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Asymptotic Meson-Meson Scattering Theory

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An asymptotic expression for the scattering amplitude for meson-meson interaction has been obtained from a theory of the Landau, Abrikosov, and Khalatnikov type by summation of an infinite number of graphs of a certain class. The problem reduces to an integral equation of a simple type, which can be solved exactly.

1. INTRODUCTION

NOT LONG AGO Landau, Abrikosov, Khalatnikov and Galanin[1,2] developed a new approach to the solution of field theory equations. While assuming that the coupling constant g_0 is small, i.e. $g_0^2 \ll 1$, and investigating the series expansion of all quantities for large momenta in powers of g_0^2 they do not however consider the quantity

$$\varkappa = g_0^2 \ln [\Lambda^2/(-p^2)] = g_0^2 (L - \xi)$$

to be small (in contrast with the assumption used in conventional perturbation theory), here

$$L = \ln (\Lambda^2/m^2) > 1, \quad \xi = \ln (-p^2/m^2) > 1,$$

where Λ is the limit for momentum cut-off and p the momentum pertinent to the problem.

The result is that the asymptotic expressions of different field theory quantities are represented by series of the type:

$$f_0(\varkappa) + g_0^2 f_1(\varkappa) + g_0^4 f_2(\varkappa) + \ldots, \tag{1}$$

where the $f_n(\varkappa)$ are closed functions. In fact, the above authors[1,2] used the integral equations of field theory to construct the $f_0(\varkappa)$ term of the zero approximation for the case of a series expansion of the Eq. (1) type for propagation functions and functions for the vertex parts in quantum electrodynamics[1] and mesodynamics[2].

Essentially the condition that $g_0^2 < 1$ is not necessary for the existence of an expansion in a series of the Eq. (1) type. As has been shown by Pomeranchuk[3], when one introduces two cut-off limits[4]

$$L_p = \ln (\Lambda_p^2/m^2), \quad L_p > L_k \gg 1$$

then in all equations and in particular in Eq. (1), g_0^2 is replaced by the quantity

$$\tilde{g}_0^2 = g_0^2 [1 + (g_0^2/\pi)(L_p - L_k)]^{-1}, \tag{2}$$

which can be as small as one likes for any g_0^2, provided that $(L_p - L_k)/\pi$ is large enough. Thus, when $(L_p - L_k)/\pi \to \infty$ the series in Eq. (1) converges as quickly as one may like, and its zero term is in fact an exact solution.*

From the standpoint of an expansion in a series like Eq. (1), the amplitude of meson-meson scattering is a quantity of the order of g_0^2 (or $\tilde{g}_0^2$ when there are two limits). Actually, the corresponding quantity for the simplest graphs in Fig. 1 has a value proportional to the product of g_0^4 and a logarithmically divergent integral of the type $\int d^4p/p^4 \sim L$ *i.e.*, a value on the order of g_0^2 [since $g_0^4 L = g_0^2(g_0^2 L)$].

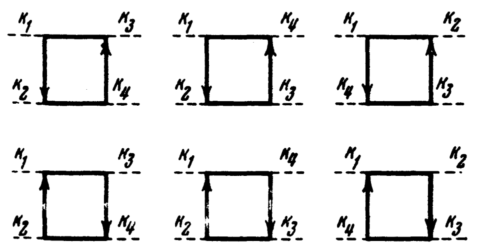

FIG. 1

Accurate computation, with inclusion of only the most important logarithmic part of the resultant integral**, shows that when the momenta are large, the sum of the contributions from the graphs in Fig. 1 depends only on the largest momentum, k, of the four momenta k_1, k_2, k_3, k_4 of the mesons, *i.e.*,

$$-k^2 = \max(-k_1^2, -k_2^2, -k_3^2, -k_4^2)$$

and can be written as

$$(g_0^2/4\pi i)\, R_0 \times (k_1, k_2, k_3, k_4) \approx (g_0^2/4\pi i)\, R_0 [g_0^2 (L - \xi)], \tag{3}$$

where for the neutral pseudoscalar theory

$$R_0 = 24\,(1 - Q^{-1/5}), \tag{3a}$$

$$Q = 1 + (5 g_0^2/4\pi)(L - \xi) \tag{4}$$

and analogously* for the symmetrical pseudoscalar theory

$$R_0 = \rho_0 \delta_c, \quad \rho_0 = \frac{16}{30}[Q^{3/5} - 1], \tag{3b}$$

with the quantity

$$\delta_c = \delta_{\xi_1\xi_2}\delta_{\xi_3\xi_4} + \delta_{\xi_1\xi_3}\delta_{\xi_2\xi_4} + \delta_{\xi_1\xi_4}\delta_{\xi_2\xi_3} \tag{5}$$

being dependent of indices (or variables) ξ_i of the isotopic spin ($i = 1, 2, 3, 4$) of the four mesons ($\xi_i = 1, 2, 3$). It is noteworthy that electrodynamics differs from the pseudoscalar meson theory in that the logarithmically divergent part is eliminated when the contributions from the graphs in Fig. 1 are summed, so that they have a value of the order of e_0^4 (rather than of the order of e_0^2 as for the $e_0^2\,(e_0^2 L)$ type).

All of the more complicated graphs for meson-meson scattering can be categorized as "reducible" or "irreducible" to simplest forms (of the type in Fig. 1), if "reducible" is taken to mean those graphs

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*This is due to the incompleteness of the present theory, according to which, as Pomeranchuk has observed[3], the renormalized charge $g^2 = g_0^2\,[1 + 5 g_0^2 L/4\pi]^{-1}$ becomes zero when $L \to \infty$. Actually, if all the quantities in series (1) are renormalized, i.e. $fn(x) = Z_n\,(g^2 L) f_n\,(g^2\,\xi)$, then we obtain $f_0(g^2\xi) + g^2 f_1(g^2\xi) + g^4 f_2(g^2\xi)$.

When $g^2 \to 0$ only the term f_0 is significant.

**If we include values for the propagation functions and vertex parts as given in Ref. 2 we have: $D(k) = d/k^2$, $G(k) = \beta/k$, $\Gamma_5 = \alpha\gamma_5$ where for the symmetrical theory $\alpha = Q^{1/5}$, $\beta = Q^{-3/10}$, $d = Q^{-4/5}$ and analogously for the neutral theory $\alpha = Q^{-1/5}$, $\beta = Q^{-1/10}$, $d = Q^{-2/5}$; Q is determined in Ref. 4.

