radius attains such a value that the energy of nucleons with the largest value of angular momentum equals the Fermi limit. This makes it possible to explain qualitatively the experimentally observed fact that the mass of the larger fission fragment is is equal for different elements. As it is well known, the mass of the lighter fragment varies within much wider limits. Evidently, for all studied fissile nuclei, the maximum values of the nucleonic angular momentum coincide prior to fission. Most probably, all of them then posses a pair of neutrons with an angular momentum of the order of 5–6. The angular momentum of these nucleons determines the cross-section of the wider end of the nucleus which subsequently forms the heavier fission fragment. It follows from this approximate quantization of the size of the heavy fragment that the variations in its mass are smaller than is the case for the lighter fragment.

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Moment of Inertia of a System of Interacting Particles

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The problem of singling out the collective degrees of freedom of a system consisting of N interacting particles is considered. It is shown that for some special states of internal motion, the energy of the system in the center of mass system can be represented as the sum of the energy of internal motion and the rotational energy. The concept of the moment of inertia of a system of N interacting particles is introduced.

INTRODUCTION

AT PRESENT it has been established that the lowest excited states of nuclei in the mass number range $150 < A < 190$ and $A > 225$ are rotational states. Such states arise in Coulombic excitation of the nuclei, in processes of radioactive decay, and also in inelastic collisions of particles with the nucleus.

An explanation of rotational states of the nucleus in the quasi-molecular model of the nucleus proposed by A. Bohr[1] is related to the motion of a wave around the nucleus. The nuclear matter is regarded as an incompressible, irrotational fluid (the hydrodynamical model). The part of the nuclear matter which participates in the rotation, according to the hydrodynamical model, is proportional to the square of the deviation of the form of the nucleus from a sphere. If we assume that the nucleus has the form of an ellipsoid of rotation with semiaxes c and a, then the moment of inertia of the nucleus is given by

$$J = {}^1\!/_5\, mA\, (c^2 - a^2)^2 / (c^2 + a^2),$$

where m is the mass of the nucleon, and A is the mass number of the nucleus. For small deviations from spherical form $J = {}^2/_5\, mAR_0^2(\Delta R / R_0)^2$, where R_0 is the radius of the sphere with volume equal to that of the nucleus, and ΔR is the difference between the larger and the smaller semiaxis. If it is assumed that the nuclear charge is uniformly distributed, then the quantity $\Delta R/R_0$ will correspond to the quadrupole moment $Q_0 = {}^4/_5\, ZR_0^2\Delta R / R_0$. in a coordinate system fixed with respect to the nucleus. Determining $\Delta R/R_0$ from experimental values of the quadrupole moments, we find values of the moments of inertia (for $R = 1.2\, A^{1/3}\, 10^{-13}$ cm.) which are 3–5 times less than the experimental valvalues[2]. In addition, it is found that the experimentally determined moments of inertia of nuclei with odd A are (up to 40 percent) larger than the moments of inertia of even-even nuclei with approximately the same deviation from spherical form[3]. Both of these facts indicate the inadequacy of the hydrodynamical model of the nucleus[2,3]. In this connection, a number of papers have appeared recently concerning the separation of collective and one-particle degrees of freedom in nuclei. In the paper by Inglis[4] the kinetic energy of rotation was obtained by studying the motion of the nucleons in the rotating self-consistent field of a three-dimensional harmonic oscillator, which deviates slightly from spherical symmetry. Similar calculations are carried out by A. Bohr and Mottelson[3] who take into account deviation from the self-consistent field due to the sum of pair interactions. In both of these papers over-determined coordinate systems are used, *i.e.*, coordinates describing the orientation of the self-consistent field and coordinates of the center of mass are introduced as additional superfluous variables. As is well known, a similar difficulty concerning the center of mass coordinates occurs in the shell models. It is usually assumed that for a large number of nucleons the superfluous coordinates change only slightly the results of investigations of internal nuclear motions. However, in investigations of the collective motions of the nucleons in the nucleus, it is necessary to study the change of just these superfluous variables, so that a special investigation of the possibility of such investigations is needed. In the papers of Tolhoek[5] and Coester[6], which are devoted to a study of collective motions in nuclei on the basis of the N-body problem, the possibility of separating collective and internal motion is assumed; however, the choice of the coordinates describing the internal degrees of freedom remains unspecified.

In the present paper we examine the question of distinguishing collective motions in a system consisting of N interacting particles. In the first section, using the example of a system consisting of three interacting spinless particles, we carry out an explicit separation of the collective degrees of freedom associated with the translational motion of the center of mass of the system and the rotation of the system. We give an expression for the square of the angular momentum of the whole system in terms of collective angular variables, and conditions are indicated under which the energy of the system can be represented as a sum of an internal energy and a rotational energy, determined by a moment of inertia which depends on the internal motion. The results obtained in the second section are applied to the case of a system consisting of one light and two heavy particles (hydrogen molecule ion). A system consisting of N particles is studied in the third section.

