determined by the product of the modulus of the matrix element squared and the statistical weight of the final state. Based on the fact that for a sufficiently large number of particles the latter factor is characterized by a sharp maximum, Fermi[1] expressed the idea that the basic outlines of the processes that lead to the formation of multiple particles are determined by statistical factors. A similar view concerning multiple processes can be extended somewhat by taking into account the effect of change of the matrix element. The latter effect, however, should be computed rather approximately, while the second, statistical factor, must be computed with maximum possible accuracy*.

It is known that the statistical term consists of three factors. The first factor $(V/8\pi^3\hbar^3)^{N-1}$ ($N$ is the number of particles) is determined by the volume $V$ in the coordinate space; the second term is determined by the laws of conservation of momentum and isotopic spin and is computed by means of standard rules of quantum mechanics; the third factor constitutes the density of states $dQ_N(E_0)/dE_0 = W_N(E_0)$ in momentum space ($Q_N(E_0)$ is the volume of the system in momentum space, determined by the laws of conservation of energy and momentum). The present work deals with the problem of computing the accurate value of $W_N(E_0)$. Only certain special values were known until now: 1) for non relativistic particles, with only approximate allowance for the law of conservation of momentum[1]; 2) an accurate expression for $W_N(E_0)$ for $N = 2$;[3] 3) an accurate expression for two extreme cases: all particles are either non relativistic or highly relativistic[4,5]; 4) an accurate expression for the case when the mass of one particle is arbitrary and the masses of the remaining two or three particles are zero[6].

Computation of the expression $W_N(E_0)$ is important not only to the statistical theory described in the above discussion but also to any future consistent theory, inasmuch as the quantity $W_N$ enters into the general expression for the transition probability.

1. We shall base the computation of the statistical weight for the mixed case ($m$ slow and $n$ relativistic particles) on the general formula:

$$W_N(E_0) = \int\int_{-\infty}^{+\infty} \cdots \int \delta\left(E_0 - \sum_{i=1}^{N} \sqrt{p_i^2 + M_i^2}\right) \delta\left(\sum_{i=1}^{N} p_{xi}\right) \times \delta\left(\sum_{i=1}^{N} p_{yi}\right) \delta\left(\sum_{i=1}^{N} p_{zi}\right) \prod_{i=1}^{N} d^3p_i, \quad (1)$$

where $\delta$ stands for the $\delta$-function. The velocity of light is $c = 1$

Assuming a kinetic energy $T_i = p_i^2/2M_i$ for slow particles and $M_i = 0$ for fast particles, and using the integral presentation of the function, we obtain*:

*Actually, the influence of the interaction between nucleons and $\pi$ mesons was approximated quite successfully by Belen'kii and Nikishov[2] by introducing the method of isobars.

*We assume for the present $M_1 = M_2 = \ldots M_m$. The generalization for different masses is carried out without difficulty (see Eq. 13 below).

$$W_{m,n}(E_0) = = (2\pi)^{-4}\int\limits_{-\infty}^{+\infty}\exp[iT_0\tau_1]\,d\tau_1\iiint\limits_{-\infty}^{+\infty}d\tau_2\,d\tau_3\,d\tau_4 \times\left\{\int\limits_{-\infty}^{+\infty}\exp\left[-i\left(\frac{p^2\tau_1}{2M}+p_x\tau_2+p_y\tau_3+p_z\tau_4\right)\right]d^3p\right\}^m \times\left\{\int\limits_{-\infty}^{+\infty}\exp[i(p\tau_1+p_x\tau_2+p_y\tau_3+p_z\tau_4)]\,d^3p\right\}^n, \quad (2)$$

where $T_0 = E_0 - mM$ is the kinetic energy of the system. Making use of the results obtained in Ref. 5, we write (2) as

$$W_{m,n}(E_0) = -2\frac{(2\pi M)^{3m/2}}{(2\pi)^3}\left[-\frac{i(1-i)}{2^{1/2}}\right]^m[8i\pi]^n \times\int\limits_{-\infty}^{+\infty}\tau_1^{n-3m/2}\exp[iT_0\tau_1]\,d\tau_1\int\limits_0^\infty\frac{\tau^2\exp[imM\tau^2/2\tau_1]}{(\tau^2-\tau_1^2)^{2n}}d\tau.$$

