Diffraction Scattering of Fast Deuterons by Nuclei

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The cross section for elastic scattering and the cross section for diffraction splitting of fast deuterons by completely black nuclei are determined. The energy distribution of the disintegration products is found. The cross section for splitting of a fast deuteron by a completely black nucleus is calculated, taking the diffraction and Coulomb interactions into account. Expressions are obtained for the cross sections for elastic deuteron scattering and splitting, taking into account the semi-transparency of the nucleus.

1. IT IS WELL KNOWN that the absorption of particles scattered by a nucleus brings about a perturbation of the incident wave and leads to elastic scattering not connected with compound-nucleus formation. In the case of point particles (for example, neutrons) with a wavelength short compared with nuclear dimensions, this scattering is analogous to the scattering of light by a completely black sphere.

The diffraction scattering of deuterons should differ by specific characteristics. In addition to purely elastic scattering, analogous to the diffraction scattering of point particles, diffraction splitting should also take place in the case of deuterons. In fact, owing to the small binding energy of the deuteron, a comparatively small change in its momentum in diffraction scattering can lead to splitting that takes place far from the nucleus. Together with the stripping reaction, the diffraction splitting of the deuteron leads to liberation of a neutron and proton, *i.e.*, increases the yield of neutrons coming from the interaction of fast deuterons with nuclei*.

The diffraction scattering of point particles by absorbing nuclei can be studied by an optical method using Huygens' principle. In order to generalize this method to the case of deuterons, we consider first the simple problem of the diffraction scattering of point particles by absorbing nuclei.

The free motion of a particle in a plane perpendicular to the wave vector of the incident particle (Z-axis) is described by the wave function

$$\psi_\varkappa = L^{-1}\exp\{i\varkappa\rho\}, \quad \int \psi_\varkappa \psi_{\varkappa'}^* d\rho = \delta_{\varkappa\varkappa'},$$

where L is the normalization length, $\varkappa$ and ρ are the projections of the wave vector and radius vector of the particle on a plane perpendicular to the Z-axis. The wave function $\psi_0 = 1/L$ corresponds to the incident particles.

The presence of the absorbing nucleus leads to disappearance of particles of this function for $\rho \leq R$ (R is the radius of the nucleus). The diffraction pattern determined by this disappearance can be obtained by expansion of the modified wave function, equal to $\Psi = \Omega(\rho)\Psi_0$, where

$$\Omega(\rho) = \begin{cases} 0, & \rho \leqslant R, \\ 1, & \rho > R, \end{cases}$$

in terms of the functions $\Psi_\varkappa$

$$\Psi = \Omega(\rho)\psi_0 = \sum_\varkappa a_\varkappa \psi_\varkappa. \tag{1}$$

The probability of diffraction scattering in which the wave vector $\varkappa$ of the scattered particle lies in the interval $d\kappa$, is connected with $a_\varkappa$ by the relation $dw = |a_\varkappa|^2 L^2 d\varkappa/(2\pi)^2$, and the corresponding differential scattering cross section is equal to

$$d\sigma = L^2 dw = |a_\varkappa|^2 L^4 d\varkappa/(2\pi)^2.$$

If K is the magnitude of the wave vector of the particle, then $\varkappa = K\sin\vartheta \approx K\vartheta$ and $d\varkappa = K^2 do$, where do is the element of solid angle. The scattering amplitude $f(\vartheta)$ is connected with the expansion coefficient $a_\varkappa$ by the relation

$$f(\vartheta) = -i\frac{L^2K}{2\pi}a_\varkappa. \tag{2}$$

From (1) follows that

$$a_\varkappa = \int \psi_\varkappa^* \Omega(\rho)\psi_0 d\rho.$$

*The possibility of diffraction splitting of the deuteron was independently established by E. Feinberg, R. Glauber and the authors[1-4].

Carrying out the integration and employing Eq. (2), we obtain the well known formula

$$f(\vartheta) = iRJ_1(KR\vartheta)/\vartheta, \quad d\sigma = R^2\vartheta^{-2}J_1^2(KR\vartheta)\, do$$

(since the diffraction considerations are valid only for small angles, we replace $\sin\vartheta$ by ϑ).

2. The treatment given for the diffraction of point particles can be generalized to the case of the diffraction scattering of weakly bound complex particles (deuterons) by completely black nuclei, if the expansion of the modified wave function for this case is carried out as before, but instead of one, two multipliers, Ω_n and Ω_p, describing the disappearance of a neutron and proton are introduced. The idea of this generalization is due to L. D. Landau.

In order to investigate the diffraction of deuterons it is necessary to take into account both the motion of their center of gravity and the relative motion of the neutron and proton in the deuteron. The motion of the center of gravity of the deuteron in a plane perpendicular to the direction of the wave vector of the incident deuteron (Z-axis) is described by the wave function $\psi_\varkappa(\rho_d) = \exp(i\varkappa\rho_d)$, where $\varkappa$ and ρ_d are the projections of the wave vector of the scattered deuteron and the radius vector of the center of gravity of the deuteron on the plane perpendicular to the Z-axis. (The normalization length L is taken hereinafter to be unity.) The functions $\psi_\varkappa(\rho_d)$ form the complete orthonormal system

$$\int \psi_\varkappa^*(\rho_d)\,\psi_\varkappa(\rho_d')\, d\varkappa\,/\,(2\pi)^2 = \delta(\rho_d - \rho_d').$$