*For example, the first graph in Fig. 1 has, to within a multiplicative factor $g_0^2/4\pi i$, the value

$$R_0 = (\rho_0/4)\,\mathrm{Sp}\,(\tau_{\xi_1}\tau_{\xi_2}\tau_{\xi_3}\tau_{\xi_4}),$$

$$\rho_0 = (g_0^2/\pi i)\int \mathrm{Sp}\,[\Gamma_5 G(p)\,\Gamma_5 G(p + k_1) \times \Gamma_5 G(p + k_1 + k_2)\,\Gamma_5 G(p - k_3)]\, d^4p$$

$$(k_1 + k_2 + k_3 + k_4 = 0).$$

We have made use of Feynman's notation where $g_0\Gamma_5$ corresponds to the point, $G(P)$ to the nucleon line, $4\pi i D(k)$ to the meson line, and $\int d^4p/(2\pi)^4$ to the integration $\int dp_0\, dp_1\, dp_2\, dp_3\,/\,(2\pi)^4$.

$$\int d^4p\,/\,(2\pi)^2 = \int dp_0 dp_1 dp_2 dp_3\,/\,(2\pi)^4.$$

By including only the logarithmic region $-k^2 \ll -p^2 \ll \Lambda^2$ or $\xi < z < L$, $z = \log(-p^2/m^2)$ in the integral and changing variables to $q = 1 + (5 g_0^2/4\pi)(L - z)$, we obtain

$$\rho_0 = (^{16}/_5)\int_1^Q \alpha^4(q)\,\beta^4(q)\,dq$$

$$= (^{16}/_5)\int_1^Q q^{-2/5}\,dq = (^{15}/_3)\,(Q^{3/5} - 1),$$

with the sum over the graphs in Fig. 1 of the quantity of the type ¼ Sp $(\tau_{\xi_1}\tau_{\xi_2}\tau_{\xi_3}\tau_{\xi_4})$ being equal to δ_c.

that consist only of nucleon squares joined by meson lines and subject to successive simplification to to one of the graphs in Fig. 1. Such simplification consists of substituting, in some sequence, one square for two joined by two meson lines in the complex graph (*e.g.*, the graphs in Fig. 2 are "reducible", while those in Fig. 3 are "irreducible").

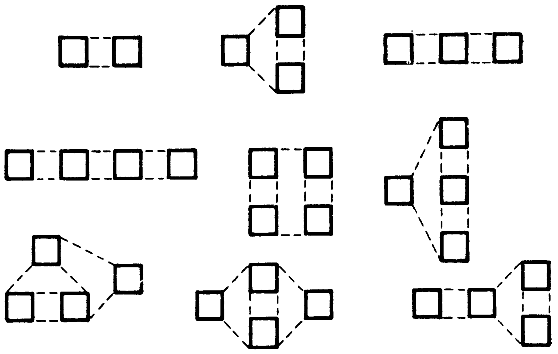

FIG. 2

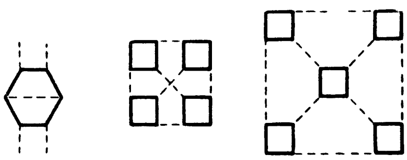

FIG. 3

It is easily seen that such substitutions do not entail any alteration in the order of magnitude of a graph (symbolically, in the sense of a location in an expansion of type (1)), because in this case the number of points on the graph is reduced by four, the number of divergent integrals by two, and the contribution from the graph changes by a factor of the form $g_0^4(L-\xi)^2 = [g_0^2(L-\xi)]^2$ which is of the order of unity. Therefore, any "reducible" graph makes a contribution of the same order

$$(g_0^2/4\pi i)\, R_n\,(k_1, k_2, k_3, k_4)$$
$$\approx (g_0^2/4\pi i)\, R_n\,[g_0^2(L-\xi)]$$

(R_n is a dimensionless function and n is the number of the "reducible" graph) as contributions (3), (3a) and (3b) from the simplest graphs in Fig. 1. Analogously, simple calculation shows that "irreducible" graphs* have a value of a higher order in g_0^2. Therefore, if the scattering amplitude is written as

$$(g_0^2/4\pi i)\, P'\,(k_1, k_2, k_3, k_4),$$

then P' will be represented, when the momenta are large, by a series of type (1)

$$P' = P\,[g_0^2(L-\xi)] + g_0^2 Q\,[g_0^2(L-\xi)] + \dots, \quad (6)$$

where the first term P of this expansion (which, as was mentioned above, gives in fact the exact value of the scattering amplitude in meson theory) is determined by the infinite sum of the contributions from all the "reducible" graphs

$$P\,(k_1, k_2, k_3, k_4) = \sum_{n=0}^{\infty} R_n\,(k_1, k_2, k_3, k_4). \quad (7)$$

Below we consider the problem of computing this sum (the so-called "parquet" problem) where it is shown that the total sum is determined from a quantity R_0 with the aid of an integral equation, which in the case of large momenta is simple in form and can be solved exactly.

When L is fixed, the magnitude of P proves to be of the same order as that of R_0 (despite the fact that the absolute value of the individual terms of the infinite alternating series, Eq. (7), increases rapidly as n increases), whereby P possesses the usual renormalization property and, in the limit as $L \to \infty$, can be renormalized without introducing interaction terms of the $\lambda\,\varphi^4$ type in the Hamiltonian.

The "parquet" problem is important for the evaluation of terms left out of the zeroth approximation [2] (in g_0^2), which proves, as Pomeranchuk has shown[4], a zero value for the renormalized charge g. In the theory advanced by Abrikosov, Galanin, and Khalatnikov[2], the equation for the vertex part operator (interaction operator) did not include (besides graphs with intersecting meson lines) graphs of the type shown in Fig. 4a and all the more complicated ones obtained from Fig. 4a by substituting

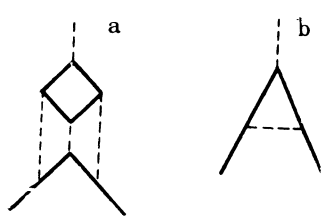

FIG. 4.

*This was noted by Landau and does not apply to electrodynamics, in which, for example (because of the elimination of the divergences in the graphs in Fig. 1), "reducible" graphs even with two squares (Fig. 5) have a value on the order of e_0^2 in comparison with the contribution from the graphs in Fig. 1.

any graph reducible to a square for a nucleon square. When $g_0^2 \ll 1$ or, $\tilde{g}_0^2 \ll 1$ in the theory with two limits, the contribution from the graph in Fig. 4a is small (it contains an extra factor $(g_0^2/4\pi i)\,R_0$, of the order of $\tilde{g}_0^2$ in comparison with the contribution by the graph in Fig. 4b which was calculated by Abrikosov, *et al.*[2]). However, the evaluation of the total contribution by all of the more complicated graphs of the type shown in Fig. 4a, but with the square replaced by a complex graph of meson-meson scattering, depends essentially on the value of the sum $P = \sum_{n=0}^{\infty} R_n$ (if this sum should turn out to be divergent, it is not permitted to neglect all the graphs of the type shown in Fig. 4a).