$$\alpha = \tau_1/2M,\quad \tau = \sqrt{\tau_2^2+\tau_3^2+\tau_4^2}. \quad (3)$$

Let us evaluate the integral:

$$I = \int\limits_0^\infty\frac{\tau^2\exp[i\tau^2\eta]}{(\tau^2-\tau_1^2)^{2n}}d\tau = \frac{1}{i}\frac{dY}{d\eta}, \qquad \eta = \frac{mM}{2\tau_1};\quad Y = \int\limits_0^\infty\frac{\exp[i\tau^2\eta]}{(\tau^2-\tau_1^2)^{2n}}d\tau. \quad (4)$$

Let us further construct a differential operator from $Y$ such that in the final analysis it is determined by the simple integral

$$\int\limits_0^\infty\exp[i\tau^2\eta]\,d\tau = (\pi/2)^{1/2}(1+i)/2\eta^{1/2}.$$

Making use of the relationship

$$\frac{d^mY}{d\eta^m} = i^m\int\limits_0^\infty\frac{\tau^{2m}\exp[i\tau^2\eta]}{(\tau^2-\tau_1^2)^{2n}}d\tau,$$

we obtain

$$\sum_{k=0}^{2n}C_{2n}^k\left(\frac{\tau_1^2}{i}\right)^k\frac{d^{(2n-k)}Y}{d\eta^{(2n-k)}} = (-1)^n\left(\frac{\pi}{2}\right)^{1/2}\frac{1+i}{2\eta^{1/2}}. \quad (5)$$

The characteristic equation corresponding to (5) is

$$\sum_{k=0}^{2n}C_{2n}^k\left(\frac{\tau_1^2}{i}\right)^k\rho^{2n-k} = (\rho-i\tau_1^2)^{2n} = 0. \quad (6)$$

Therefore, $\rho = i\tau_1^2$ is the root of a characteristic equation of order $2n$; the general solution of the homogeneous equation corresponding to (5) is

$$Y_0 = P(\eta)\exp\{i\tau_1^2\eta\}, \quad (7)$$

where $P(\eta)$ is a polynomial of degree $2n-1$. We write the solution of the non-homogeneous equation in the form

$$Y = D(\eta)\exp\{i\tau_1^2\eta\}.$$

For the function $D(\eta)$ we have the following equation:

$$d^{2n}D(\eta)/d\eta^{2n} = (-1)^n2^{-3/2}\pi^{1/2}(1+i)\eta^{-1/2}\exp[-i\tau_1^2\eta]. \quad (8)$$

From this

$$D(\eta) = (-1)^n\left(\frac{\pi}{2}\right)^{1/2}\frac{1+i}{2}\underbrace{\iint\cdots\int}_{2n}\frac{\exp[-i\tau_1^2\eta]}{\eta^{1/2}}d\eta + D_1\eta^{2n-1}+D_2\eta^{2n-2}+\ldots+D_{2n}, \quad (9)$$

where $D_1, D_2 \ldots D_{2n}$ are constants.

Analysis of the asymptotic behavior of the function $Y$ shows readily that $Y(\eta)\to 0$ when $\eta\to\infty$. Therefore, $D_1 = D_2 = \ldots = D_{2n} = 0$.

Let us expand the integral (9) in powers of $\eta^{-1/2}$. After simple but tedious transformations we obtain:

$$D(\eta) = (-1)^n\left(\frac{\pi}{2}\right)^{1/2}\frac{\exp[-i\tau_1^2\eta]}{2(-1/2)!} \times\sum_{k=0}^\infty C_{k+2n-1}^k\frac{(k-1/2)!}{\eta^{k+1/2}(i\tau_1^2)^{k+2n}} \quad (10)$$

and finally

$$Y = -\left(\frac{\pi}{2}\right)^{1/2}\frac{1+i}{2i(-1/2)!} \sum_{k=0}^\infty C_{k+2n-1}^k\frac{(k+1/2)!\,i^{-k}}{(mM/2)^{k+3/2}\tau_1^{4n+k-3/2}}. \quad (11)$$