1. SYSTEM CONSISTING OF THREE PARTICLES OF EQUAL MASS

We consider three spinless particles of equal mass m, interacting with central forces of an arbitrary type. Let $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ be the radius vectors locating the position of these particles in space. We go into the center of mass system (c.m.s.) xyz by introducing new coordinates according to:

$$ {}^1/_3\,(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3) = \mathbf{R}, \quad -\mathbf{r}_1 + {}^1/_2\,(\mathbf{r}_2 + \mathbf{r}_3) = \mathbf{r}, $$
$$ -\mathbf{r}_2 + \mathbf{r}_3 = \boldsymbol{\rho} = 2\mathbf{q} / \sqrt{3}. \tag{1.1} $$

In the c.m.s. the kinetic energy operator

$$ T = T_q + T_r, \quad T_q = -(\hbar^2 / 2\mu)\,\Delta_q, \quad T_r = -(\hbar^2 / 2\mu)\,\Delta_r \tag{1.2} $$

is the sum of the kinetic energy operators of two equivalent particles of mass $\mu = 2m/3$, which completely describe the behavior of the system of three particles in the xyz coordinate system. Instead of the vectors $\mathbf{q}$, $\mathbf{r}$ we introduce polar coordinates. Then

$$T_q = -\frac{\hbar^2}{2\mu q^2}\left\{\frac{\partial}{\partial q}\left(q^2\frac{\partial}{\partial q}\right) - L_q^2\right\},$$
$$T_r = -\frac{\hbar^2}{2\mu r^2}\left\{\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - L_r^2\right\},$$
$$L_q = -i\,[\mathbf{q}\nabla_q], \quad L_r = -i\,[\mathbf{r}\nabla_r],$$
$$L_j^2 = -\left\{\frac{1}{\sin\vartheta_j}\frac{\partial}{\partial\vartheta_j}\left(\sin\vartheta_j\frac{\partial}{\partial\vartheta_j}\right) + \frac{1}{\sin^2\vartheta_j}\frac{\partial^2}{\partial\varphi_j^2}\right\},$$
$$\{L_j\}_z = -i\frac{\partial}{\partial\varphi_j}, \quad j = r, q.$$

We introduce a new, moving coordinate system (ξ, η, ζ) related to our particles in such a way that the ζ-axis coincides with the direction of the vector q, and the ξ, ζ-plane coincides with the plane of the vectors r, q. In Fig. 1

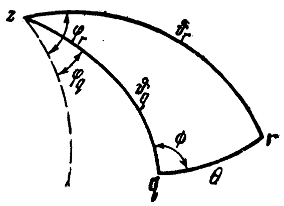

FIG. 1.

the direction of the z axis and the vectors q and r are indicated by the points z, q, r on the surface of a unit sphere. The broken line represents the intersection of this surface with the xz plane. The polar angles φ_q and δ_q of the vector q, and the angle ϕ between the planes going through the axes z, ζ and ζ, ξ, completely determine the position of the system. In addition, let θ be the angle between the vectors q and r. The angles φ, ϑ, ϕ, θ are determined by the angles φ_q, $\vartheta_q \phi_r$, ϑ_r with the help of the relations

$$\varphi = \varphi_q, \quad \vartheta = \vartheta_q, \quad \cos\theta = \sin\vartheta_q \sin\vartheta_r \cos(\varphi_q - \varphi_r)$$
$$+ \cos\vartheta_r\cos\vartheta_q, \quad \sin\theta\sin\phi = \sin\vartheta_r \sin(\varphi_q - \varphi_r),$$
$$\sin\theta\cos\phi$$
$$= \cos\vartheta_q \sin\vartheta_r\cos(\varphi_q - \varphi_r) - \cos\vartheta_r\sin\vartheta_q.$$

The potential energy of the system as a function of the distances between the particles will depend only on the coordinates q, r, θ., which we shall call the internal coordinates

$$V = V\left(\sqrt{r^2 + q^2/3 + 2qr\cos\theta/\sqrt{3}}, \quad 2q/\sqrt{3}, \right.$$
$$\left.\sqrt{r^2 + q^2/3 - 2qr\cos\theta/\sqrt{3}}\right). \qquad (1.3)$$

We shall call the angles $\varphi\vartheta\phi$ determining the orientation of the system $\xi\eta\zeta$ the external or collective coordinates.

The operator corresponding to the total angular momentum of the entire system has the form

$$\hbar\mathbf{L} = -i\hbar\,\{\mathbf{q}\times\nabla_r + \mathbf{r}\times\nabla_r\}. \qquad (1.4)$$

In the new variables we have for the operator corresponding to the square of the angular momentum (1.4).

$$\hbar^2L^2 = -\hbar^2\left\{\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right)\right.$$
$$\left.+\frac{1}{\sin^2\vartheta}\left(\frac{\partial^2}{\partial\varphi^2} + 2\cos\vartheta\frac{\partial^2}{\partial\varphi\partial\phi} + \frac{\partial^2}{\partial\phi^2}\right)\right\}, \qquad (1.5)$$

$$\hbar L_z = -ih\partial/\partial\varphi. \qquad (1.6)$$

In going over to the variables r, q, θ, ϑ, φ, ϕ, the $H = T + V$ of the entire system is expressed by the equation

$$H = \mathcal{K} + V + \hbar^2L^2/2\mu q^2 - \pi, \qquad (1.7)$$

$$\mathcal{K} = -\frac{\hbar^2}{2\mu}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right.$$
$$+\frac{1}{q^2}\frac{\partial}{\partial q}\left(q^2\frac{\partial}{\partial q}\right) - \frac{2}{q^2}\frac{\partial^2}{\partial\phi^2}$$
$$\left.+\left(\frac{1}{q^2}+\frac{1}{r^2}\right)\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\right\}, \qquad (1.8)$$

$$\pi = \frac{\hbar^2}{\mu q^2}\left\{\left[\frac{\sin\phi}{\sin\vartheta}\frac{\partial}{\partial\varphi} - \cos\phi\frac{\partial}{\partial\vartheta} + \cot\vartheta\sin\phi\frac{\partial}{\partial\phi}\right]\frac{\partial}{\partial\theta}\right.$$
$$\left.+\cot\theta\left[\cot\vartheta\cos\phi\frac{\partial}{\partial\phi} + \sin\phi\frac{\partial}{\partial\vartheta} + \frac{\cos\phi}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right]\frac{\partial}{\partial\phi}\right\}, \qquad (1.9)$$

the other operators involved in (1.7) were defined earlier.