The relative motion of the particles in the deuteron is described by the wave function

$$\varphi_0(r) = \sqrt{\alpha/2\pi}\; e^{-\alpha r}/r, \quad \alpha = 1/2R_d$$

(R_d is the radius of the deuteron). The relative motion of the neutron and proton freed as a result of the splitting of the deuteron is described by the wave function

$$\varphi_f(\mathbf{r}) = e^{i\mathbf{f}\mathbf{r}} + \frac{a}{r}e^{-ifr},$$

where $\hbar\mathbf{f}$ is the momentum of relative motion of the particles and $a = -1/(\alpha - if)$, the neutron-proton scattering length for the S-state. Here it is assumed that the interaction between neutron and proton exists only in S-states. The sum of the plane wave and incoming spherical waves corresponds to the production of particles. The functions $\varphi_f(\mathbf{r})$ together with the functions $\varphi_0(r)$, describing the bound state of the system, form a complete set of orthonormal functions satisfying the relation

$$\varphi_0(r)\,\varphi_0(r') + \int \varphi_f(\mathbf{r})\,\varphi_f(\mathbf{r}')\, d\mathbf{f}/(2\pi)^3 = \delta(\mathbf{r} - \mathbf{r}').$$

Because the deuteron is a weakly bound system in which the neutron and proton spend most of their time outside the range of nuclear force, the pattern for the diffraction of deuterons by a completely black nucleus is determined by the expansion of the modified wave function $\Psi = \Omega_n\Omega_p\psi_0(\rho_d)\varphi_0(r)$ in terms of the complete set of functions $\psi_\varkappa(\rho_d)\varphi_0(r)$ $\psi_\varkappa(\rho_d)\varphi_f(\mathbf{r})$:

$$\Psi = \sum_\varkappa A_\varkappa\psi_\varkappa(\rho_d)\,\varphi_0(r) + \sum_{\varkappa,\,f} A_{\varkappa f}\psi_\varkappa(\rho_d)\,\varphi_f(\mathbf{r}), \tag{3}$$

where $A_\varkappa$ and $A_{\varkappa_f}$ are the probability amplitudes for diffraction scattering and diffraction splitting of the deuteron. From (3) it follows that

$$\begin{aligned} A_\varkappa &= \iint \varphi_0(r)\,\psi_\varkappa^*(\rho_d)\,\Omega_n\Omega_p\psi_0(\rho_d)\,\varphi_0(r)\, d\rho_d d\mathbf{r} \\ &= -\iint \varphi_0(r)\,\psi_\varkappa^*(\rho_d)\,\{\omega_n + \omega_p - \omega_n\omega_p\} \\ &\qquad \times \psi_0(\rho_d)\,\varphi_0(r)\, d\rho_d d\mathbf{r}, \end{aligned} \tag{4}$$

$$\begin{aligned} A_{\varkappa f} &= \iint \varphi_f^*(\mathbf{r})\,\psi_\varkappa^*(\rho_d)\,\Omega_n\Omega_p\psi_0(\rho_d)\,\varphi_0(r)\, d\rho_d d\mathbf{r} \\ &= -\iint \varphi_f^*(\mathbf{r})\,\psi_\varkappa^*(\rho_d)\,\{\omega_n + \omega_p - \omega_n\omega_p\} \\ &\qquad \times \psi_0(\rho_d)\cdot\varphi_0(r)\, d\rho_d d\mathbf{r}, \end{aligned} \tag{5}$$

where $\omega = 1 - \Omega$ (in writing the last equalities, we used the orthogonality of the functions φ_0 and φ_f).

3. Employing the expansion

$$\omega(\rho) = \frac{1}{2\pi}\int \frac{RJ_1(gR)}{g}\exp\{i\mathbf{g}\rho\}\, d\mathbf{g} \tag{6}$$

and the formula

$$\int \varphi_0^2(r)\exp(i\varkappa\rho/2)\, d\mathbf{r} = \frac{4\alpha}{\varkappa}\tan^{-1}\frac{\varkappa}{4\alpha}, \tag{7}$$

it is possible to express the amplitude for elastic scattering $f(\vartheta)$, connected with $A_\varkappa$ by the relation (2), in the form

$$f(\vartheta) = iK\left\{2\frac{4\alpha}{\varkappa}\tan^{-1}\frac{\varkappa}{4\alpha}\frac{RJ_1(\varkappa R)}{\varkappa} - \right.$$
$$\left. -\frac{1}{2\pi}\int\frac{4\alpha}{|2\mathbf{g}-\boldsymbol{\varkappa}|}\tan^{-1}\frac{|2\mathbf{g}-\boldsymbol{\varkappa}|}{4\alpha}\frac{RJ_1(gR)}{g}\frac{RI_1(|\boldsymbol{\varkappa}-\mathbf{g}|R)}{|\boldsymbol{\varkappa}-\mathbf{g}|}d\mathbf{g}\right\} \tag{8}$$

The differential cross section for elastic scattering is equal to

$$d\sigma_e = R^2\left|2\frac{2p}{\varkappa'}\tan^{-1}\frac{\varkappa'}{2p}\cdot\frac{J_1(\varkappa')}{\varkappa'} - \frac{1}{2\pi}\int\frac{2p}{|2\mathbf{g}'-\boldsymbol{\varkappa}|}\tan^{-1}\frac{|2\mathbf{g}'-\boldsymbol{\varkappa}'|}{2p}\frac{J_1(g')}{g'}\right.$$
$$\left.\times\frac{J_1(|\boldsymbol{\varkappa}'-\mathbf{g}'|)}{|\boldsymbol{\varkappa}'-\mathbf{g}'|}d\mathbf{g}'\right|^2 d\boldsymbol{\varkappa}', \tag{9}$$

where $\varkappa' = \varkappa R$, $\mathbf{g}' = \mathbf{g}R$, $p = R/R_d$.