Inserting into (3) and computing the residue at $\tau_1 = 0$ we get:

$$W_{m,n}(E_0) = = 2^{3n}\pi^n(2\pi)^{3(m-1)/2}M^{3m/2}(mM)^{-3/2}T_0^{3n+3(m-1)/2-1} \times\sum_{k=0}^\infty C_{k+2n-1}^k\frac{(-1)^k(2k+1)!!}{(k+3n+(3m/2)-5/2)!}\left(\frac{T_0}{mM}\right)^k. \quad (12)$$

The series obtained can be readily generalized to include particles of different masses:

$$W_{m,n}(E_0) = 2^{3n}\pi^n (2\pi)^{3(m-1)/2} \left(\prod_{i=1}^{m} M_i\right)^{3/2} \left(\sum_{i=1}^{m} M_i\right)^{-3/2} T_0^{3n+3(m-1)/2-1}$$
$$\times \sum_{k=0}^{\infty} C_{k+2n-1}^{k} \frac{(-1)^k (2k+1)!!}{(k+3n+(3m/2)-5/2)!} \left(T_0 \Big/ \sum_{i=1}^{m} M_i\right)^k; \quad (13)$$
$$T_0 = E_0 - \sum_{i=1}^{m} M_i.$$

It should be noted that the first term of expansion (12) was obtained by Fermi[1]. However, it follows from the nature of the series (12) that it is not well converging (especially at $T_0 \sim mM/2$), and therefore the first term represents the entire series only very roughly.

2. In this section we shall give an accurate expression for the statistical weight in the general case. We shall rely on the expression obtained by Lepore and Stuart[4] (in the final state there are formed $N$ particles of arbitrary mass):

$$W_N(E_0) = \frac{(2\pi^2)^N}{(2\pi)^3} \left(\prod_{i=1}^{N} M_i^2\right) \int_{-\infty}^{+\infty} d\tau_1 \tau_1^N e^{i\tau_1 E_0}$$
$$\times \frac{\tau^2 d\tau}{(\tau_1^2-\tau^2)^N} \prod_{j=1}^{N} \{H_2^{(2)}[M_j(\tau_1^2-\tau^2)^{1/2}]\}, \quad (14)$$

where integration is to be understood in the sense of Cauchy *i.e.*, the integration path is taken to be a straight line parallel to the real axis and approaching it from below*; $H_2^{(2)}$ is the Hankel function.

Let us introduce new variables:

$$\tau_1 = \frac{y+z}{E_0}; \quad \tau = \frac{y-z}{2}; \quad \nu_j = \frac{M_j}{E_0};$$

then

$$W_N(E_0) =$$
$$\frac{\pi^{2N-3}}{2^{N+2}} E_0^{3N-4} \left(\prod_{i=1}^{N} \nu_i^2\right) \sum_{k=0}^{N} \sum_{l=0}^{2} (-1)^l C_N^k C_2^l \quad (15)$$
$$\times \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y^{k+l-N} z^{2-k-l} e^{i(y+z)} \prod_{j=1}^{N} \{H_2^{(2)}[2\nu_j (zy)^{1/2}]\}.$$

Since the Hankel function approaches zero as the argument approaches infinity, it is permissible, according to the Jordan lemma, to close the integration path from above by an arc of infinitely large radius with a cut at point $i\infty$.

We next make use of the series form of the Hankel function:

$$H_2^{(2)}[2\nu_j(yz)^{1/2}] = J_2[2\nu_j(yz)^{1/2}]$$
$$-(2i/\pi) J_2[2\nu_j(yz)^{1/2}] \ln[\nu_j(yz)^{1/2}]$$
$$+\frac{i}{\pi}\left[1+\frac{1}{\nu_j^2 yz}\right] - \frac{i}{\pi}\sum_{t=0}^{\infty} \frac{(-1)^t [\nu_j(yz)^{1/2}]^{2t+2}}{t!(2+t)!}$$
$$\left[2C - \sum_{r=1}^{t+r}\frac{1}{r} - \sum_{r=1}^{t}\frac{1}{r}\right], \quad (16)$$

where $J_2$ is the Bessel function and $C = 0.577$ is Euler's constant. Inasmuch as $\nu_j < 1$, it is appropriate to use these quantities as small parameters in which integral (16) is expanded. By comparing series (16) with (15) we expect the expansion of (15) to be of the following form:

$$W_N(E_0) = \left(\frac{\pi}{2}\right)^{N-1} E_0^{3N-4} \times$$
$$\left\{\sum_{k=0}^{\infty} W_{kN}^{(0)} + \sum_{k=0}^{\infty} W_{kN}^{(1)} + \sum_{k=0}^{\infty} W_{kN}^{(2)} + \ldots + \sum_{k=0}^{N} W_{kN}^{(N)}\right\} \quad (17)$$

where the terms $W_{kN}^{(l)}$ are proportional to $\nu^{2k}\ln^l \nu$.

*A relatively simple derivation of Eq. (14), different from that presented in Ref. 4, can be obtained. Write equation (1) in the form:

$$W_N(E_0) = (2\pi)^{-4} \int_{-\infty}^{+\infty} e^{-iE_0\tau_1} \iiint_{-\infty}^{+\infty} d\tau_2 d\tau_3 d\tau_4$$
$$\times \prod_{j=1}^{N} \left\{\iiint_{-\infty}^{+\infty} \exp[i(\tau_1 \sqrt{p_j^2+M_j^2} + (\tau p_j))]\, dp_{jx} dp_{jy} dp_{jz}\right\}.$$

Using the singular functions $\Delta^{(1)}$, $\Delta$ introduced by Dirac[7], it can be shown that

$$\iiint_{-\infty}^{+\infty} \exp[\tau_1 \sqrt{p_j^2+M_j^2} + (\tau p_j))]\, dp_{jx} dp_{jy} dp_{jz}$$
$$= \frac{(2\pi)^3 d}{i\, d\tau_1}[\Delta^{(1)}(M_j, \tau_1^2-\tau^2) + i\Delta(M_j, \tau_1^2-\tau^2)].$$

Using further the presentation of $\Delta^{(1)}$, $\Delta$ in terms of Hankel functions (see, for example, Ref. 8) we obtain (14).

To determine $W_{kN}^{(j)}$ it is necessary to select from the expansion (16), multiplied by $N$, the terms which contribute to $W_{kN}^{(j)}$, substitute into (15), as was done in Ref. 5 for the computation of $W_{0N}^{(0)}$, and determine find the indicated coefficients by computing the residues at $y = z = 0$. To determine certain coefficients it is necessary to evaluate integrals of the type

$$I_1(k, l) = \int y^{-k} e^{iy} \ln^l y \, dy. \tag{18}$$

The transformed integration contour starts from $i\infty - \delta$ ($\delta \to +0$), follows the imaginary axis, encircles zero from below and returns to point $i\infty + \delta$. It is easy to note, that

$$I_1(k, l) = \frac{d^l}{dk^l} \int e^{iy - k \ln y} dy = (-1)^k 2\pi \frac{d^l}{dk^l} \frac{i^k}{\Gamma(k)}, \tag{19}$$

where $\Gamma$ is the gamma function.

For further computations it is convenient to represent (19) by means of the well-known logarithmic derivatives of the $\Gamma$-function, which are in turn expressed through the Rieman $\zeta$ function (see, for example, Ref. 9). Thus,

$$(d/dk)/\Gamma(k) = -\psi(k)/\Gamma(k). \tag{20}$$

Making use of the expression

$$\psi(k) = -C + \sum_{r=1}^{k-1} \frac{1}{r}, \tag{21}$$

it is easy to reduce this problem also to an evaluation of residues. In particular, at $l = 1$

$$I(k, 1) = 2\pi \frac{d}{dk} \frac{i^k}{\Gamma(k)} = 2\pi i^k \left\{ \frac{\Gamma'(k)}{\Gamma^2(k)} - \frac{\pi i}{2} \frac{1}{\Gamma(k)} \right\}. \tag{22}$$

If $k = 1, 2, 3, \ldots$,

$$\frac{\Gamma'(k)}{\Gamma^2(k)} = \frac{1}{\Gamma(k)} \frac{\Gamma'(k)}{\Gamma(k)} = \frac{1}{\Gamma(k)} \left\{ -C + \sum_{r=1}^{k-1} \frac{1}{r} \right\}. \tag{23}$$

If $k = 0, -1, -2, -3, \ldots$,

$$\Gamma'(k)/\Gamma^2(k) = (-1)^{k+1}\Gamma(1-k); \quad 1/\Gamma(k) = 0. \tag{24}$$

In this manner, the problem is, in principle, completely solved.