It is easy to see that the operator corresponding to the square of the angular momentum (1.5) and its projection (1.6) commute with the complete Hamiltonian of the system (1.7). Consequently, the quantities corresponding to them will be integrals of motion. The square of the total angular momentum operator and its projection have eigenvalues and eigenfunctions given by the equations

$$\hat{L}^2 D^L_{MK} = L(L+1)D^L_{MK}, \qquad (1.10)$$

$$\hat{L}_z D^L_{MK} = M D^L_{MK}, \quad M, K = 0, \pm 1 \pm \ldots \pm L,$$
$$D^L_{MK} = e^{iM\varphi} d^L_{MK}(\vartheta) e^{iK\varphi},$$
$$d^L_{MK}(\vartheta) = \sum_{\varkappa} (-1)^{\varkappa} \frac{\sqrt{(L+M)!(L-M)!(L+K)!(L-K)!}}{(L-K-\varkappa)!(L+M-\varkappa)!\varkappa!(\varkappa+M-K)!} \times \cos^{2L+M-K-2\varkappa}\left(\frac{\vartheta}{2}\right)\sin^{2\varkappa+M-K}\left(\frac{\vartheta}{2}\right). \tag{1.11}$$

The functions D^L_{MK} are the irreducible representations of the three-dimensional rotation group, first introduced by Wigner[7]. They form a unitary matrix and satisfy the orthogonality relations

$$\int_0^{\pi} \sin\vartheta d\vartheta \int_0^{2\pi} d\varphi \int_0^{2\pi} d\psi D^L_{MK} D^{*l}_{mK'} = \frac{8\pi^2}{2L+1}\delta_{Ll}\delta_{Mm}\delta_{KK'}.$$

For $M = 0$ or $K = 0$, the functions D^L_{MK} reduce to the spherical functions

$$D^L_{M0} = \sqrt{\frac{4\pi}{2L+1}} Y_{LM}(\varphi, \vartheta),$$
$$D^L_{0K} = \sqrt{\frac{4\pi}{2L+1}} Y_{LK}(\psi, \vartheta). \tag{1.12}$$

The stationary states of the system of three particles are determined by the Schroedinger equation

$$(H - E)\psi = 0. \tag{1.13}$$

Let us consider the states with definite values of the integrals of motion L and M. The wave functions of such a state can be represented in the form

$$\psi^L_M = \sqrt{\frac{2L+1}{8\pi^2}} \sum_K D^L_{MK}(\varphi, \vartheta, \psi)\varphi_K(r, q, \theta). \tag{1.14}$$

In particular $\psi^0_0 = \varphi_0(r, q, \theta)$ for $L = 0$, *i.e.*, the properties of the system in the s-state do not depend on its spatial orientation and are determined only by the internal coordinates r, q, θ. The wave function $\varphi_0(r, q, \theta)$ satisfies the equation

$$\left\{-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{q^2}\frac{\partial}{\partial q}\left(q^2\frac{\partial}{\partial q}\right) + \left(\frac{1}{q^2}+\frac{1}{r^2}\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right] + V - E\right\}\varphi_0(r, q, \theta) = 0. \tag{1.15}$$

We substitute (1.14) into (1.13), multiply the result by $\sqrt{(2L+1)/8\sigma^2}D^{*L}_{MK'}$ and integrate over the external variables. Then we obtain the system of equations

$$\left\{\frac{\hbar^2}{2\mu q^2}[L(L+1) - K(K+1)] + \mathscr{L}(K) - E\right\} \times \varphi_K = \sum_{K'} (K|\pi|K')\varphi_{K'}, \tag{1.16}$$

$$(K|\pi|K') = \frac{2L+1}{8\pi^2}\int D^L_{MK'}\pi D^{*L}_{MK}d\Omega, \tag{1.17}$$

$$\mathscr{L}(K) = -\frac{\hbar^2}{2\mu}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{q^2}\frac{\partial}{\partial q}\left(q^2\frac{\partial}{\partial q}\right) + \left(\frac{1}{q^2}+\frac{1}{r^2}\right) \times \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{K^2}{\sin^2\theta}\right] - \frac{K(1-K)}{q^2}\right\} + V. \tag{1.18}$$

The operator of internal motion (1.18) depends only on the absolute value of K. The diagonal elements of the matrix (1.17) are equal to zero.

If we omit the right side of (1.16), we get the system of independent equations

$$\{(\hbar^2/2\mu q^2)[L(L+1) - K(K+1)] + \mathscr{L}(K) - E\}\varphi_K(r, q, \theta) = 0. \tag{1.19}$$

The system of equations (1.19) will be a good approximation to (1.16) if two conditions are met: a) the system of three particles has axial symmetry in the coordinate system fixed with respect to these particles, and b) the ζ axis of this coordinate system coincides with the axis of symmetry*.