To avoid the difficulty of computing the sum $P = \sum_{n=0}^{\infty} R_n$ in order to evaluate the contribution by all graphs of the type shown in Fig. 4a, Pomeranchuk analyzed a special type of limit process for point interaction, in which $L_k/(L_p - L_k) \to 0$ as $L_k \to \infty$. In this case, all the complex graphs of meson-meson scattering (including the "reducible" ones) yield in the limit a contribution infinitely small in comparison with the simplest graphs shown in Fig. 1. For example, the contribution from graphs of the type a, b, c shown in Fig. 5 contains, in comparison with Eq. (3), twice the factor $\tilde{g}_0^2 = \pi/(L_p - L_k) \to 0$ and two divergent integrals over the nucleon and meson momenta that give the factor $L_p\,L_k$, so that the total is a factor of the order of

$$L_p L_k\,(\tilde{g}_0^2)^2 \sim L_p L_k\,/\,(L_p - L_k)^2 \sim L_k\,/\,L_p \to 0,$$

which goes to zero when $L_k \to \infty$. In the limit $P = R_0$, i.e. the sum of the contributions from all of the complex graphs of the type shown in Fig. 4a coincides with the contribution from a single simple graph of the type shown in Fig. 4a.

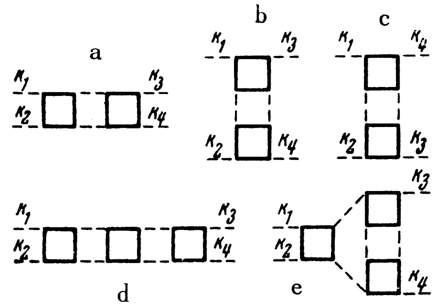

FIG. 5.

The conclusion obtained below concerning the finiteness of P is in agreement with Pomeranchuk's result which shows that the prediction that renormalized charge becomes zero in the pseudoscalar theory holds true not only for a special type of transition to the limit, *i.e.*, where $L_p/(L_p - L_k) \to 0$, but also for a more general case (if the sum P is a quantity of the same order as R_0, then the contribution from all graphs of the type shown in Fig. 4a is a quantity of the order of $\tilde{g}_0^2$ in comparison with the contribution from Fig. 4b and it may be disregarded if $\tilde{g}_0^2 \ll 1$, *i.e.*, if $[\pi/(L_p - L_k)]$ is sufficiently small as $L_k \to \infty$.

2. THE EXACT INTEGRAL EQUATION

We shall show that the sum (7) of the contribution by all of the "reducible" graphs* satisfies an exact integral equation whose form depends only on the value $R_0(k_1, k_2, k_3, k_4)$ of the contribution from the simplest graphs in Fig. 1.

First we shall introduce the concept of reducible and irreducible graphs and derive a simple general relation (analogous to the Dyson-Schwinger equations for propagation functions or to the Bethe-Salpeter equation), which is satisfied by the total contribution $P'(k_1, k_2, k_3, k_4)$, Eq. (6), of all general "reducible" and "irreducible" graphs.

Let us examine an arbitrary meson-meson scattering graph and call it reducible as regards the separation of meson lines ("ends") k_1 and k_2 from k_3 and k_4, if it can be divided, at least in one certain manner, into two parts connected with one another by only two meson lines, with the division made in such a way that lines k_1 and k_2 approach one part and k_3 and k_4 the other (*e.g.*, the graphs in Fig. 5a, d,e, are reducible). We shall call graphs that do not possess this property irreducible relative to "separation" of k_1, k_2 from k_3, k_4 (*e.g.*, the graphs in Fig. 1 and Fig. 5b, c).

One and the same graph, depending on how the meson lines approach it, can be reducible or irreducible relative to the separation of k_1, k_2 from k_3, k_4 (the graph in Fig. 5a is reducible, while 5b and 5c are irreducible).

Let us designate by $R'(k_1, k_2, k_3, k_4)$ the sum of

*Henceforth, to simplify the notation, we shall neglect the factor $(g_0^2/4\pi i)$, *i.e.*, we shall assume the contribution by the graph for meson-meson scattering to be a value determined in accordance with Feynman's rules and multiplied by $(4\pi i/g_0^2)$.

the contributions from all of the graphs generally irreducible in the indicated sense, and by $F'(k_1, k_2, k_3, k_4)$ the sum of the contributions from all of the reducible ones. Since any graph is either reducible or irreducible, it is obvious that

$$P'(k_1, k_2, k_3, k_4) = R'(k_1, k_2; k_3, k_4) + F'(k_1, k_2; k_3, k_4). \tag{8}$$

The quantity P' is symmetrical for any transposition of the meson lines, and the values of R' and F' in Eq. (8) are unchanged if k_1 and k_2 or k_3 and k_4 are transposed or if k_1 and k_2 are interchanged with k_3 and k_4 (because they include a contribution from all of the graphs). For example, R' includes a contribution from both the graph in Fig. 5b and that in 5c (the latter differs from 5b in the transposition of k_3 and k_4), so that one may write

$$R'(k_1, k_2; k_3, k_4) = R_1'(k_1, k_2; k_3, k_4) + R_1'(k_1, k_2; k_4, k_3), \tag{9}$$

where the quantity $R_1'(k_1, k_2; k_3, k_4)$ includes the irreducible graphs only for arrangement of lines k_3 and k_4 (thus, R' includes both graphs in Fig. 5b, 5c, while $R_1'(k_1, k_2; k_3, k_4)$ includes only one of them, *e.g.*, Fig. 5b).

In this case, it is not difficult to see that P' and R' are connected by the following integral equation

$$P'(k_1, k_2, k_3, k_4) = R'(k_1, k_2, k_3, k_4) - (g_0^2/2\pi i)\int R'(k_1, k_2; l, l') \times D(l)\, D(l')\, P'(-l, -l', k_3, k_4)\, d^4l, \tag{10}$$

where $-l' = l + k_1 + k_2$, D is the meson propagation function, and $d^4l = dl_0 dl_1 dl_2 dl_3/4\pi^2$.

In order to check the correctness of this relation, let us examine an arbitrary reducible graph (relative to the separation from k_1, k_2 and from k_3, k_4) and select the point of separation such that the part adjoining k_1, k_2 is already reducible as regards the separation of k_1, k_2 from meson lines l and l', which connect both parts of the graph. Let us use $\rho_n'(k_1, k_2; l, l')$ to designate the contribution from the part of the (irreducible) graph adjoining lines k_1, k_2 (we assume all graphs to be renumbered, with index n indicating the graph number in the part adjoining k_1, k_2) and $\sigma_m'(-l, -l', k_3, k_4)$ to designate the contribution from the part (of number m) adjacent to k_3 and k_4.* Then the form of the reducible graph under consideration will be determined by the two numbers n and m, and the corresponding contribution $f_{nm}(k_1, k_2; k_3, k_4)$ can be written as

$$f_{nm}'(k_1, k_2; k_3, k_4) = -(g_0^2/\pi i)\int \rho_n'(k_1, k_2; l, l')\, D(l)\, D(l') \times \sigma_m'(-l, -l', k_3, k_4)\, d^4l. \tag{11}$$

The factor $-(g_0^2/\pi i)d^4l$ is the product of the quantities $g_0^2(4\pi i$ from the definition of the contribution from the graph (*cf.*, footnote, p. 634), $(4\pi i)^2$ from the two meson lines, and $d^4l/4\pi^2$ from the integration over the meson momentum.