In the limiting case of large p this formula greatly simplifies:

$$d\sigma_e = 2\pi R^2\left\{\left(\frac{2p}{\varkappa'}\tan^{-1}\frac{\varkappa'}{2p}\right)^2\frac{J_1^2(\varkappa')}{\varkappa'}\right.$$
$$\left.+\frac{1}{2p}J_1(\varkappa')J_0(\varkappa')\right\}d\varkappa', \quad \varkappa' \ll p, \quad p \gg 1. \tag{10}$$

To obtain the integral cross section for elastic scattering we use the completeness of the functions $\psi_\varkappa$. From Eq. (4) it follows that

$$\sigma_e = \int I^2(\rho_d)\,d\rho_d,$$
$$I(\rho_d) = \int\{\omega_n + \omega_p - \omega_n\omega_p\}\varphi_0^2(r)\,d\mathbf{r}.$$

If $p \gg 1$, the term in the cross section σ_e, coming from the region $\rho_d < R$, is equal to πR^2 to within an accuracy of terms of the order $1/p^2$.

In the region $\rho_d > R$, $\omega_n\omega_p = 0$ and therefore

$$I(\rho_d) = \frac{1}{\pi}\int d\mathbf{g}\exp\{i\mathbf{g}\boldsymbol{\rho}_d\}\frac{RJ_1(gR)}{g}$$
$$\times\int\varphi_0^2(r)\exp\{i\mathbf{g}\mathbf{r}/2\}/d\mathbf{r}$$
$$= 2\int_0^\infty\frac{2p}{g}\tan^{-1}\frac{g}{2p}J_1(g)J_0(g\rho_d/R)\,dg, \quad \rho_d > R.$$

Because

$$\frac{\tan^{-1}a}{a} = \int_0^1\frac{dy}{1+a^2y^2},$$
$$\int_0^\infty\frac{J_1(bx)J_0(ax)}{k^2+x^2}dx = \frac{1}{k}I_1(bk)K_0(ak)$$

then

$$I(\rho_d) = 4p\int_0^1\frac{dy}{y}I_1\left(\frac{2p}{y}\right)K_0\left(\frac{\rho_d}{R}\frac{2p}{y}\right), \quad \rho_d > R.$$

Employing asymptotic expressions for $I_1(x)$ and $K_0(x)$ for $x \gg 1$ we obtain

$$I(\rho_d) = \sqrt{\frac{R}{\rho_d}}\int_1^\infty\frac{d\xi}{\xi^2}e^{-4\alpha b\xi}, \quad b = \rho_d - R, \quad p \gg 1,$$

and, consequently, the term in σ_e coming from the region $\rho_d > R$ is equal to

$$2\pi R\int_0^\infty db\left|\int_1^\infty e^{-4\alpha b\xi}\frac{d\xi}{\xi^2}\right|^2 = \frac{2\pi}{3}(1-\ln 2)RR_d.$$

Thus, the integral cross section for elastic scattering is equal to

$$\sigma_e = \pi R^2 + \frac{2\pi}{3}(1-\ln 2)RR_d, \quad R_d \ll R. \tag{11}$$

We note that, integrating (10) with respect to $\varkappa'$, we would obtain as correction to the main term πR^2 a coefficient two times smaller than in (11)[2]. This is connected with the fact that large $\varkappa'$ have an appreciable role in the correction to πR^2.

4. The cross section for differential splitting is connected with the amplitude $A_{\varkappa\mathbf{f}}$ by the relation

$$d\sigma_d = |A_{\varkappa\mathbf{f}}|^2(2\pi)^{-2}d\boldsymbol{\varkappa}(2\pi)^{-3}d\mathbf{f}. \tag{12}$$

Using Eq. (6) we put $A_{\varkappa\mathbf{f}}$ into the form

$$A_{\varkappa\mathbf{f}} = -\frac{(2\pi)^{3/2}R}{\alpha^{5/2}}a_{\mathbf{zu}} =$$
$$-\frac{(2\pi)^{3/2}R}{\alpha^{5/2}}\left\{\frac{J_1(pz)}{z}[\Phi(\mathbf{u},\mathbf{z}) + \Phi(\mathbf{u},-\mathbf{z})]\right.$$
$$\left.-\frac{1}{2\pi}\int d\mathbf{g}\frac{J_1(g)}{g}\cdot\frac{J_1(p)|\mathbf{z}-\mathbf{g}/p|)}{|\mathbf{z}-\mathbf{g}/p|}\Phi\left(\mathbf{u},\frac{2\mathbf{g}}{p}-\mathbf{z}\right)\right\}, \tag{13}$$

$$\Phi(\mathbf{u},\mathbf{z}) = \frac{1}{4\pi}\int\frac{e^{-x}}{x}\left(e^{-i\mathbf{u}\mathbf{x}} - \frac{1}{1+iu}\cdot\frac{e^{iux}}{x}\right)e^{i\mathbf{z}\mathbf{x}}d\mathbf{x}, \tag{14}$$

where $z = x/2\alpha$, $u = f/\alpha$ and $x = \alpha r$.

For the integral cross section for diffraction splitting we obtain the following expression:

$$\sigma_d = \frac{R^2}{\pi^2}\iint dz d\mathbf{u}\left|\frac{J_1(pz)}{z}\left[\Phi(\mathbf{u},\mathbf{z}) + \Phi(\mathbf{u}, -\mathbf{z})\right] - \frac{1}{2\pi}\int d\mathbf{g}\frac{J_1(g)}{g}\frac{J_1(p\,|\,\mathbf{z}-\mathbf{g}/p\,|)}{|\,\mathbf{z}-\mathbf{g}/p\,|}\Phi\left(\mathbf{u}, \frac{2\mathbf{g}}{p}-\mathbf{z}\right)\right|^2. \quad (15)$$

If $p \gg 1$, then

$$\sigma_d = \frac{2}{3} RR_d\int_0^\infty uI(u)\,du,$$

where $I(u)$ gives the energy distribution of the products of the splitting and has the form

$$I(u) = \frac{3}{(1+u^2)^2}\left[1 + \frac{2u}{1+u^2} - \arcsin\frac{u}{\sqrt{1+u^2}}\right] - 16(1-\ln 2)\frac{u}{(1-u^2)^3}. \quad (16)$$

The integral cross section for splitting is equal to

$$\sigma_d = \frac{\pi}{3}\left(2\ln 2 - \frac{1}{2}\right)RR_d,\ R_d \ll R,\ \lambda \ll R_d. \quad (17)$$

This formula agrees with the formula obtained by Glauber[4]. (We note that in the expression for σ_d obtained in Ref. 2 the region of large $\varkappa$ was not correctly taken into account. Therefore the numerical coefficient of RR_d in Ref. 2 is equal to 1.25, differing from the correct coefficient which is 0.96.)