Naturally, the series (17) is convenient for the computation of $W_N(E_0)$, if the kinetic energy $T_0 < \Sigma M_i$. Otherwise it is necessary to use too many terms of series (17) which, of course, complicates the computation. Therefore, if $T_0 < \Sigma M_i$, it is necessary to use expression (12)*.

We shall give next the values of the first several terms of series (17). For convenience we introduce the following symbols for the quantities found in the expressions for the first terms of series (17):

$$D_N^{(A)} = (-1)^A \sum_{r=0}^{N} C_N^r (N - 2r + 1) / [2N - r - (A+1)]! \, [N + r - (A+2)]! =$$
$$= (-1)^A [4N - 2(A+2)]! \, [2N - (A+1)] / [3N - 2(A+2)]! \{[2N - (A+1)]!\}^2;$$

$$F_N^{(A)} = (-1)^A \sum_{r=0}^{N} C_N^r \left\{ \frac{\alpha[2N - r - A]}{[N + r - (A+3)]!} - \frac{\alpha[2N - r - (A+1)]}{[N + r - (A+2)]!} + \right.$$
$$\left. + \frac{\alpha[N + r - (A+2)]}{[2N - r - (A+1)]!} - \frac{\alpha[N + r - (A+1)]}{[2N - r - (A+2)]!} \right\} = D_N^{(A)} \{2[4N - 2(A+2)]! \times$$
$$\times \alpha[4N - 3 - 2A] - 2[3N - 2(A+2)]! \, \alpha[3N - 2A - 3] - [2N - (A+1)]! \times$$
$$\times \alpha[2N - A] - [2N - (A+2)]! \, \alpha[2N - A - 1]\},$$

*If $T_0 \ll mM$, it is permissible to use only the first term of series (11) or (12) or to make use of their generalized representation, in the case when terms proportional to $\mu^2$ are computed for relativistic particles:

$$W_{m,n}(E_0) = 2^{3n} \pi^n (2\pi)^{3(m-1)/2} M^{3m/2} (mM)^{-3/2} T_0^{3n + 3(m-1)/2 - 1}$$
$$\times \left\{ \sum_{k=0}^{\infty} C_{k+2n-1}^k \frac{(-1)^k (2k+1)!!}{(k + 3n + 3m/2 - 5/2)!} \left(\frac{T_0}{mM}\right)^k + \frac{n}{4} \left(\frac{\mu}{T_0}\right)^2 \sum C_{k+2n-2}^k \frac{(-1)^k (2k+1)!! \, (T_0/mM)^k}{(k + 3n + 3m/2 - 9/2)!} \right\}.$$

where $A$ is a positive integer;

$$\alpha(z)=\begin{cases}(-1)^{z+1}|z|! & z=0,\ -1,-2,\ldots\\ 0 & z=1\\ \dfrac{1}{(z-1)!}\sum\limits_{r=1}^{z-1}\dfrac{1}{r} & z=2,3,4,\ldots\end{cases}$$

A. *All particles have the same mass:*

$$\begin{gathered}W_{0N}^{(0)}=D_N^{(0)},\quad W_{1N}^{(0)}=N\nu^2D_N^{(1)},\quad W_{2N}^{(1)}=N\nu^4\ln\frac{1}{\nu}D_N^{(2)},\\ W_{2N}^{(0)}=\nu^4\Big\{\Big[\frac{3}{4}N+\frac{N(N-1)}{2}\Big]D_N^{(2)}+\frac{N}{2}F_N^{(2)}\Big\},\\ W_{3N}^{(1)}=\nu^6\ln\frac{1}{\nu}\Big[N(N-1)-\frac{N}{3}\Big]D_N^{(3)},\\ W_{3N}^{(0)}=\nu^6\Big\{\Big[\frac{N(N-1)(N-2)}{6}+\frac{3}{4}N(N-1)-\frac{17}{36}N\Big]D_N^{(3)}\\ +\Big[\frac{N(N-1)}{2}-\frac{N}{6}\Big]F_N^{(3)}\Big\},\\ W_{4N}^{(2)}=\nu^8\ln^2\frac{1}{\nu}\ \frac{N(N-1)}{2}D_N^{(4)},\\ W_{0N}^{(1)}=W_{1N}^{(1)}=W_{0N}^{(2)}=\ldots=W_{3N}^{(2)}=0.\end{gathered}\tag{25}$$