We shall assume that both these conditions are fulfilled. In this approximation the number K is a good quantum number; its value determines the projection of the angular momentum on the ζ axis. For $K = L$, equation (1.19) goes over into the equation

*The multiple-valued nature of the choice of the system ξ, η, ζ is important for the symmetry properties of the wave function, and will be considered in a subsequent paper, where systems of particles with spin will be studied.

$$[\mathscr{L}(K)-\varepsilon_K]\,\varphi(r,q,\theta)=0. \qquad (1.20)$$

Solving equation (1.20) we obtain a series of energy levels. We number these levels in order of increasing index a, which takes on the values 0, 1, 2, ..., and designate the corresponding wave functions by φ_{aK}. In particular, the wave function φ_{0K} corresponds to the lowest energy.

We introduce the concept of the moment of inertia of the system in the state φ_{0K} with the help of the relation

$$J_{0K}=\left[\frac{1}{\mu}\int\varphi^*_{0K}\,\frac{1}{q^2}\,\varphi_{0K}\,d\tau\right]^{-1}. \qquad (1.21)$$

If the inequality

$$\hbar^2/2J_{0K}<\varepsilon_{1K}-\varepsilon_{0K}, \qquad (1.22)$$

is fulfilled, then, according to (1.19), for a given value of K, the energy of the system corresponding to the state of lowest energy of internal motion can be represented approximately in the form of a sum of the internal energy ε_{0K} and the rotational energy with $L > K$.

$$E_{0KL}=\varepsilon_{0K}+(\hbar^2/2J_{0K})\{L(L+1)-K(K+1)\}. \qquad (1.23)$$

If inequality (1.22) is not satisfied, a division of the energy into internal and rotational energy is impossible. The representation of the energy in the form (1.23) is approximate. If we take further approximations into consideration, we can find a relation between the internal motion of the system of particles and the rotation of the system as a whole.

Thus, in states of a system of particles which exhibit an axis of symmetry coinciding with the ζ axis, the problem of determining the energy levels of the system of particles can be divided into two parts: first the energy levels of the system of particles are determined by solving Eq. (1.20) for a given value of K, and then the motions of the entire system (rotation) are studied for a given state φ_{aK} and different values of the total angular momentum $L > K$ of the system. The rotational angular momentum is $\mathbf{R} = \mathbf{L} - \mathbf{K}$. Since only values of $L \geqq K$ are possible in (1.23), the projection of the angular momentum on the axis of symmetry ζ must always be zero. In other words, a system of particles with axial symmetry in a state of internal motion described by the function φ_{aK} can rotate as a whole only about an axis perpendicular to the axis of symmetry of the system.

2. SYSTEM CONSISTING OF THREE PARTICLES OF DIFFERENT MASS

The example just considered of a system of three particles of equal mass has only methodological interest, since in this case the states with the lowest internal energy do not have a sharply distinguished axial symmetry. In order to deal with a system of three particles which satisfy the above-mentioned conditions for the possibility of distinguishing rotational energy, we study a system consisting of two particles of identical mass ($m_2 = m_3 = m$) and a third particle of considerably smaller mass $m_1 = am$, where $a \sim 10^{-3}$.

If the position of the particles is described by vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, then the transition to the center of mass system is effected by the coordinate transformation

$$\mathbf{r}_3-\mathbf{r}_2=\boldsymbol{\rho}=2\sqrt{\frac{\alpha}{2+\alpha}}\,\mathbf{q},\quad -\mathbf{r}_1+\frac{\mathbf{r}_2+\mathbf{r}_3}{2}=\mathbf{r},$$
$$\frac{\mathbf{r}_3+\mathbf{r}_2+\alpha\mathbf{r}_1}{2+\alpha}=\mathbf{R}. \qquad (2.1)$$

In the c.m.s. the kinetic energy has the form

$$T=-(\hbar^2/2\mu)(\Delta_q+\Delta_r), \qquad (2.2)$$
$$\mu=2\alpha m/(2+\alpha). \qquad (2.3)$$

Thus, all the results of the preceding section can be retained, if by q and μ we understand the quantities defined by (2.1) and (2.3). Since we are considering the case $a \ll 1$, we have approximately

$$\mu=\alpha m,\quad \rho=q\sqrt{2\alpha}. \qquad (2.4)$$

Since both particles lie in the direction of the vector q, this direction will coincide with the direction of the axis of symmetry of the system (for $a \ll 1$).

For the case $K \doteq 0$, Eq. (1.20) has the form

$$\left\{-\frac{\hbar^2}{2\alpha m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{2\alpha}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial}{\partial\rho}\right)+\left(\frac{2\alpha}{\rho^2}+\frac{1}{r^2}\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]+V-\varepsilon_0\right\}\varphi_0(r,\rho,\theta)=0, \qquad (2.5)$$

$$V=V(\rho)+V\left(\sqrt{r^2+(\rho^2/4)+r\rho\cos\theta}\right)+V\left(\sqrt{r^2+(\rho^2/4)-r\rho\cos\theta}\right). \qquad (2.6)$$

in terms of the variables r, ρ, θ.