5. In addition to diffraction scattering and diffraction splitting of the deuteron, the reaction of splitting of the neutron and proton and absorption of both by the nucleus is also possible. The cross sections for the first two processes are given, for $p \gg 1$ (Ref. 5) by

$$\sigma_n = \sigma_p = \pi RR_d/2,\ R_d \ll R. \quad (18)$$

Since the cross section of absorption of a single particle by the nucleus is equal to πR^2, and the cross section for the process in which one particle of the deuteron hits the nucleus and the other passes outside the nucleus is equal to $\pi RR_d/2$, the cross section for absorption of both particles is equal to

$$\sigma_a = \pi R^2 - \pi RR_d/2,\ R_d \ll R. \quad (19)$$

The total cross section of all processes σ_t can be determined knowing the elastic scattering amplitude at zero angle[6]

$$\sigma_t = 4\pi\lambda\,\mathrm{Im}\,f(0). \quad (20)$$

For point particles $f(0) = iKR^2/2$ and $\sigma_t = 2\pi R^2$. In the case of deuterons the scattering amplitude at zero angle is equal to

$$f(0) = i\frac{K}{2\pi}\iint\varphi_0^2(r)\{\omega_n + \omega_p - \omega_n\omega_p\}\,d\rho_d d\mathbf{r},$$

and the total cross section σ_t is determined by the expression

$$\sigma_t = 2\iint\varphi_0^2(r)\{\omega_n + \omega_p - \omega_n\omega_p\}\,d\rho_d d\mathbf{r}. \quad (21)$$

Using Eqs. (6) and (7), we obtain

$$\sigma_t = 4\pi R^2\left\{1 - \int_0^\infty \frac{p}{\xi}\tan^{-1}\frac{\xi}{p}\cdot\frac{J_1^2(\xi)}{\xi}d\xi\right\}, \quad (22)$$

$$p = \frac{R}{R_d}.$$

For $p \to \infty$

$$\sigma_t = 4\pi R^2\left\{1 - \int_0^\infty\frac{J_1^2(\xi)}{\xi}d\xi\right\} = 2\pi R^2.$$

In order to find the correction to this quantity for $R_d \ll R$, we calculate the difference of the integrals

$$\int_0^\infty \frac{p}{\xi}\tan^{-1}\frac{\xi}{p}\cdot\frac{J_1^2(\xi)}{\xi}d\xi - \int_0^\infty\frac{J_1^2(\xi)}{\xi}d\xi = \int_0^\infty\frac{J_1^2(pz)}{z}\left\{\frac{\tan^{-1}z}{z} - 1\right\}dz \equiv \delta_p.$$

In the latter integral the region of small z is not important, large pz playing the main role; therefore, the well known asymptotic representation of the Bessel function can be employed. Setting here $\sin^2(pz - \pi/4) \cong 1/2$, we obtain

$$\delta_p = \frac{1}{\pi p}\int_0^\infty(\tan^{-1}z - z)\frac{dz}{z^3} = -\frac{1}{4p},\ p \gg 1.$$

Thus, for $p \gg 1$

$$\int_0^\infty\frac{p}{\xi}\tan^{-1}\frac{\xi}{p}\frac{J_1^2(\xi)}{\xi}d\xi = \frac{1}{2} - \frac{1}{4p},$$

and, consequently, the total cross section of all processes is equal to

$$\sigma_t = 2\pi R^2 + \pi RR_d,\ R_d \ll R. \quad (23)$$

For an arbitrary value of p, one has the following relations

$$\sigma_e + \sigma_d = {}^1/_2\sigma_t, \quad \sigma_n + \sigma_p + \sigma_a = {}^1/_2\sigma_t. \qquad (24)$$

In fact, substituting the expression (5) into Eq. (12) instead of $A_{\varkappa \mathbf{f}}$ and integrating over $\varkappa$ and $\mathbf{f}$, we find

$$\sigma_e + \sigma_d = \iint \varphi_0^2(r)\{\omega_n + \omega_p - \omega_n\omega_p\}\, d\rho_d d\mathbf{r}.$$

Comparing this expression with Eq. (20) we obtain Eq. (24).

6. We show now how to take into account the Coulomb interaction in the treatment of diffraction. In the scattering by absorbing nuclei of fast charged particles, whose energies E are considerably above the Coulomb barrier ze^2/R, we should, obviously, take the factor $\Omega(\rho)$ equal to

$$\Omega^z(\rho) = \begin{cases} 0 & \rho < R, \\ e^{2i\eta(\rho)} & \rho > R, \end{cases}$$

where $\eta(\rho)$ is the scattering phase in the Coulomb field at infinity; for $KR \gg 1$ it is equal to

$$\eta(\rho) = n \ln(K\rho), \quad n = Ze^2/\hbar v$$

(v is the velocity of the particle at infinity).

The diffraction scattering and splitting of fast deuterons by completely black nuclei with account taken of the Coulomb interaction is determined by expansion of the modified deuteron wave function

$$\Psi^z = \Omega(\rho_n)\,\Omega^z(\rho_p)\,\psi_0(\rho_d)\,\varphi_0(r)$$
$$= \sum_{\varkappa} A_\varkappa^z \psi_\varkappa(\rho_d)\varphi_0(r) + \sum_{\varkappa,\mathbf{f}} A_{\varkappa\mathbf{f}}^z \psi_\varkappa(\rho_d)\varphi_{\mathbf{f}}(\mathbf{r}), \qquad (25)$$

where the expansion coefficients $A_\varkappa^z$ and $A_{\varkappa\mathbf{f}}^z$ are the probability amplitudes for elastic diffraction scattering and diffraction splitting of the deuteron.