Let us now sum both sides of Eq. (11) over all numbers n, m (*i.e.*, over all of the graphs); it is now obvious that

$$\sum_m \sigma_m'(-l, -l', k_3, k_4) = P'(-l, -l', k_3, k_4), \tag{12}$$

since any particular graph may be adjacent to k_3 and k_4. By the same token

$$\sum_n \rho_n'(k_1, k_2; l, l') = R'(k_1, k_2; l, l'), \tag{13}$$

since only irreducible graphs are, by definition, adjacent to k_1 and k_2. Furthermore, it is evident that

$$\sum_{n,m} f_{nm}'(k_1, k_2; k_3, k_4) = 2F'(k_1, k_2; k_3, k_4); \tag{14}$$

the factor 2 appears because half of all the graphs from the sum in Eq. (13) that enter into $R_1'(k_1, k_2; l, l')$ will, on account of the symmetry of P' in l and l', give exactly the same set of all reducible graphs as will the other half of the sum in Eq. (12), which enters into $R_1'(k_1, k_2,; l', l)$.

By utilizing equalities (12) to (14) we obtain, after summing both halves members of (11) over n and m,

*Since k or l always represent the momentum *entering* into any graph, then $k_1 + k_2 + l + l' = -l - l' + k_3 + k_4 = 0$, while the momenta l and l', which enter into the part adjacent to k_1 and k_2, *come out* of the other part and are written in σ_m' with a minus sign.

$$F'(k_1, k_2; k_3, k_4) = -(g_2^0/2\pi i)\int R'(k_1, k_2; l, l') \times D(l)\, D(l')\, P'(-l, -l', k_3, k_4)\, d^4 l. \quad (15)$$

It is obvious that Eq. (15) together with Eq. (8) gives Eq. (10).

Now let us consider only the "reducible" graphs and designate by P, R and F the corresponding sums determined by them only. With respect to these graphs, all of the considerations that led to Eq. (8), (15) and (10) can be repeated literally; in an analogous fashion we obtain

$$P(k_1, k_2, k_3, k_4) = R(k_1, k_2; k_3, k_4) + F(k_1, k_2; k_3, k_4), \quad (16)$$

$$F(k_1, k_2; k_3, k_4) = -(g_0^2/2\pi i)\int R(k_1, k_2; l, l')\, D(l) \times D(l')\, P(-l, -l', k_3, k_4)\, d^4 l.$$

These relations permit one to solve the problem stated above, if one bears in mind that the "reducible" graph, if it is not the simplest one (Fig. 1), is necessarily reducible as regards the separation of any one pair of meson lines from another,

k_1, k_2 from k_3, k_4, or k_1, k_3 from k_2, k_4, or k_1, k_4 from k_2, k_3.

Actually, the "reduction" process makes it possible to bring any reducible graph to the form of Fig. 5a, 5b or 5c, where that which has been said becomes obvious.* In accordance with this, all reducible complex graphs fall into three classes, and their contribution is contained either in $F(k_1, k_2; k_3, k_4)$, in $F(k_1, k_3; k_2, k_4)$, or in $F(k_1, k_4; k_2, k_3)$ (the contribution from each reducible graph belongs in one and only one of these three functions, because if a graph is reducible as regards the separation of any one pair of lines from another, it is then irreducible to any other division of meson lines into pairs. This is made clear by any analysis of the graphs a, b, c in Fig. 5 to which any complex reducible graph may be "reduced"). Therefore, by including the independent contribution R_0 from the simplest graphs in Fig. 1 we obtain

$$P(k_1, k_2, k_3, k_4) = R_0(k_1, k_2, k_3, k_4) + F(k_1, k_2; k_3, k_4) + F(k_1, k_3; k_2, k_4) + F(k_1, k_4; k_2, k_3). \quad (17)$$

Taken with Eq. (16) this yields

$$R(k_1, k_2; k_3, k_4) = R_0(k_1, k_2, k_3, k_4) + F(k_1, k_3; k_2, k_4) + F(k_1, k_4; k_2, k_3), \quad (18)$$

$$F(k_1, k_2; k_3, k_4) = -(g_0^2/2\pi i)\int [R_0(k_1, k_2, l, l') + F(k_1, l; k_2, l') + F(k_1, l'; k_2, l)] \times D(l)\, D(l')\, [R_0(-l, -l', k_3, k_4) + F(-l, k_3; -l', k_4) + F(-l, k_4; -l', k_3) + F(-l, -l'; k_3, k_4)]\, d^4 l. \quad (19)$$

Transposing k_2 and k_3 or k_2 and k_4 in both sides of Eq. (19), we obtain two more analogous equations and thus have a system of three integral equations, which identically determine the functions

$F(k_1, k_2; k_3, k_4)$, $F(k_1, k_3; k_2, k_4)$ and $F(k_1, k_4; k_2, k_3)$,

in terms of the known quantity $R_0(k_1, k_2,; k_3, k_4)$

3. THE INTEGRAL EQUATION FOR THE CASE OF LARGE MOMENTA

a) Consequences of Eq. (19).

If the meson momenta k_i are large, as in the case of the neutral theory, Eq. (19) can be reduced to the form

$$\varphi(\xi, \eta, \zeta) = \frac{-g_0^2}{8\pi}\int_\eta^\xi [R_0(\xi) + 2F(\xi)]\,[R_0(\zeta) + 2F(\zeta) + \varphi(\lambda, \eta, \zeta)]\, d^2(\lambda)\, d\lambda$$
$$- \frac{g_0^2}{8\pi}\int_\zeta^\xi [R_0(\xi) + 2F(\xi)]\,[R_0(\lambda) + 2F(\lambda) + \varphi(\lambda, \eta, \zeta)]\, d^2(\lambda)\, d\lambda \quad (20)$$
$$- \frac{g_0^2}{8\pi}\int_\xi^L [R_0(\lambda) + 2F(\lambda)][R_0(\lambda) + 2F(\lambda) + \varphi(\lambda, \eta, \zeta)]\, d^2(\lambda)\, d\lambda.$$

*By carrying out the "reduction" process in reverse we can restore to the original form both parts of the graph which were obtained from each square of the graph (e.g., Fig. 5a). However, these parts will be joined by only two meson lines.

here

$$\xi = \ln(-k_{\mathrm{I}}^2/m^2),\quad \zeta = \ln(-k_{\mathrm{II}}^2, m^2),\quad \eta = \ln[-(k_1+k_2)^2/m^2],$$
$$\lambda = \ln(-l^2/m^2),\quad -k_{\mathrm{I}}^2 = \max(-k_1^2,\ -k_2^2,\ -(k_1+k_2)^2),$$
$$-k_{\mathrm{II}}^2 = \max(-k_3^2,\ -k_4^2,\ -(k_3+k_4)^2),$$

for which the most general case $\xi > \zeta > \eta$ is treated, when $F(k_1, k_2; k_3, k_4)$ is dependent on all three quantities ξ, η, ζ, *i.e.*,

$$F(k_1, k_2; k_3, k_4) = \varphi(\xi, \eta, \zeta) \tag{21}$$

and when, according to Eq. (17) and Eq. (18),

$$P(k_1, k_2; k_3, k_4) = R_0(\xi) + 2F(\xi) + \varphi(\xi, \eta, \zeta), \tag{22}$$
$$R(k_1, k_2; k_3, k_4) = R_0(\xi) + 2F(\xi),$$

with*

$$F(\xi) = \varphi(\xi, \xi, \xi) \approx F(k_1, k_3; k_2, k_4)$$
$$\approx F(k_1, k_4; k_2, k_3).$$