Since the small coefficient a appears in the derivatives with respect to ρ, the solution of (2.6) can be accomplished in two steps. First we define the lowest energy of the system for fixed values of ρ (the adiabatic approximation), *i.e.*, we solve the equation

$$\left\{-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\left(\frac{2\alpha}{\rho^2}+\frac{1}{r^2}\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right.\right.$$
$$\left.\left.-\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]-V-E_0(\rho)\right\}f_0(r,\theta)=0. \quad (2.7)$$

Solving (2.7) we obtain the energy as a function of the parameter ρ. Then setting

$$\varphi_0(r,\rho,\theta)=f_0(r,\theta)\,u(\rho)/\rho, \quad (2.8)$$

and using (2.7), we obtain from (2.5) an equation which determines the function $u(\rho)$.

$$[-(\hbar^2/m)\,d^2u/d\rho^2+E_0(\rho)-\varepsilon]\,u(\rho)=0. \quad (2.9)$$

If we designate by ρ_0 the value of ρ for which $E_0(\rho)$ has a minimum, then, expanding $E_0(\rho)$ in powers of the difference $(\rho - \rho_0)$ we obtain

$$E_0(\rho)=E_0+\frac{m\omega^2}{4}(\rho-\rho_0)^2,$$
$$\frac{m\omega^2}{4}=\frac{1}{2}\left(\frac{\partial^2E}{\partial\rho^2}\right)_{\rho=\rho_0} \quad (2.10)$$

to within an accuracy of terms of the second order. Equation (2.9) reduces to the equation of a one-dimensional harmonic oscillator. Therefore we can immediately write for the energy and eigenfunctions

$$\varepsilon_\alpha=E_0+\hbar\omega(\alpha+{}^1/_2),$$
$$u_\alpha(\rho)=(m\omega/2\hbar)^{1/4}e^{-x^2/2}H_\alpha(x),$$
$$x=(\rho-\rho_0)(m\omega/2\hbar)^{1/2}.$$

We are interested in the case $a = 0$, where

$$u_0(\rho)=(m\omega/2\hbar)^{1/4}e^{-x^2/2}. \quad (2.11)$$

Setting (2.8) in (1.20), and taking into consideration (2.4), we obtain

$$J_{00}=\left[\frac{2}{m}\int u_0^2(\rho)\,\rho^{-2}\,d\rho\right]^{-1}\approx\frac{m\rho_0^2}{2},$$

i.e., we obtain the usual expression for the moment of inertia of two bodies of mass m, a distance ρ_0 apart, with respect to the axis which is the perpendicular bisector of the line joining them.

3. SYSTEM OF N INTERACTING PARTICLES

We consider a system of N identical particles of equal mass, interacting with central forces. Let $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$ be the radius vectors of these particles. We go into the c.m.s. by introducing Jacobi coordinates

$$\frac{1}{N}(\mathbf{r}_1+\mathbf{r}_2+\ldots+\mathbf{r}_N)=\mathbf{R},$$
$$-\mathbf{r}_1+\frac{1}{N-1}(\mathbf{r}_2+\mathbf{r}_3+\ldots\mathbf{r}_N)=\boldsymbol{\rho}_1, \quad (3.1)$$
$$-\mathbf{r}_2+\frac{1}{N-2}(\mathbf{r}_3+\mathbf{r}_4+\ldots+\mathbf{r}_N)=\boldsymbol{\rho}_2,\ldots,$$
$$-\mathbf{r}_{N-1}+\mathbf{r}_N=\boldsymbol{\rho}_{N-1}.$$

The kinetic energy operator in the c.m.s. is

$$T=-\frac{\hbar^2}{2\mu}\sum_{i=1}^{N-1}\frac{(N+1-i)(N-1)}{N(N-i)}\Delta_{\rho_i}, \quad (3.2)$$
$$\mu=(N-1)\,m/N. \quad (3.3)$$

For convenience we modify the length of the radius vectors in accordance with the following relations

$$\mathbf{q}_i=\boldsymbol{\rho}_i\,[N(N-i)/(N+1-i)(N-1)]^{1/2}, \quad (3.4)$$

Then the kinetic energy operator is

$$T=-\frac{\hbar^2}{2\mu}\sum_{i=1}^{N-1}\Delta_i, \quad \Delta_i=\Delta_{q_i}. \quad (3.5)$$

We go over from the vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_{N-1}$ to polar coordinates $q_1, \vartheta_1, \varphi_1, \ldots, q_{N-1}, \vartheta_{N-1}, \varphi_{N-1}$ with respect to a center of mass coordinate system with a fixed direction of the polar axis. In these coordinates (3.5) has the form

$$T=-\frac{\hbar^2}{2\mu}\sum_{i=1}^{N-1}q_i^{-2}\left\{\frac{\partial}{\partial q_i}\left(q_i^2\frac{\partial}{\partial q_i}\right)+L_i^2\right\}, \quad (3.6)$$

where the operator corresponding to the square of the angular momentum of the i'th particle is defined by the equation

$$\hbar^2L_i^2\equiv\{-i\hbar\,[q_i\nabla_i]\}^2$$
$$=-\hbar^2\left\{\frac{1}{\sin\vartheta_i}\frac{\partial}{\partial\vartheta_i}\left(\sin\vartheta_i\frac{\partial}{\partial\vartheta_i}\right)+\frac{1}{\sin^2\vartheta_i}\frac{\partial^2}{\partial\varphi_i^2}\right\}. \quad (3.7)$$