The amplitude for elastic scattering, which is connected with $A_\varkappa^z$ by Eq. (2), is equal to

$$f(\vartheta) = -iK\left\{\frac{4\alpha}{\varkappa}\tan^{-1}\frac{\varkappa}{4\alpha}\int_R^\infty e^{2i\eta(\rho)}J_0(\varkappa\rho)\,\rho\,d\rho - \int\frac{4\alpha}{|2\mathbf{g}-\varkappa|}\tan^{-1}\frac{|2\mathbf{g}-\varkappa|}{4\alpha}\cdot\frac{RJ_1(|\varkappa-\mathbf{g}|R)}{|\varkappa-\mathbf{g}|}\theta(g)\,d\mathbf{g}\right\},$$

$$\theta(g) = \frac{1}{2\pi}\int_R^\infty e^{2i\eta(\rho)}J_0(g\rho)\,\rho\,d\rho. \qquad (26)$$

In the limiting case $R_d \ll R$, the scattering amplitude has the form

$$f(\vartheta) = -iK\left\{\frac{4\alpha}{\varkappa}\tan^{-1}\frac{\varkappa}{4\alpha}\int_R^\infty e^{2i\eta(\rho)}J_0(\varkappa\rho)\,\rho\,d\rho - \frac{RJ_0(\varkappa R)}{8\alpha}e^{2i\eta(R)}\right\} =$$
$$= i\lambda\left\{\frac{2p}{l_0\vartheta}\tan^{-1}\frac{l_0\vartheta}{2p}\left[l_0^{2in+1}\frac{J_1(l_0\vartheta)}{\vartheta} + 2in\vartheta^{-2in-2}\int_{l_0\vartheta}^\infty J_1(\zeta)\zeta^{2in}\,d\zeta\right] + \frac{l_0^{2in+2}}{4p}J_0(l_0\vartheta)\right\}, \qquad (27)$$
$$l_0 = KR, \quad E \gg Ze^2/R, \quad p \gg 1.$$

The differential cross section for elastic deuteron scattering is equal to

$$\sigma(\vartheta) = |f(\vartheta)|^2.$$

If $n \ll 1$ and $1 \ll p \ll l_0$, then

$$\sigma(\vartheta) = 4n^2\lambda^2/\vartheta^4, \quad \vartheta \ll \sqrt{2n}/l_0,$$
$$\sigma(\vartheta) = l_0^2\lambda^2 J_1^2(l_0\vartheta)/\vartheta^2, \quad \sqrt{2n}/l_0 \ll \vartheta \ll 2p/l_0,$$
$$\sigma(\vartheta) = (l_0^3\lambda^2/8\pi p^2)\cos^2\left(l_0\vartheta - \frac{\pi}{4}\right)\Big/\vartheta, \qquad (28)$$
$$2p/l_0 \ll \vartheta \ll 1.$$

Thus, in the case $n \ll 1$ and $1 \ll p \ll l_0$ purely Coulomb scattering occurs only in the angular region $\vartheta \ll \sqrt{2n}/l_0$. In the angular region $\sqrt{2n}/l_0 \ll \vartheta \ll 2p/l_0$ the scattering of deuterons has just the same character as the diffraction scattering of neutral particles. Finally, in diffraction scattering of deuterons in the angular region $2p/l_0 \ll \vartheta \ll 1$, their spatial structure shows up.

If $n \gg 1$ and $n \ll p \ll l_0$, then

$$\sigma(\vartheta) = 4n^2\lambda^2/\vartheta^4, \quad \vartheta \ll 2n/l_0,$$
$$\sigma(\vartheta) = (2l_0\lambda^2/\pi)\sin^2\left(l_0\vartheta - \frac{\pi}{4}\right)\Big/\vartheta^3,$$
$$2n/l_0 \ll \vartheta \ll 2p/l_0, \qquad (29)$$
$$\sigma(\vartheta) = (l_0^3\lambda^2/8\pi p^2)\cos^2\left(l_0\vartheta - \frac{\pi}{4}\right)\Big/\vartheta,$$
$$2p/l_0 \ll \vartheta \ll 1.$$

Thus, in this case the region of purely Coulomb scattering broadens in comparison with the preceding case, right up to angles of the order of $2n/l_0$. For $\vartheta \sim 2n/l_0$, a sharp decrease in the scattering, by a factor of the order of n takes place[7]. In the angular interval $2n/l_0 \ll \vartheta \ll 2p/l_0$ the deuteron scattering has the same character as diffraction scattering of point neutral particles. The spatial structure of the deuterons shows up in the angular interval $2p/l_0 \ll \vartheta \ll 1$.

If, finally, $1 \ll p \ll n \ll l_0$, then

$$\begin{gathered}\sigma(\vartheta) = 4n^2\lambda\!\!\!^{-2}/\vartheta^4, \quad \vartheta \ll 2p/l_0,\\ \sigma(\vartheta) = 4\pi^2p^2n^2\lambda\!\!\!^{-2}/l_0^2\vartheta^6,\\ 2p/l_0 \ll \vartheta \ll 2(\pi^3n^2p^4)^{1/5}/l_0,\\ \sigma(\vartheta) = (l_0^3\lambda\!\!\!^{-2}/8\pi p^2)\cos^2\left(l_0\vartheta - \frac{\pi}{4}\right)\Big/\vartheta,\\ 2(\pi^3n^2p^4)^{1/5}/l_0 \ll \vartheta \ll 1.\end{gathered} \tag{30}$$

We see that in this case the region of Coulomb scattering does not extend to angles $2n/l_0$, but to angles of the order of $2p/l_0$. The region of diffraction scattering of point particles vanishes, in general. The finite size of the deuteron begins to show up at angles of the order of $2p/l_0$. In the angular interval $2p/l_0 \ll \vartheta \ll 2(\pi^3n^2p^4)^{1/5}/l_0$ the cross section falls off as $1/\vartheta^6$, and then changes as $1/\vartheta$.