If, however, all the momenta k_i and all of their sums $k_i + k_j$ are quantities of the same order, then $\xi = \eta = \zeta$ and

$$P(k_1, k_2; k_3, k_4) = R_0(\xi) + 3F(\xi). \tag{23}$$

It is assumed everywhere that the quantities P and F depend, like R_0, logarithmically on the momenta when the momenta are large [this is confirmed by the solution of Eq. (20)]. When Eq. (20) was obtained from Eq. (19), the only region included was $\eta < \lambda < L$, in which the integral over l is logarithmic and for which $D(l)D(l')d^4 l \approx (i/4)d^2(\lambda)d\lambda$. In the integration over λ, it was taken into account that in region a (see Fig. 6) where $\gamma \le \lambda \le \zeta$, the quantities

$$R_0(k_1, k_2; l, l') + F(k_1, l; k_2, l')$$
$$+ F(k_1, l', k_2, l),\quad R_0(-l, -l'; k_3, k_4)$$
$$+ F(-l, k_3; -l', k_4) + F(-l, k_4; -l', k_3)$$

which appear under the integral in Eq. (19) do not depend on λ; in region b, where $\zeta \le \lambda \le \xi$, the first of these quantities does not depend on λ, but in region c, where $\xi \le \lambda \le L$, both are functions of λ.

*It is easy to see that when k_2 and k_3 are transposed, i.e. when there is a transition from $F(k_1, k_2; k_3, k_4)$ to $F(k_1, k_3; k_2, k_4)$ the quantities ξ', η', ζ', which correspond to the new arrangement of momenta, are all of the same order equal to ξ. By the same token the quantities corresponding to $F(k_1, k_4; k_2, k_3)$ are $\xi'' = \eta'' = \zeta'' = \xi$.

If we put in Eq. (20) $\eta = \zeta$ and $\eta = \zeta = \xi$ we obtain two equations whose simultaneous solution determines the functions $F(\xi)$ and $\varphi(\xi, \eta, \eta) = P(\xi, \eta)$,

$$P(\xi, \eta) = -(g_0^2/8\pi)\int_\eta^\xi [R_0(\xi) + 2F(\xi)][R_0(\lambda) + 2F(\lambda) + \Phi(\lambda, \eta)]\, d^2(\lambda)\, d\lambda - (g_0^2/8\pi)\int_\xi^L [R_0(\lambda) + 2F(\lambda)][R_0(\lambda) + 2F(\lambda) + \Phi(\lambda, \eta)]\, d^2(\lambda)\, d\lambda,$$
$$F(\xi) = -(g_0^2/8\pi)\int_\xi^L [R_0(\lambda) + 2F(\lambda)][R_0(\lambda) + 2F(\lambda) + \Phi(\lambda, \xi)]\, d^2(\lambda)\, d\lambda. \tag{24}$$

It is evident from these equations that the value $F(\xi)$ of the function $F(k_1, k_2; k_3, k_4)$ is, in the case where all the momenta are of the same order of magnitude, essentially dependent on the quantity $\Phi(\lambda, \xi)$ i.e. on the value of $F(-l, -l'; k_3, k_4)$ for the case when the two momenta l and l' are very large in comparison with their sum ($|l + l'| = |k_3 + k_4|$). This circumstance is a troublesome peculiarity of Eq. (19) and complicates considerably the subsequent computations.

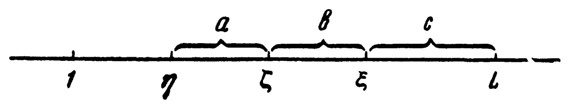

FIG. 6.

In the case of the symmetrical theory the equations look even more awkward. In this case the isotopic meson spin variables, a_i, can be left out of the calculations if one desires asymptotic solutions for Eq. (19) of the form,

$$F(k_1, k_2; k_3, k_4) \approx \Phi(\xi, \eta)\,\delta_c + \Phi_1(\xi, \eta)\,\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4},$$
$$(\eta = \zeta);$$
$$F(k_1, k_2; k_3, k_4) = F(\xi)\,\delta_c + F_1(\xi)\,\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4},$$
$$(\xi = \eta = \zeta),$$

where $F(\xi) = \Phi(\xi, \xi)$, $F_1(\xi) = \Phi_1(\xi, \xi)$. Then, according to eq. (17) and Eq. (18)

$$P(k_1, k_2, k_3, k_4) = \Pi(\xi, \eta)\,\delta_c + \Pi_1(\xi, \eta)\,\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4},$$

$$P(k_1, k_2, k_3, k_4) = P(\xi)\,\delta_c, \quad (\xi = \eta), \tag{25}$$

where $P(\xi) = \Pi(\xi, \xi)$, $\Pi_1(\xi, \xi) = 0$ (the latter is due to the fact that P, which is symmetrical over all coordinates of all the mesons, can only contain the factor δ_c for those cases where all the momenta are of the same order).

Substituting the above expression for F in Eq. (19) and equating the coefficients of δ_c and $\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}$ which occur in both sides of the equation, we obtain, just as in the case of the neutral theory,

$$\begin{aligned}
\Phi(x, y) = &-\frac{1}{3}[\rho_0(x) + 2F(x)]\int_x^y [\rho_0(z) + 2F(z) + F_1(z) + \Phi(z, y)]\frac{dz}{z^2} \\
&-\frac{1}{3}\int_1^x [\rho_0(z) + 2F(z) + F_1(z)][\rho_0(z) + 2F(z) + F_1(z) + \Phi(z, y)]\frac{dz}{z^2}, \\
\Phi_1(x, y) = &-\frac{1}{3}\int_x^y \Big\{\frac{5}{2}[\rho_0(x) + 2F(x)][\rho_0(z) + 2F(z) + \Phi(z, y) \\
&+ \Phi_1(z, y)] + F_1(x)[\Phi_1(z, y) - F_1(z)]\Big\}\frac{dz}{z^2} - \frac{1}{3}\int_1^x \Big\{\frac{5}{2}[\rho_0(z) + 2F(z)] \\
&\times [\rho_0(z) + 2F(z) + \Phi(z, y) + \Phi_1(z, y)] + F_1(z)[\Phi_1(z, y) - F_1(z)]\Big\}\frac{dz}{z^2}.
\end{aligned} \tag{26}$$

Here instead of ξ, η and λ we have used more convenient variables, viz.