We introduce a new coordinate system ξ, η, ζ, associated with the system of particles in such a way that the plane $\xi\eta$ coincides with the plane of the vectors $\mathbf{q}_1$, $\mathbf{q}_2$, and the ζ axis lies along the vector $\mathbf{q}_1$. The position of this coordinate system is determined by the polar angles φ_1, ϑ_1 of the vector $\mathbf{q}_1$ and the angle ϕ_2 between the planes $Z\zeta$ and $\zeta\xi$. If φ_1, ϑ_1, ϕ_2 are given, then the position of the vector $\mathbf{q}_2$ is completely specified by the angle θ_2 between the vectors $\mathbf{q}_1$ and $\mathbf{q}_2$. The position of the remaining vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots \mathbf{q}_{N-1}$ in the $\xi\eta\zeta$ system is determined by the angles $\alpha_i = \phi_i - \phi_2$ for $i \geqslant 3$. In Fig. 2 the intersection of this plane with the plane xz is represented on the surface of a unit sphere by the dotted line. The points z, q_1, q_2, q_3 characterize the position of the z axis and of the radius vectors $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ drawn from the center of the sphere. The arcs ϑ_1, ϑ_2, ϑ_3 correspond to the polar angles of these vectors with respect to the z axis.

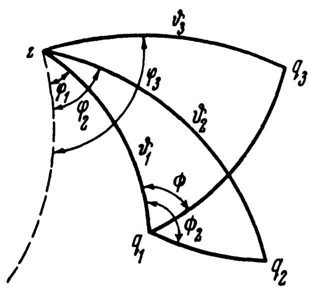

FIG. 2

We go from the angles $\varphi_1, \vartheta_1, \varphi_2, \ldots {}_{N-1}$ to the angles φ, ϑ, ϕ_2, θ_2, ... θ_{N-1} with the help of the relations

$$\vartheta = \vartheta_1,\ \varphi = \varphi_1,\ \cos\theta_j = \sin\vartheta_1 \sin\vartheta_j \cos(\varphi_1 - \varphi_j) + \cos\vartheta_1 \cos\vartheta_j,$$
$$\sin\theta_j \sin\phi_j = \sin\vartheta_j \sin(\varphi_1 - \varphi),$$
$$\sin\theta_j \cos\phi_j = \cos\vartheta_1 \sin\vartheta_j \cos(\varphi_1 - \varphi_j) - \sin\vartheta_1 \cos\vartheta_j.\ j = 2,3,\ldots N-1. \tag{3.8}$$

The operator corresponding to the total angular momentum of the entire system has the form

$$\hbar \mathbf{L} = -i\hbar \sum_{i=1}^{N-1} [q_i\ \nabla_i]. \tag{3.9}$$

In the new variables (3.8) the projections of the total angular momentum (3.9) on the axes x, y, z have the form

$$L_x = -i\left\{\sin\varphi \frac{\partial}{\partial\vartheta} + \cot\vartheta \cos\vartheta \frac{\partial}{\partial\varphi} + \frac{\cos\varphi}{\sin\vartheta} \sum_{j=2}^{N-1} \frac{\partial}{\partial\phi_j}\right\},$$
$$L_y = -i\left\{\cos\varphi \frac{\partial}{\partial\vartheta} - \cot\vartheta \sin\varphi \frac{\partial}{\partial\varphi} - \frac{\sin\varphi}{\sin\vartheta} \sum_{j=2}^{N-1} \frac{\partial}{\partial\phi_j}\right\},\ L_z = -i\frac{\partial}{\partial\varphi}. \tag{3.10}$$

We introduce the new notation $\phi = \phi_2$ and $\alpha_j = \phi_j - \phi_2$, if $j \geqslant 3$, then

$$\frac{\partial}{\partial\phi_2} = \frac{\partial}{\partial\phi} - \sum_{j\geqslant 3}^{N-1} \frac{\partial}{\partial\alpha_j},\quad \frac{\partial}{\partial\phi_j} = \frac{\partial}{\partial\alpha_j},\quad i \geqslant 3,$$

and the components L_x, L_y of the total angular momentum simplify to

$$L_x = -i\left\{\sin\varphi \frac{\partial}{\partial\vartheta} + \cot\vartheta \cos\varphi \frac{\partial}{\partial\varphi} + \frac{\cos\varphi}{\sin\vartheta} \frac{\partial}{\partial\phi}\right\},$$
$$L_y = -i\left\{\cos\varphi \frac{\partial}{\partial\vartheta} - \cot\vartheta \sin\varphi \frac{\partial}{\partial\varphi} - \frac{\sin\varphi}{\sin\vartheta} \frac{\partial}{\partial\phi}\right\}. \tag{3.11}$$

The operator corresponding to the square of the total angular momentum depends only on the external (collective) angles φ, ϑ, ϕ

$$\hbar^2 L^2 = -h^2\left\{\frac{1}{\sin\varphi} \frac{\partial}{\partial\vartheta}\left(\sin\vartheta \frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\left[\frac{\partial}{\partial\varphi^2} + 2\cos\vartheta \frac{\partial^2}{\partial\varphi\partial\phi} + \frac{\partial}{\partial\phi^2}\right]\right\} \tag{3.12}$$

and, of course, has the same form as the total angular momentum operator (1.5) of the system of three particles.

