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Covariant Equation for Two Annihilating Particles

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The functional-derivative technique is used to investigate the annihilation (or production) of two interacting particles which may also exist in a bound state. Covariant equations have been found for the Green function (probability amplitude) which describes the annihilation of an electron and a positron into two quanta as well as for the Green function of the reverse process. The equations thus obtained have been used to solve the problem of interaction between the electron and positron during pair production (or annihilation) with account of radiative corrections.

RELATIVISTICALLY invariant equations for bound states were obtained by various authors[1-5]. Not enough attention, however, was paid to equations that take into account a possible annihilation of particles. In the present work the functional-derivative technique is applied to the solution of the problem concerning the annihilation (or production) of two interacting particles which may also exist in a bound state. While up to now functional equations were derived for the probability amplitudes (Green functions) describing transitions not accompanied by any change in the number of particles, in the present case functional equations have been set up for the probability amplitudes (Green functions) describing the annihilation or production of particles. The resulting equations are, therefore, of a different form. Starting with these equations, it is easy to obtain the wave equation of positronium, the possible annihilation of the electron and positron being taken into the account[6]. Such generalization of the method of functional derivatives to problems involving a change in the number of particles during the studied process enables us to calculate with any desired accuracy the probability of a two-photon (and in general, n-photon) annihilation of particles existing in a bound state. The results of previous works[7-9] dealing with the annihilation of two interacting particles in the S and P states are essentially reproduced if we limit ourselves to the first non-vanishing approximation. The contribution of Coulomb interaction in pair production is also accounted[10]. The proposed method, however, makes it also possible to find the radiative corrections for the above processes (*cf.*, Ref. 11 and 12). The investigation of radiative corrections for the probability of photoproduction and annihilation of positronium confirms the results of Ref. 13 with respect to the infra-red divergence in bound states of the particles.

1. DERIVATION OF THE EQUATION FOR THE GREEN FUNCTION OF TWO PARTICLES ANNIHILATING INTO TWO QUANTA

The Green function $G_2(x_1 x_2, \xi\xi')$ describing the transmutation of two photons into an electron-positron pair (and the two-photon annihilation of the particles as well) is defined, according to Ref. 14, in the following way:

$$G_2(x_1x_2, \xi\xi') \equiv \frac{\delta^2 G(x_1x_2)}{\delta J(\xi)\,\delta J(\xi')}\bigg|_{J=0}$$
$$= i\,\langle \psi(x_1)\,\overline{\psi}(x_2)\,A(\xi)\,A(\xi')\rangle - i\,\langle \psi(x_1)\,\overline{\psi}(x_2)\rangle\,\langle A(\xi)\,A(\xi')\rangle, \qquad (1)$$

where $G(x_1x_2)$ is the Green function of one particle[1], $J(x)$ is the external current, and $\delta/\delta J(x)$ denotes the functional derivative with respect to the current[1]. $\psi(x)$ and $A(\xi)$ are the operators of the free fields of electrons and photons respectively, and the brackets $\langle\ldots\rangle$ should be understood to mean, for example,

$$\langle \psi(x_1)\,\overline{\psi}(x_2)\rangle = [T(\psi(x_1)\,\overline{\psi}(x_2)\,S)]_{\text{vac}}S^{-1}_{\text{vac}}$$
$$= [ST(\psi(x_1)\,\overline{\psi}(x_2))]_{\text{vac}}S^{-1}_{\text{vac}}, \qquad (1')$$

where the subscript "vac" indicates that the corresponding expression is averaged over the state of the vacuum. The index T denotes the T-product of operators standing within the parentheses and, operators in the Heisenberg representation are everywhere in boldface. Furthermore,

$$S = T\left(\exp\left\{-i\int H_{\text{int}}(x)\,d^4x\right\}\right),$$
$$H_{\text{int}}(x) = (-J(x) + j(x))\,A(x), \qquad (1'')$$

$$j_\mu(x) = \frac{e}{2}\,\gamma^\mu_{\alpha\beta}\,(\overline{\psi}_\alpha(x)\,\psi_\beta(x) - \psi_\beta(x)\,\overline{\psi}_\alpha(x)), \qquad (1''')$$

while $\overline{\psi} = \psi^*\gamma^0$, $\gamma^0 = \beta$ and $\gamma^{1,2,3} = \beta\alpha^{1,2,3}$. Besides, the system of units in which $\hbar = c = 1$ is always used and the following summation rule is adopted: $ab = a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3$.

In order that the electron and positron (with coordinates x_1 and x_2 respectively) enter the theory symmetrically, we shall go over in Eq. (1) to the charge-conjugate field with respect to variable x_2, *i.e.*, we shall exchange $\overline{\psi}(x_2)$ by $\psi'(x_2)$, so that the Green function G_{ep} describing the transmutation of two photons into an electron and a positron will be now of the following form:

$$G_{ep}(x_1x_2, \xi\xi') = i\,\langle \psi(x_1)\,\psi'(x_2)\,A(\xi)\,A(\xi')\rangle$$
$$- i\,\langle \psi(x_1)\,\psi'(x_2)\rangle\,\langle A(\xi)\,A(\xi')\rangle, \qquad (2)$$

where $\psi'(x_2)$ is the field operator, charge conjugate with $\psi(x_2)$:

$$\psi'_\sigma(x) = C_{\sigma\rho}\overline{\psi}_\rho(x), \quad \psi_\sigma(x) = C_{\sigma\rho}\overline{\psi}'_\rho(x),$$
$$C^{*T} = C^{-1}, \quad C^T = -C, \quad C\gamma^T = -\gamma C.$$