$$x = [1 + (5g_0^2/4\pi)(L - \xi)]^{1/5} \tag{27b}$$

and, accordingly, for y and z. If we assume $x = y$ in Eq. (26), we obtain two more equations (analogous to the second equation in Eq. (24); consequently we have a system of four equations for the four functions Φ, Φ_1 F, and F_1. It is not difficult to solve Eq. (26) provided that $x - 1 > 1$ (this provision corresponds to the conventional perturbation theory) and that $x > 1$. Omitting the computations, the results are

$$\Pi(x, y) = \begin{cases} \frac{16}{3}(x-1)\Big\{1 - \frac{88}{27}(x-1)^2 - \frac{8}{3}[(y-1)^2 - (x-1)^2] + \cdots\Big\}, \\ \qquad\qquad x - 1 < 1, \quad y - 1 < 1; \\ \frac{16}{11}x\Big\{1 + \frac{5}{3}[x^{40/30}y^{-40/33} - x^{16/33}y^{-16/33}] + \cdots\Big\}, \quad x > 1, \ y > 1, \end{cases}$$

$$\Pi_1(x, y) = \begin{cases} -\frac{5}{12}\left(\frac{16}{3}\right)^2(x-1)\{[(y-1)^2 - (x-1)^2] + \cdots\}, \\ \qquad\qquad x - 1 < 1, \ y - 1 < 1; \\ \frac{16}{11}\{x[x^{16/33}y^{-16/33} - 1] + \cdots\}, \quad x > 1, \ y > 1, \end{cases}$$

whence

$$P(x) = \Pi(x, x) = \begin{cases} \frac{16}{3}(x-1)\Big\{1 - \frac{88}{27}(x-1)^2 + \cdots\Big\}, & x - 1 < 1; \\ \frac{16}{11}x + \cdots, & x > 1 \end{cases} \tag{28}$$

b) Direct Integral Equation for $P(\xi)$.

We shall show that one can obtain an equation for $P(\xi)$ and $F(\xi)$ directly, and in this fashion avoid the case where two of the four momenta are very large and eliminate completely functions like $\Phi(\xi, \eta)$ and $\varphi(\xi, \nu, \zeta)$. For this purpose we shall derive once more an equation which, like Eq. (19), determines the function $F(k_1, k_2; k_3, k_4)$, although we shall assume from the outset that all momenta and their sums are large quantities and of the same order of magnitude. We shall consider only the asymptotic value of all the functions and include only the most essential region—the region where all the integrals vary logarithmically.

Let us examine an arbitrary reducible (as regards the separation of k_1, k_2 from k_3, k_4) graph and write the function belonging to it in a form analogous to Eq. (11). Then in the part adjacent to lines k_3 and k_4 a graph may be reducible, as well as irreducible as regards the separation of lines l and l' from k_3, k_4. In the first case, *i.e.*, if the graph is reducible, we again present it in the form of an integral like Eq. (11) and so on until no irreducible graph remains in the part adjacent to k_3 and k_4. Consequently, the graph in question is broken down into a series of irreducible ones (Fig. 7) connected to each other by pairs of meson lines. We shall designate the momenta in these lines by $l_1, l_1', l_2, l_2'; \dots ; l_N, l_N'$ (with $l_i = l_i' + k_1 + k_2$, $i = 1, 2, \dots, N$) and use $\rho_{n_0}, \rho_{n_1}, \dots, \rho_{n_N}$ to designate the values of the contributions from the irreducible graphs in Fig. 7 (here n_i is the number of the corresponding graphs in the i-th irreducible part). Then, in analogy with Eq. (11), we can write

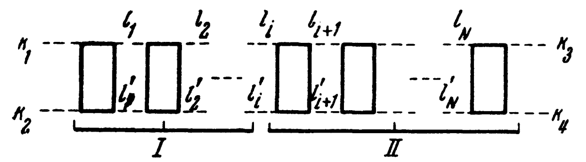

FIG. 7.

$$\sigma_{n_0, n_1, \dots, n_N}(k_1, k_2; k_3, k_4) = \iint \cdots \int \rho_{n_0}(k_1, k_2; l_1, l_1') A(l_1) d^4 l_1$$
$$\times \rho_{n_1}(-l_1', -l_1'; l_2, l_2') A(l_2) d^4 l_2 \cdots A(l_N) d^4 l_N \rho_{n_N}(-l_N, -l_N'; k_3, k_4);$$
$$A(l_i) = -(g_0^2/\pi i) D(l_i) D(l_i'),$$

or, if we include only the region of logarithmic variation of l_i (for which $A(l_i)d^4 l_i = -(g_0^2/4\pi) d^2(\lambda_i) d\lambda_i$ [see footnote* following Eq. (7)], we obtain for the case where all momenta k_1, k_2, k_3, k_4 and all of their sums are of the same magnitude, *i.e.*, where $\sigma = \sigma(\xi)$,

$$\sigma_{n_0, n_1, \dots, n_N}(\xi) = (-g_0^2/4\pi)^N \int_\xi^L d\lambda_1 \int_\xi^L d\lambda_2 \cdots \int_\xi^L d\lambda_N \rho_{n_0}(\xi, \lambda_1) d^2(\lambda_1) \rho_{n_1}(\lambda_1, \lambda_2) \times d^2(\lambda_2) \cdots d^2(\lambda_N) \rho_{n_N}(\lambda_N, \xi). \tag{29}$$

Here $\rho_{n_i}(\lambda_i, \lambda_{i+1})$ stands for the value of $\rho_{n_i}(-l_i, -l_i', l_{i+1}, l_{i+1}')$ when l_i and l_{i+1} are large. It is essential that ρ_{n_i} be in fact dependent only on the largest of quantities λ_i, λ_{i+1}, because ρ_{n_i} is the contribution from the graph which is irreducible as regards the separation of $-l_i, -l_i'$ from l_{i+1}, l_{i+1} (the contribution from graphs of this type is contained in $F(-l_i, l_{i+1}'; -l_i', l_{i+1})$ or in $F(-l_i, l_{i+1}; l_i', l_{i+1})$ and, in agreement with what has been said above, when l_i and l_i' are large these F functions depend only on the largest momentum.* This last assertion can be easily checked directly on simple graphs with two, three, and more squares. Thus, the integrand in Eq. (29) does not, in fact, depend on ξ, *i.e.*, when $\lambda_1 \geq \xi$ and $\lambda_N \gg \xi$, then

$$\rho_{n_0}(\xi, \lambda_1) = \rho_{n_0}(\lambda_1), \quad \rho_{n_N}(\lambda_N, \xi) = \rho_{n_N}(\lambda_N).$$

The region of integration over $\lambda_1, \lambda_2, \dots, \lambda_N$ in Eq. (29) can be broken down into N regions, in any one of which one of the variables, say λ_i, is smaller than the others; accordingly Eq. (29) is presented as a sum of N integrals

$$\sigma_{n_0, n_1, \dots, n_N}(\xi) = -\sum_{i=1}^{N} \frac{g_0^2}{4\pi} \int_\xi^L \sigma_{n_0, n_1, \dots, n_{i-1}}(\lambda_i)\, \sigma_{n_i, n_{i+1}, \dots, n_N}(\lambda_i)\, d\lambda_i, \tag{30}$$