The potential energy of a system of N particles interacting with central forces depends only on the absolute values of the vectors $\mathbf{q}_i$ and the cosines of the angles between them. The cosines of the angles between $\mathbf{q}_1$ and all the other vectors $\mathbf{q}_2$, ... $\mathbf{q}_{N-1}$ are $\cos(\mathbf{q}_1 \mathbf{q}_j) = \cos\theta_j$, $j \geqslant 2$, respectively. The cosines of the angles between the other pairs of vectors are

$$\cos(\widehat{\mathbf{q}_i\mathbf{q}_j}) = \sin\theta_j \sin\theta_j \cos(\phi_j - \phi_i) + \cos\theta_i \cos\theta_j,\ i,j \geqslant 2.$$

Consequently, the potential energy depends on the $3N-6$ internal variables $\{r_j\}\equiv q_1, q_2, \dots q_{N-1}$, $\theta_2 \dots \theta_{N-1}$, $\alpha_3 \dots \alpha_{N-1}$, and does not depend on the collective variables φ, ϑ, ϕ. Therefore the operator corresponding to the square of the total angular momentum of the system (3.12) and its projections on the Z axis commute with the total Hamiltonian of the system.

In the new variables the kinetic energy operator of the system has the form

$$T=-\frac{\hbar^2}{2\mu}\Big\{\sum_{i=1}^{N-1}\frac{1}{q_i^2}\Big[\frac{\partial}{\partial q_i}\Big(q_i^2\frac{\partial}{\partial q_i}\Big)\Big] + \sum_{i=2}^{N-1}\Big[\Big(\frac{1}{q_i^2}+\frac{1}{q_1^2}\Big)R_i+\frac{1}{q_1^2}P_i\Big]\Big\}+\frac{\hbar^2}{2\mu q_1^2}L^2-\tau,$$

$$R_2=-\Big\{\frac{1}{\sin\theta^2}\frac{\partial}{\partial\theta^2}\Big(\sin\theta_2\frac{\partial}{\partial\theta_2}\Big)+\frac{1}{\sin^2\theta_2}\Big(\frac{\partial}{\partial\phi}-\sum_{i=3}^{N-1}\frac{\partial}{\partial\alpha_i}\Big)^2\Big\},$$

$$R_i=-\Big\{\frac{1}{\sin\theta_i}\frac{\partial}{\partial\theta_i}\Big(\sin\theta_i\frac{\partial}{\partial\theta_i}\Big)+\frac{1}{\sin^2\theta_i}\frac{\partial}{\partial\alpha_1^2}\Big\},\quad i\geqslant 3,$$

$$\begin{aligned}P_i=&-2\frac{\partial^2}{\partial\phi^2}+\sum_{\substack{i\\ i+j\geqslant 2}}\sum_{j}\Big\{\cos(\alpha_i-\alpha_j)\frac{\partial}{\partial\theta_i\partial\theta_j}+[\cot\theta_i\cot\theta_j\cos(\alpha_i-\alpha_j)-1]\\ &\times\frac{\partial^2}{\partial\alpha_i\partial\alpha_j}+2\cot\theta_i\sin(\alpha_i-\alpha_j)\frac{\partial^2}{\partial\theta_j\partial\alpha_i}\Big\}+2\sum_{i=3}^{N-1}\cos\alpha_i\frac{\partial^2}{\partial\theta_i\partial\theta_2}\\ &+\cot\theta_i\cot\theta_2\cos\alpha_i\frac{\partial}{\partial\phi}\Big(\frac{\partial}{\partial\phi}-\sum_{j=3}^{N-1}\frac{\partial}{\partial\alpha_j}\Big)\\ &+\cot\theta_2\sin\alpha_i\frac{\partial}{\partial\theta_i}\Big(\frac{\partial}{\partial\phi}-\sum_{j=3}^{N-1}\frac{\partial}{\partial\alpha_j}\Big)\Big].\end{aligned}\tag{3.13}$$

$$\begin{aligned}\tau=&\frac{\hbar^2}{\mu q_1^2}\sum_{i=3}^{N-1}\Big\{\cot\vartheta\Big[\cot\theta_i\cos(\alpha_i+\phi)\frac{\partial}{\partial\alpha_i}+\sin(\alpha_i+\phi)\frac{\partial}{\partial\theta_i}\Big]\frac{\partial}{\partial\phi}\\ &+\frac{1}{\sin\vartheta}\Big[\sin(\alpha_i+\phi)\frac{\partial}{\partial\theta_i}+\cot\theta_i\cos(\alpha_i+\phi)\frac{\partial}{\partial\alpha_i}\Big]\frac{\partial}{\partial\varphi}+\Big[\cot\theta_i\sin(\alpha_i+\phi)\frac{\partial}{\partial\alpha_i}\\ &-\cos(\alpha_i+\phi)\frac{\partial}{\partial\theta_i}\Big]\frac{\partial}{\partial\vartheta}\Big\}+\frac{\hbar^2}{2\mu q_1^2}\Big\{\cot\theta_2\Big[\frac{\cos\phi}{\sin\vartheta}\frac{\partial}{\partial\varphi}\Big(\frac{\partial}{\partial\phi}-\sum_{j=3}^{N-1}\frac{\partial}{\partial\alpha_j}\Big)\\ &+\cot\vartheta\cos\phi\frac{\partial^2}{\partial\phi^2}+\sin\phi\frac{\partial^2}{\partial\phi\partial\vartheta}\Big]+\Big[\frac{\sin\phi}{\sin\vartheta}\frac{\partial}{\partial\varphi}-\cos\phi\frac{\partial}{\partial\vartheta}+\cot\vartheta\sin\phi\frac{\partial}{\partial\phi}\Big]\frac{\partial}{\partial\theta_2}.\end{aligned}$$