B. The derived expressions can be readily generalized for the case when all *particles have different masses.**

$$\begin{gathered}W_{1N}^{(0)}=\Big(\sum_{i=1}^{N}\nu_i^2\Big)D_N^{(1)},\quad W_{2N}^{(0)}=\Big(\frac{3}{4}\sum_{i=1}^{N}\nu_i^4+\frac{1}{4}\sum_{\substack{i=1\\ i\neq j}}^{N}\sum_{j=1}^{N}\nu_j^2\nu_j^2\Big)D_N^{(2)}+\frac{1}{2}\Big(\sum_{i=1}^{N}\nu_i^4\Big)F_N^{(2)},\\ W_{2N}^{(1)}=\Big(\sum_{i=1}^{N}\nu_i^4\ln\frac{1}{\nu_i}\Big)D_N^{(2)},\quad W_{3N}^{(1)}=\Big(\frac{1}{2}\sideset{}{'}\sum_{i,j=1}^{N}\nu_i^4\nu_j^2\ln\frac{1}{\nu_i}-\frac{1}{3}\sum_{i=1}^{N}\nu_i^6\ln\frac{1}{\nu_i}\Big)D_N^{(3)},\\ W_{3N}^{(0)}=\Big(\frac{1}{36}\sum_{i,j,k=1}^{N}\nu_i^2\nu_j^2\nu_k^2+\frac{3}{8}\sideset{}{'}\sum_{i,j=1}^{N}\nu_i^4\nu_j^2-\frac{17}{36}\sideset{}{'}\sum_{i=1}^{N}\nu_i^6\Big)D_N^{(3)}\\ +\Big(\frac{1}{4}\sideset{}{'}\sum_{i,j=1}^{N}\nu_i^4\nu_j^2-\frac{1}{6}\sum_{i=1}^{N}\nu_i^6\Big)F_N^{(3)},\\ W_{4N}^{(2)}=\Big(\frac{1}{2}\sideset{}{'}\sum_{i,j=1}^{N}\nu_i^4\nu_j^4\ln\frac{1}{\nu_i}\cdot\ln\frac{1}{\nu_i}\Big)D_N^{(4)};\end{gathered}\tag{26}$$

where the primed summation sign indicates that the term $(i=j=k)$ is omitted.

3. The equations derived can be used to compare the predictions of the statistical theory with experimental data on multiple production of particles. Leaving the detailed comparison for a later paper, we are presenting now only a brief discussion of the nature of the obtained results.

The expressions obtained were compared with certain special values of $W_N(E_0)$ computed by other

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*The value of $W_{0N}^{(0)}$ does not depend on $\nu$ and is the same in all cases: the equations of the last line also apply in either case.

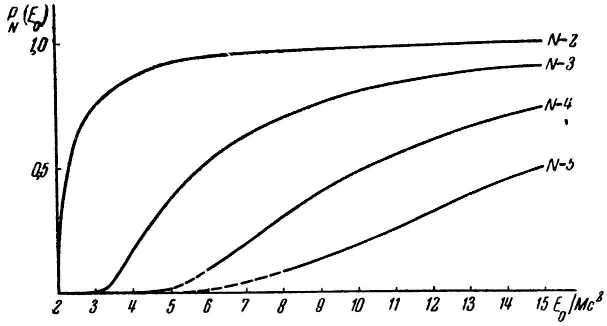


FIG. 1. $P_N(E_0)$ for the case of $N$ particles of equal mass M. ($E_0$ is the total energy of all the particles, center of mass system).

methods, namely: $W_2(E_0)^3$ and $W_3(E_0)$, $W_4(E_0)$ for the condition $M_1 \neq 0$ and $M_2 = M_2 = M_4 = 0$[6]. Identical corresponding coefficients were obtained in all cases.