7. We turn now to consideration of the splitting of fast deuterons, taking into account the Coulomb interaction.

The amplitude $A^z_{\varkappa f}$ in the expansion is determined by the relation

$$\begin{aligned}A^z_{\varkappa f} &= \iint \exp\{-i\varkappa\rho_d\}\ \varphi^*_f(\mathbf{r})\,\Omega(\rho_n)\,\Omega^z(\rho_p)\,d\rho_d\,d\mathbf{r}\\ &= e^{2i\eta(R)}\Big\{A_{\varkappa f} + 2\pi\int_R^\infty [e^{2i\{\eta(\rho)-\eta(R)\}}\\ &\quad - 1]\,J_0(\varkappa\rho)\,\rho d\rho \int e^{-i\varkappa\mathbf{r}/2}\,\varphi^*_f(\mathbf{r})\,\varphi_0(r)\,d\mathbf{r}\\ &\quad - \int d\mathbf{g}\int_R^\infty [e^{2i\{\eta(\rho)-\eta(R)\}} - 1]\,J_0(g\rho)\,\rho\,d\rho\\ &\quad \times \frac{RJ_1(|\varkappa - \mathbf{g}|R)}{|\varkappa - \mathbf{g}|}\cdot\int e^{-i(\mathbf{g}-\varkappa/2)\mathbf{r}}\varphi^*_f(\mathbf{r})\,\varphi_0(r)\,d\mathbf{r}\Big\}\end{aligned} \tag{31}$$

The differential cross section for splitting of a fast deuteron is equal to

$$d\sigma_f = |A^z_{\varkappa f}|^2(2\pi)^{-2}d\varkappa\cdot(2\pi)^{-3}d\mathbf{f}. \tag{32}$$

For $p \gg 1$ we have

$$\begin{aligned}A^z_{\varkappa f} \approx e^{2i\eta(R)}\Big\{A_{\varkappa f} - 2\pi\frac{2in}{\varkappa}\int_R^\infty J_1(\varkappa\rho)\,e^{2i\{\eta(\rho)-\eta(R)\}}\,d\rho\\ \times \int e^{-i\varkappa\mathbf{r}/2}\varphi^*_f(\mathbf{r})\,\varphi_0(r)\,d\mathbf{r}\Big\},\end{aligned} \tag{33}$$

and the integral cross section for deuteron splitting is equal to

$$\sigma_f = \frac{R^2}{\pi^2}\iint dz\,d\mathbf{u}\left|a_{z\mathbf{u}} + \frac{2in}{p}\frac{(pz)^{-2in}}{z^2}\int_{pz}^\infty J_1(\zeta)\,\zeta^{2in}d\zeta\cdot\Phi(\mathbf{u}, z)\right|^2 \tag{34}$$

The expression for σ_f diverges logarithmically for $z \to 0$. This is connected with the Coulomb character of the interaction, which leads to divergence of the elastic scattering cross section at small angles, *i.e.*, at small z. In so far as the deuteron is assumed to be practically unbound, this sort of divergence obviously ought to take place also for the splitting cross section. In reality, for splitting of the deuteron to take place, z must exceed some minimum value z_m. This value can be determined, if we take into account the fact that the change of energy of the deuteron connected with the change of its momentum, which is approximately equal to $\hbar^2K\kappa/2M$, must exceed the binding energy of the deuteron $\epsilon = \hbar^2\alpha^2/M$. From this it follows that $z_m \approx \alpha/K = \lambda\!\!\!^{-}/2R_d$.

Just this value of the lower limit of the z integration can be obtained in the following way. The treatment of deuteron splitting is valid for sufficiently large z satisfying the inequality $z > z'$ ($1 \gg z' \gg \alpha/K$), since in this region of z the deuteron can be considered approximately as an unbound system. On the other hand, small z corresponds to large values of the impact parameter, for which splitting of the deuteron can be considered using perturbation theory. Going to the system in which the center of gravity of the deuteron before collision is at rest, and employing the perturbation energy in the form

$$V(t) = Ze^2/[\rho_p^2 + (z_p - vt)^2]^{1/2},$$

it is easy to obtain the following expression for the part of the cross section for splitting in a Coulomb field which corresponds to large values of the impact parameter

$$d\sigma_c' = \frac{n^2}{2\pi^3}\left|\int e^{-i\mathbf{K}'\mathbf{r}/2}\varphi_{\mathbf{f}}(\mathbf{r})\,\varphi_0(\mathbf{r})\,d\mathbf{r}\right|^2$$

$$\delta\left\{\frac{\mathbf{K}'\mathbf{v}-\omega}{v}\right\}\frac{d\mathbf{K}'}{K'^4}\,d\mathbf{f},$$

where $\mathbf{K}'$ is the wave vector of the center of gravity of the deuteron after collision and $\hbar\omega = \epsilon + \hbar^2 f^2/M + \hbar^2 K^2/4M$. Getting rid of the δ-function by integration over the angle between the vectors $\mathbf{K}'$ and $\mathbf{v}$ and carrying out the integration over $\mathbf{f}$ we obtain

$$\sigma_c' = 8\pi n^2 R_d^2 \int\limits_{z_m}^{z'} z^{-5}\left[z^2 - 4\left(\tan^{-1}\frac{z}{2}\right)^2\right]dz, \qquad (35)$$

$$z = \frac{K'}{2\alpha},$$

where $z_m \approx \alpha/K$ and the upper limit z' is chosen such that perturbation theory can be employed.