The stationary states of the system of N particles are determined by the Schroedinger equation (1.13). The wave function of the state with definite values of the integrals of motion L and M can now be written in the form

$$\psi_M^L=\sqrt{\frac{2L+1}{8\pi^2}}\sum_K D_{MK}^L(\vartheta\,\varphi\,\phi)\,\varphi_K(\{r_i\}).\tag{3.14}$$

Substituting (3.14) in (1.13), we obtain the system of equations

$$\Big\{\frac{\hbar^2}{2\mu q_1^2}[L(L+1)-K(K+1)]+\mathscr{L}_N(K)-E\Big\}\varphi_K=\sum_{K'}(K|\tau|K')\varphi_{K'},\tag{3.15}$$

where

$$(K|\tau|K')=\frac{2L+1}{8\pi^2}\int D_{MK}^{*L}\,\tau\,D_{MK'}^L d\Omega$$

is a matrix with vanishing non-diagonal elements;

$$\mathscr{L}_N(K) = -\hbar^2(2L+1)(16\mu\pi^2)^{-1}\int D^{*L}_{MK}\left\{\sum_{i=1}^{N-1}\frac{1}{q_i^2}\left[\frac{\partial}{\partial q_i}\left(q_i^2\frac{\partial}{\partial q_i}\right)\right]\right.$$
$$\left.+\sum_{i=2}^{N-1}\left[\left(\frac{1}{q_i^2}+\frac{1}{q_1^2}\right)R_i+\frac{1}{q_1^2}P_i\right]\right\}D^L_{MK}d\Omega+\frac{\hbar^2K(K+1)}{2\mu q_i^2}$$

is the Hamiltonian operator of the "internal" motion. For systems such that K is a good quantum number, the right side of (3.15) can also be omitted here, and we obtain a system of independent equations for each value of K. For $L = K$ equation (3.15) reduces to

$$\{\mathscr{L}_N(K) - \varepsilon_{\alpha K}\}\,\varphi_{\alpha K} = 0, \qquad (3.16)$$

which determines the "internal" state of the system N interacting particles. Furthermore, we can carry out the same considerations as in Sec. 1, and obtain the moment of inertia of the system of N interacting particles, corresponding to the internal state of motion φ_{0K}:

$$J_{0K}=\left[\frac{1}{\mu}\int\varphi^*_{\alpha K}\,q_1^{-2}\,\varphi_{\alpha K}\,d\tau\right]^{-1}. \qquad (3.17)$$

The index a characterizes the quantum numbers which, together with K, determine the internal state of the system of N particles. If inequality (1.22) is satisfied, we can represent the energy approximately in the form of a sum of the internal energy and the energy of rotation

$$E_{0KL} = \varepsilon_{0K} + h^2[L(L+1) - K(K+1)]/2J_{0K}\,. \qquad (3.18)$$

The normalized wave function describing the rotation of the system has the form

$$\Phi^L_{MK}(\varphi\vartheta\phi) = \sqrt{\frac{2L+1}{8\pi^2}}D^L_{MK}(\varphi\vartheta\phi). \qquad (3.19)$$

For $K = 0$, according to (1.12), this function reduces to the usual spherical function

$$\Phi^L_{M0}(\varphi\vartheta\phi) = Y_{LM}(\varphi\vartheta).$$

The complete wave function of a system with energy (3.18) is the product of the function (3.19) of the collective degrees of freedom and the wave function (3.19) of the collective degrees of freedom and the wave function $\varphi_{\alpha K}(\{r_i\})$ describing the internal motion in the system

$$\Psi^L_{MK\alpha} = \Phi^L_{MK}(\varphi\vartheta\phi)\,\varphi_{\alpha K}(\{r_i\}). \qquad (3.20)$$

The separation of the total energy of the system into rotational energy and internal energy, and the representation of the wave function in the form of a product of functions of internal and collective coordinates is possible only in systems with marked axial symmetry and where the inequality (1.22) is satisfied, *i.e.*, in the case where the rotation takes place slowly enough so that it does not substantially change the internal structure of the system of N particles. As the frequency of rotation $\omega = \hbar J^2_{0K}\sqrt{L(L+1)-K(K+1)}$ increases, the particles will not be able to follow adiabatically the change of orientation of the mean field. Centrifugal and Coriolis forces will arise in the system, leading to an interaction between the rotational motion and the internal motion of the particles. If the frequencies of rotation are small with respect to the frequency $|\varepsilon_{\alpha K} - \varepsilon_{\alpha+1,K}|/\hbar$, corresponding to transitions to excited states near $K\alpha$, then according to (3.17) each state of internal motion has its own moment of inertia. This conclusion does not agree with the remark by Coester[6] that the moment of inertia does not depend on the internal wave function. In systems which deviate slightly from spherical symmetry, which were studied in the papers of Inglis[4] and Coester[6], the projection of the total angular momentum on the axis of symmetry is not conserved, and the representation of the wave function in the form of the simple product (3.21) is not justified.

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