*In contrast to $F(-l_i, -l_i'; l_{i+1}, l_{i+1}') = \varphi(\lambda_i, \eta, \lambda_{i+1})$ which depends on three quantities: $\lambda_i, \eta, \lambda_{i+1}$.

where the quantity

$$\sigma_{n_0, n_1, \ldots, n_{i-1}}(\lambda_i) = \left(-\frac{g_0^2}{4\pi}\right)^{i-1} \int_{\lambda_i}^{L} d\lambda_1 \ldots \int_{\lambda_i}^{L} d\lambda_{i-1} \rho_{n_0}(\lambda_1) d^2(\lambda_1) \rho_{n_1}(\lambda_1, \lambda_2) \ldots \ldots d^2(\lambda_{i-1}) \rho_{n_{i-1}}(\lambda_{i-1}) \quad (31)$$

is determined in a manner exactly analogous with Eq. (29), *i.e.*, it may be regarded as the contribution from the part of the reducible graph under consideration that adjoins lines k_1, k_2 and $l_i\, l_i'$ (part *I* in Fig. 7), which would be made if the momenta k_1 and k_2 were of the same order as l_i, l_i'. Analogously $\sigma_{n_i, n_{i-1}, \ldots, n_N}(\lambda_i)$ may be regarded as the contribution that would come from part II in Fig. 7 if k_3 and k_4 were of the same order as l_i, l_i'.

It is not difficult to see that, in analogy with Eq. (12)

$$\sum_{i=1}^{\infty} \sum_{n_0, n_1, \ldots, n_{i-1}} 2^{-(i-1)} \sigma_{n_0, n_1, \ldots, n_{i-1}}(\lambda) = P(\lambda),$$

where $P(\lambda)$ is the desired value for the sum of all the reducible graphs provided that all the momenta are of the same order. In exactly the same way as in Eq. (14) we obtain

$$\sum_{N-i=1}^{\infty} \sum_{n_i, n_{i-1}, \ldots, n_N} 2^{-(N-i)} \sigma_{n_i, n_{i+1}, \ldots, n_N}(\lambda) = P(\lambda).$$

The negative powers of 2 in the left side of these equations are due to the fact that when we sum over all possible irreducible parts, *i.e.*, over n_i ($j = 0, \ldots, N$), we obtain (in the last case) 2^N identical resulting graphs.

Using these equalities, we obtain, by the summation of Eq. (29) over all types (or numbers) of n_i graphs in each irreducible part and over the number N of these parts,

$$F(\xi) = -(g_0^2/8\pi) \int_{\xi}^{L} P(\lambda) d^2(\lambda) P(\lambda) d\lambda. \quad (32)$$

For the neutral pseudoscalar theory we obtain according to Eq. (32) a simple integral equation,

$$P(\xi) = R_0(\xi) - \frac{3g_0^2}{8\pi} \int_{\xi}^{L} P^2(\lambda) d^2(\lambda) d\lambda, \quad (33a)$$

where R_0 is determined in Eq. (3a).

Introducing [as in Eq. (27b)] a more convenient variable

$$x = [1 + (5g_0^2/4\pi)(L - \xi)]^{-1/5} = Q^{-1/5}, \quad (27a)$$

we can write Eq. (33a) as

$$P(x) = 24(1 = x) - \frac{3}{2} \int_{x}^{1} P^2(z) \frac{dz}{z^2}. \quad (34a)$$

Differentiation with respect to x converts this equation into a simple differential one, which is easily solved. A simple computation gives us

$$P(x) = \frac{\sqrt{145} + 1}{3} \times x \frac{1 - x^{\sqrt{145}}}{1 + (\sqrt{145} + 1)\, x^{\sqrt{145}} / (\sqrt{145} - 1)}. \quad (35a)$$

In the case of the symmetrical theory Eq. (25) should be substituted in Eq. (32) and summed over the indices of the isotopic spin; this will give

$$F(k_1, k_2; k_3, k_4) \approx [2\delta_c + 5\delta_{\xi_1, \xi_2}\delta_{\xi_3, \xi_4}] \times \left(\frac{-g_0^2}{8\pi}\right) \int_{\xi}^{L} P^2(\lambda) d^2(\lambda) d\lambda,$$

where [see footnote* following Eq. (2)] $d(\lambda) = Q^{-2/5}$. In conformance with Eq. (17), we then obtain (upon dividing through δ_c)

$$P(\xi) = \rho_0(\xi) - \frac{11g_0^2}{8\pi} \int_{\xi}^{L} P^2(\lambda) d^2(\lambda) d\lambda, \quad (33b)$$

$$P(x) = \frac{16}{3}(x - 1) - \frac{11}{6} \int_{1}^{x} P^2(z) \frac{dz}{z^2}, \quad (34b)$$

or, if we introduce the variable x, make use of Eq. (3b), and integrate, we obtain

$$P(x) = \frac{16x}{11} \frac{1 - x^{-19/3}}{1 + (8/11)\, x^{-19/3}} \quad (35b)$$

Thus, the total sum $P(x)$ of the reducible graphs is a finite quantity of the same order as the contribution R_0 from the simplest graphs in Fig. 1 (when $Q \to \infty$, Eq. (35b) for example, differs from Eq. (28) only in the factor 3/11).

Eq. (33a) and (33b) could, it would seem, also be derived mathematically directly from Eq. (24) to (26). However, we were unable to do this. The value of Eq. (35b) for $x - 1 < 1$ and $x > 1$ coincide with the value of Eq. (28) derived directly from Eq. (26).

4. THE RENORMALIZATION PROPERTIES OF THE AMPLITUDE P FOR MESON-MESON SCATTERING

In the conventional scheme for charge renormalization the amplitude for meson-meson scattering

$$(g_0^2/4\pi i)\,P\,(k_1,\,k_2,\,k_3,\,k_4),$$

which corresponds to graphs with four external meson lines, is multiplied by a factor Z_3^2 (where $D = Z_3\,D_c$), $\sqrt{Z_3}$ for each external meson line. Thus every renormalized expression will contain the quantity

$$Z_3^2\,(g_0^2/4\pi i)\,P.$$

Since, according to Ref. 2, $g^2 = (g_0^2/Q_0)$ and $Z_3 = d\,(0) = Q_0^{-2/5}$ for the neutral theory and $Z_3 = Q^{-4/5}$ for the symmetrical theory, we obtain in both cases

$$Z_3^2\,(g_0^2/4\pi i)\,P\,(x) = (g^2/4\pi i)\,(P\,(x)\,/\,x_0),$$

$$Q_0 = 1 + (5g_0^2/4\pi)\,L$$

where, according to Eq. (27a), (27b) $x_0 = Q_0^{-1/5}$ for the neutral theory and $Q_0^{3/5}$ for the symmetrical one. Thus, after renormalization of the charge, instead of $P\,(x)$ all of the equations will contain

$$P_c\,(x) = P\,(x)\,/\,x_0.$$

If we set $x/x_0 = x_c$ where $x_c = Q^{-1/5}$ for the neutral theory and $x_c = Q_c^{3/5}$ for the symmetrical theory, and where

$$Q_c = 1 - (5g^2\,/\,4\pi)\,\xi$$

we see that $Q \to \infty$ when $L \to \infty$ and

$$P_c\,(x) = P\,(x_c),$$

where P is given by Eq. (35a) or (35b) with $Q \to \infty$ (*i.e.*, $P = (\sqrt{145} + 1/3)\,x_c$ for Eq. (35a) and $P = (16/11)x_c$ for Eq. (35b)).