In order to obtain the total cross section for splitting it is necessary to add Eq. (34), in which the integration over z is carried out from z' to infinity, to Eq. (35). Since both these expressions behave as $\ln z$ for small z, the sum will not contain z' and leads to the expression (34) in which $z_m = \alpha/K$ is taken as lower limit.

Carrying out the integration in Eq. (34) over u and the angle determining the direction of the vector $\mathbf{z}$ we obtain

$$\sigma_f = \sigma_d + \sigma_c + \sigma_{\text{int}},$$

$$\sigma_c = 8\pi n^2 R^2 p^2 \int\limits_{pz_m}^{\infty}\left\{\frac{x^2}{p^2} - 4\left(\tan^{-1}\frac{x}{2p}\right)^2\right\}\left|\int\limits_x^{\infty} J_1(\zeta)\,\zeta^{2in}d\zeta\right|^2\frac{dx}{x^5}, \qquad (36)$$

$$\sigma_{\text{int}} = 4\pi R^2 p^2\,\mathrm{Re}\int\limits_0^{\infty}\left\{\frac{x^2}{p^2} - 4\left(\tan^{-1}\frac{x}{2p}\right)^2\right\}\frac{J_1(x)}{x}\,2inx^{-2in-2}\int\limits_x^{\infty} J_1(\zeta)\,\zeta^{2in}d\zeta\,\frac{dx}{x},$$

where σ_d is the cross section found earlier for diffraction splitting of the deuteron with neglect of Coulomb interaction, and σ_c is the cross section for splitting of the deuteron due to the Coulomb field of the nucleus. The quantity σ_{int} determines the part of the splitting cross section coming from interference of diffraction and Coulomb scattering. In the expressions for σ_d and σ_{int} we set the lower limit of the z-integration equal to zero, since these expressions do not diverge as $z_m \to 0$.

It is easy to see that the interference term σ_{int} is equal to zero for $p \gg 1$. In fact,

$$\sigma_{\text{int}} = \frac{4\pi R_d^2}{3}\,\mathrm{Re}\left\{in\int\limits_1^{\infty} dx\,x^{2in}\int\limits_0^{\infty} dy\,yJ_1(y)\,J_1(xy)\right\}$$

$$= \frac{4\pi R_d^2}{3}\,\mathrm{Re}\left\{in\int\limits_1^{\infty} dx\,x^{2in}\,\delta(x-1)\right\} = 0.$$

Thus, for $p \gg 1$, the integral cross section for splitting is equal to the sum of the cross sections for diffraction splitting and splitting coming from the Coulomb field of a black nucleus. It can be shown that the interference term is the order of p times smaller than σ_c.

We consider in detail the cross section for splitting of the deuteron in a Coulomb field for $p \gg 1$, defined by the general formula Eq. (36). It can be evaluated in the two limiting cases of small and large n. If $n \ll 1$, then

$$\int\limits_x^{\infty} J_1(\zeta)\,\zeta^{2in}\,d\zeta \approx \int\limits_x^{\infty} J_1(\zeta)\,d\zeta = J_0(x),$$

$$\sigma_c = \frac{4\pi}{3}n^2 R_d^2 \int\limits_{pz_m}^{\infty}\frac{J_0^2(x)}{x}\,dx, \quad n \ll 1, \quad p \gg 1.$$

Integrating by parts we find

$$\int\limits_{pz_m}^{\infty}\frac{J_0^2(x)}{x}\,dx = J_0^2(pz_m)\ln\frac{1}{pz_m} + A,$$

where A is the order of unity. Thus, the cross section for splitting of fast deuterons coming from the Coulomb field is equal, for $n \ll 1$, to

$$\sigma_c = \frac{4\pi}{3}n^2 R_d^2\left\{\ln\frac{1}{pz_m} + A\right\}$$
$$\approx \frac{4\pi}{3}n^2 R_d^2 \ln\frac{R_d^2}{R\lambda\!\!\!^{-}}, \quad n \ll 1, \qquad (37)$$

(the last formula is valid if $pz_m \ll 1$). This expression coincides with the result obtained by Dancoff[8], and Mullin and Guth[9], using perturbation theory.

We look now at the case of large n. Noting that

$$\int_x^\infty J_1(\zeta)\,\zeta^{2in}\,d\zeta = 2^{2in}\frac{\Gamma(1+in)}{\Gamma(1-in)} - \int_0^x J_1(\zeta)\,\zeta^{2in}\,d\zeta,$$

and using the asymptotic expansion

$$\int_0^x J_1(\zeta)\,\zeta^{2in}\,d\zeta = \frac{1}{2in}J_1(x)\,x^{2in+1} + \frac{1}{4n^2}J_0(x)\,x^{2in+2} + O(n^{-3}),$$

we obtain, neglecting terms of order $0(n^{-3})$,

$$\sigma_c = 8\pi n^2 R_d^2 \int_{z_m}^\infty \frac{z^2 - 4\left(\tan^{-1}\frac{z}{2}\right)^2}{z^5}\,dz - 2\pi R^2 \int_0^\infty \frac{z^2-4\left(\tan^{-1}\frac{z}{2}\right)^2}{z^3} J_1^2(pz)\,dz = \frac{4\pi}{3} n^2 R_d^2\left\{\ln\frac{2}{z_m} + 1 - \frac{3}{4}\frac{\pi^2}{8}\right\} - \frac{\pi}{3}\left(\ln 2 + \frac{1}{2}\right) R R_d. \qquad (38)$$

The second term in this expression is of the same order of magnitude as the cross section for diffraction splitting. In order that the asymptotic expansion can be used this term must be small compared with the first; in other words, the inequality $n^2 \gg p$ should be fulfilled. We see that in this case the cross section for diffraction splitting is a small correction to the cross section coming from the Coulomb field. The total cross section for splitting is determined by the formula (accurate to within a constant factor in the logarithm):

$$\sigma_f = (4\pi/3)\,n^2 R_d^2 \ln(R_d/\lambda\!\!\!^{-}), \quad n \gg 1. \qquad (39)$$