In conclusion we present results of certain numerical computations. Since $W_N(E_0)$ is a rapidly increasing function of $E_0$, it is convenient to introduce the dimensionless quantity

$$P_N(E_0) = W_N(E_0) / W_N^{\mathrm{rel}}(E_0),$$

where $W_N^{\mathrm{rel}}(E_0) = (\pi/2)^{N-1} D_N^{(0)} E_0^{3N-4}$ is the extremal relativistic expression for $W_N(E_0)$; obtained earlier[4,5]. Fig. 1 shows $P_N(E_0)$ for the case of $N$

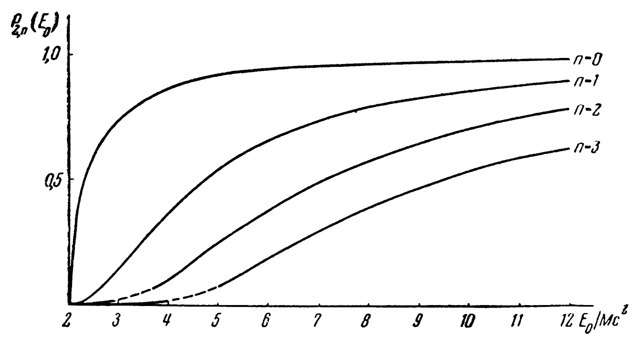


FIG. 2. $P_{2,n}(E_0)$ for the case of two nucleons and $n$ $\pi$-mesons. ($M$ denotes the nucleon mass, $E_0$ the total energy of all particles, center of mass system)

particles of equal mass $M$; Fig. 2 shows $P_{2,n}(E_0)$ for the case of two nucleons and $n$ $\pi$-mesons. The first section of each curve is plotted from Eq. (12), the remaining portion is drawn according to Eq. (17), using 8 to 12 terms; the dotted lines show the sections of the curves which were interpolated between the two equations.

A simple analysis of these curves shows that in a very important case when the condition $E_0 \sim \sum_{i=1}^{N} M_i$, is satisfied, the use of the extremal expressions may lead to completely incorrect results.

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*Note added in proof.* It was determined, after sending this paper to press, that series (13) can be summed and written in closed form. Unable to present here the derived expressions in general form we will give only the most interesting case of two nucleons and $n$ $\pi$-mesons ($M = 1$):

$$W_{2,n}(E_0) = \frac{2^{3n+3}\pi^{n+1}}{(2n-1)!}\Big\{\arctan\sqrt{E_0-2}$$

$$\times\sum_{k=1}^{3n} p_{3n-k}(E_0-2)^{3n-k} - \sqrt{E_0-2}\cdot\sum_{l=0}^{3n-2} q_l (E_0-2)^l\Big\},$$

$$\text{where } p_{3n-k} = \frac{(-1)^{k-1}(2k-1)!\,(2n-{}^3/_2-k)!}{[(k-1)!]^2\,(3n-k)!\,2^{2k}\sqrt{\pi}};$$

$$q_l = \sum_{i=0}(-1)^i\,\frac{p_{l-i}}{2i+1}.$$

[1] E. Fermi, Progr. Theor. Phys. **5**, 570 (1950).

[2] S. Z. Belen'kii and A. I. Mikishov, J. Exptl. Theoret. Phys. (U.S.S.R.) **28**, 744 (1955).

[3] M. I. Podgoretskii and I. L. Rosental', J. Exptl. Theoret. Phys. (U.S.S.R.) **27**, 129 (1954).

[4] J. V. Lepore and R. N. Stuart, Phys. Rev. **94**, 1724 (1954).

[5] I. L. Rozental', J. Exptl. Theoret. Phys. (U.S.S.R.) **27**, 1 (1955).

[6] R. Milburn, Rev. Mod. Phys. **27**, 1 (1955).

[7] P. Dirac, Proc. Camb. Phil. Soc. **30**, 150 (1934).

[8] A. I. Akhiezer and V. B. Berestetski, *Quantum Electrodynamics*, GITTL (1953)

[9] E. G. Whittaker and G. H. Watson, Modern Analysis II, GTTI, (1934).

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