Thus, when $L \to \infty$ the quantities (35a) and (35b) are automatically renormalized without introducing any counter terms proportional to φ^4 in the Hamiltonian.

In principle it is not difficult to think of a case where the Hamiltonian would, in additional to the usual interaction terms, have a term proportional to φ^4 [or $(\varphi_a\varphi_a)^2$ in the case of the symmetrical theory, with a as the index for the isotopic spin].

At the same it is obvious that there will be a change in the term R_0, which corresponds to the simplest graphs in Fig. 1. A constant λ, which is proportional to the coefficient of the φ^4 term is added to Eq. (3a) and (3b), *i.e.*,

$$R_0 = 24\,(1 - x) + \lambda, \quad \rho_0 = (16/3)\,(x - 1) + \lambda.$$

Consequently, instead of (34a) and 34b) we obtain

$$P\,(\lambda,\,x) = 24\,(1 - x) + \lambda - \frac{3}{2}\int_x^1 P^2\,(\lambda,\,z)\,z^{-2}\,dz, \tag{36a}$$

$$P\,(\lambda,\,x) = \frac{16}{3}\,(x - 1) + \lambda - \frac{11}{6}\int_1^x P^2\,(\lambda,\,z)\,z^{-2}\,dz \tag{36b}$$

($P\,(\lambda,\,x)$ is the scattering amplitude when a direct interaction is present characterized by the constant λ). These equations, as in the case of Eq. (35a) and (35b), are simple to solve. By way of example let us take the case of the symmetrical theory. The solution to Eq. (36b), as is easily seen, has the form

$$P\,(\lambda,\,x) = \frac{16x}{11}\,\frac{A - x^{-19/3}}{A + (8/11)\,x^{-19/3}}\,, \qquad A = \frac{1 + \lambda/2}{1 - 11\lambda\,/\,16} \tag{37}$$

[when $\lambda = 0$, $A = 1$ and Eq. (37) coincides with Eq. (36b)]. It should be noted that the quantity $P\,(\lambda,\,x)$ is renormalized for any λ (as are the functions d, β, a found in Abrikosov, *et al.*[2]) if the renormalization entails change in λ. Actually, according to Eq. (37), if we have that $x = x_0\,x_c$, we directly obtain

$$P\,(\lambda, x) = x_0\,P\,(\lambda_c,\,x_c), \tag{38}$$

where the right half is the same function as Eq. (37), and where λ_c and λ are connected by the equality

$$\frac{1 + \lambda_c/2}{1 - (11/16)\,\lambda_c} = x_0^{19/3}\,\frac{1 + \lambda/2}{1 - (11/16)\,\lambda}\,. \tag{39}$$

Note that according to Eq. (36) or Eq. (37), $P(\lambda, 1) = \lambda$ i.e. $\lambda_c = P(\lambda_c, 1)$ determines the magnitude of the interaction observed in experiments with mesons at low energies, when $\xi = 0$, $x_c = 1$. If this quantity is considered known and one considers Eq. (38) as specifying λ in terms of λ_c, g_0^2 and L, then Eq. (38) and Eq. (39) constitute the usual means for renormalizing the amplitude of meson-meson scattering, which, as is apparent, also occurs outside the framework of the perturbation theory.

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Nonlinearity of the Field in Conformal Reciprocity Theory

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The first version of Born and Infeld's nonlinear electrodynamics and the variability of the gravitational constant are deduced from the conformally covariant gravitational equations derived from a certain generalized reciprocity law which is based on group theory and yields a nonlocal field theory. A correspondence principle is established between relativity theory and reciprocity theory.

I. FIELD EQUATIONS

1. LET k BE Einstein's gravitational constant, $ds^2 = g_{ik}dx^i dx^k$ be the element of interval with the metric $g_{ik} = g_{ki}$, $g = \| g_{ik} \|$ $(i, k = 1, \ldots, 4)$. Let Γ_{ik}, $\Gamma = \Gamma_r^r$ be the contracted curvature tensor and the curvature scalar constructed from Weyl's conformal connection[1] Γ_{kl}^i, Riemannian in the metric $u_{ik} = \Psi g_{ik}$, and of weight zero with respect to the g_{ik} ($\Psi = E^{-2\psi}$ is of weight -1 with respect to the g_{ik}). Let $p_{ik} = (\partial p_k/\partial x^i - \partial p_i/\partial x^k)$ be the electromagnetic field of absolute magnitude $P/\sqrt{2} = P'\sqrt{k/2}$. Let $P_{ki} = P_{ik} = p_{ir}p_k^r/P$, $P = P_r^r$,

$$L_{ik} = \Gamma\Gamma_{ik} + PP_{ik},\quad L = L_r^r = \Gamma^2 + P^2, \quad (1)$$

$$Q_{ik} = \Gamma Q'_{ik} = \Gamma(\Gamma_{ik} - {}^1/_4\Gamma g_{ik}),$$
$$S_{ik} = PS'_{ik} = P(P_{ik} - {}^1/_4 P g_{ik}). \quad (2)$$

The Γ_{ik}, P_{ik} (as well as the p_{ik}) depend only on the ratios of the g_{ik} (and therefore on those of the of the u_{ik}). Variation of $L\sqrt{g}$ with respect to g_{ik} and p_i gives[2] the gravitational $Q_i^k\sqrt{g}$ and electromagnetic $S_i^k\sqrt{g}$ energy-momentum tensor densities of weight zero and current-charge vector densities $s^i\sqrt{g}$ and $s'^i\sqrt{g}$ of weight zero. This leads,[2,3] to the conformal covariant equations which satisfy the reciprocity principle[2-12]

$$Q_{ik} = -S_{ik} \text{ or } L_{ik} = {}^1\!/_4\, L g_{ik}, \quad (3)$$

$$\partial(p^{ik}\sqrt{g})/\partial x^k = s^i\sqrt{g}. \quad (4)$$

when $\partial(\Gamma^2\sqrt{g})/\partial p_i = 0$, Equations (4) become

$$\partial(p^{ik}\sqrt{g})/\partial x^k = 0; \quad (4')$$

Eq. (3) and (4) describe the gravitational, and electromagnetic fields respectively.