8. In the high-energy region when the mean free path of the particle in nuclear matter becomes comparable with nuclear dimensions, the nucleus cannot be considered completely black, and must be viewed as a semi-transparent body characterized by a complex absorption coefficient $b = b_1 - ib_2$, where b_1 is the absorption coefficient of nuclear matter and $b_2 = 2(\nu - 1)K$, where ν is the index of refraction of nuclear matter[10]. In the investigation of diffraction problems in this case, the expansion of the modified wave function can be used as before, if we consider the factor $\Omega(\rho)$ equal, for uncharged particles, to

$$\Omega(\rho) = \begin{cases} \exp\left(-b\sqrt{R^2-\rho^2}\right), & \rho \leqslant R \\ 1 & \rho > R. \end{cases}$$

Application of these formulae to the scattering of fast neutral particles leads to the well known expression for the amplitude of elastic scattering[10]

$$f(\vartheta) = -\frac{iK}{2\pi}\int \Omega(\rho)\,e^{-i\varkappa\rho}\,d\rho = iK\int_0^R \left(1 - \exp\{-b\sqrt{R^2-\rho^2}\}\right) J_0(\varkappa\rho)\,\rho\,d\rho,$$

which, in the limiting cases of large and small absorption has the form

$$f(\vartheta) = iK\left\{\frac{RJ_1(\varkappa R)}{\varkappa} - \frac{J_0(\varkappa R)}{b^2}\right\}, \quad \varkappa \ll R\,|b|^2,$$
$$f(\vartheta) = iR\frac{Kb}{\varkappa^2}\left\{\frac{\sin\varkappa R}{\varkappa R} - \cos\varkappa R\right\}, \quad \varkappa \gg R\,|b|^2. \qquad (40)$$

The following expression can be obtained for the amplitude of elastic diffraction scattering of fact deuterons by semi-transparent nuclei with no account of the Coulomb interaction

$$f(\vartheta) = 2\pi i K R^2\left\{\frac{2p}{\varkappa'}\tan^{-1}\frac{\varkappa'}{2p}\,[\chi_n(\varkappa') + \chi_p(\varkappa')] - \int \frac{2p}{|2g' - \varkappa'|}\tan^{-1}\frac{|2g'-\varkappa'|}{2p} \times \chi_n(g')\,\chi_p(|\varkappa' - g'|)\,dg'\right\},$$
$$\chi_{n,p}(\varkappa') = \frac{1}{2\pi}\int_0^1 \left(1 - \exp\{-b_{n,p}R\sqrt{1-y^2}\}\right) \times J_0(\varkappa' y)\,y\,dy \qquad (41)$$

(b_n and b_p are the complex absorption coefficients of neutrons and protons). For $p \gg 1$

$$f(\vartheta) = iKR^2\left\{\frac{2p}{\varkappa'}\tan^{-1}\frac{\varkappa'}{2p} \times \int_0^1 \left(1 - \exp\{-BR\sqrt{1-y^2}\}\right) J_0(\varkappa' y)\,y\,dy - 2\pi\int\left(\frac{\tan^{-1}\xi/p}{\xi/p} - 1\right)\chi_n(\xi)\,\chi_p(|\xi - \varkappa'|)\,d\xi\right\}, \qquad (42)$$

where B is the complex absorption coefficient for deuterons equal to $B = B_1 - iB_2 = b_n + b_p$.

The total cross section for all processes is given by the relation

$$\sigma_t = 4\pi R^2\,\mathrm{Re}\left\{\int_0^1 \left(1 - \exp\{-BR\sqrt{1-y^2}\}\right) y\,dy - 4\pi^2\int_0^\infty \left\{\frac{\tan^{-1}(\xi/p)}{\xi/p} - 1\right\}\chi_n(\xi)\,\chi_p(\xi)\,\xi\,d\xi\right\}. \qquad (43)$$

In the case of large absorption of particles ($|b_n|^2R^2 \gg p$, $|b_p|^2R^2 \gg p$) the total cross section and the cross sections of elastic scattering and diffraction splitting of the deuteron are given by the relations:

$$\sigma_t = 2\pi R^2\left\{1 - 2\frac{B_1^2 - B_2^2}{(B_1^2+B_2^2)^2 R^2}\right\} + \pi R R_d.$$

$$\sigma_e = \pi R^2\left\{1 + \frac{1}{2B_1^2R^2} - \frac{4(B_1^2 - B_2^2)}{(B_1^2 + B_2^2)^2R^2}\right\} + \frac{2\pi}{3}(1 - \ln 2)RR_d,$$

$$\sigma_d = \frac{\pi}{3}\left(2\ln 2 - \frac{1}{2}\right)RR_d.$$

In the case of small absorption of particles ($|b_n|^2R^2 \ll p$, $|b_p|^2R^2 \ll p$) the cross sections are given by the formulae:

$$\sigma_t = \sigma_t^0 + \frac{2\pi}{3} R_d^2 \operatorname{Re}(b_n b_p) R^2 \ln\frac{R}{R_d},$$

$$\sigma_t^0 = \frac{4\pi}{3} R^3 B_1$$

$$\sigma_e = \sigma_e^0 - \frac{\pi}{6} R_d^2 |B|^2 R^2 \ln\frac{R}{R_d},$$

$$\sigma_e^0 = \frac{\pi}{2} R^4 |B|^2$$

$$\sigma_d = \frac{\pi}{6} R^2 |B|^2 R_d^2 \ln\frac{R}{R_d},$$

where σ_t^0 is the total cross section for all processes and σ_e^0 is the cross section for the elastic scattering of point particles, the complex absorption coefficient of which is the sum of complex absorption coefficients of neutron and proton.

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176