We shall define the Green function $\overline{G}_{ep}$ describing the reverse process, *i.e.*, the annihilation of an electron and a positron with the emission of two quanta, in the following way:

$$\overline{G}_{ep}(\xi\xi', x_2x_1) = i\,\langle A(\xi)\,A(\xi')\,\overline{\psi}'(x_2)\,\overline{\psi}(x_1)\rangle$$
$$- i\,\langle A(\xi)\,A(\xi')\rangle\,\langle \overline{\psi}'(x_2)\,\overline{\psi}(x_1)\rangle. \qquad (3)$$

It can be easily seen that all the relations obtained for $G_{ep}(x_1x_2, \xi\xi')$ will be fulfilled for the function $\overline{G}_{ep}(\xi\xi', x_2x_1)$ if we make the following substitutions:

$$\gamma \to \gamma^T, \quad C \to C^{-1}, \quad p \to -p \qquad (4)$$

(the last one for the momentum of both the electron and the positron). We shall limit ourselves therefore to the derivation of the equation for G_{ep} only, since from this will follow automatically, taking into account relation (4), the equation for $\overline{G}_{ep}$.

In the following it is convenient to make use of the matrix notation of Karplus and Klein, which consists in the following: the set of all coordinates and the spinor indices of a particle will be denoted by one number, while ξ (or ξ', $\xi''\ldots$) will be denote the set of all coordinates and projections of the photon polarization vector. The matrix index will be represented as the argument of a function, being a number in the case of a particle and ξ (or ξ', $\xi''\ldots$) in the case of a photon. Summation is understood in case of repeating arguments (for the spin indices and the projections of the polarization vectors) and integration in the case of coordinate variables. In this notation, the functions $\gamma(\xi, 12)$ and $C(12)$ have the following meanings:

$$\gamma(\xi, 12) \equiv \gamma^{\nu_\xi}_{\alpha_1\alpha_2}\,\delta(\xi - x_1)\,\delta(x_1 - x_2),$$
$$C(12) \equiv C_{\alpha_1\alpha_2}\delta(x_1 - x_2).$$

To find the equation satisfied by the function $G_{ep}(12, \xi\xi')$ we shall introduce, following Schwinger[1], the auxiliary function $J(\xi)$ of the external sources of the photon field [$J(\xi)$ does not contain field operators], *i.e.*, we shall assume again that the interaction operator H_{int} is of the form $H_{\text{int}} = (-J + j)A$. All the Green functions determined above will then represent functionals of the sources J, depending on them through the operator $S(1'')$. It should be assumed henceforth that $J = 0$ when the Green function is applied to the calculation of real physical processes.

If we make use now of the fact that in the Heisenberg representation the electron and positron operators fulfill the Dirac equation, respectively,*

$$(p(11') - e\gamma(\bar{\xi}, 11')\mathbf{A}(\bar{\xi}) - m\delta(11'))\psi(1') = 0, \tag{5}$$

$$(p(22') + e\gamma(\bar{\xi}, 22')\mathbf{A}(\bar{\xi}) - m\delta(22'))\psi'(2') = 0, \tag{5'}$$

we can write the functional derivative equation for $G_{ep}(12, \xi\xi')$. In fact, applying the operator $p(11') - m\delta(11')$ to the function G_{ep} and taking into account relations (5) and (5′) we obtain

$$(p(11') - m\delta(11'))G_{ep}(1'2, \xi\xi')$$
$$= ie\gamma(\bar{\xi}, 11')\langle A(\bar{\xi})\psi(1')\psi'(2)A(\xi)A(\xi')\rangle \tag{6}$$
$$- ie\gamma(\bar{\xi}, 11')\langle A(\bar{\xi})\psi(1')\psi'(2)\rangle\langle A(\xi)A(\xi')\rangle.$$

Making use of the self-evident equalities

$$\frac{\delta}{\delta J(\bar{\xi})}\langle\psi(1)\psi'(2)\rangle = i\langle A(\bar{\xi})\psi(1)\psi'(2)\rangle - i\langle\psi(1)\psi'(2)\rangle\langle A(\bar{\xi})\rangle,$$

$$\frac{\delta}{\delta J(\bar{\xi})}\langle\psi(1)\psi'(2)A(\xi)A(\xi')\rangle \tag{7}$$
$$= i\langle A(\bar{\xi})\psi(1)\psi'(2)A(\xi)A(\xi')\rangle$$
$$- i\langle\psi(1)\psi'(2)A(\xi)A(\xi')\rangle\langle A(\bar{\xi})\rangle$$

we can write Eq. (6) in the following form:

$$\mathcal{F}^{e}(11')G_{ep}(1'2, \xi\xi')$$
$$= e\gamma(\bar{\xi}, 11')\langle\psi(1')\psi'(2)\rangle\frac{\delta}{\delta J(\bar{\xi})}\langle A(\xi)A(\xi')\rangle, \tag{8}$$

where the following notation has been introduced:

$$\mathcal{F}^{e}(11') \equiv p(11') - e\gamma(\bar{\xi}, 11')\langle A(\bar{\xi})\rangle - m\delta(11')$$
$$+ ie\gamma(\bar{\xi}, 11')\,\delta/\delta J(\bar{\xi}). \tag{9}$$

If we apply from the left the operator $F^{p}(22')$, which differs from

$$\mathcal{F}^{p}(22') \equiv p(22') + e\gamma(\bar{\xi}, 22')\langle A(\bar{\xi})\rangle$$
$$- m\delta(22') - ie\gamma(\bar{\xi}, 22')\,\delta/\delta J(\bar{\xi})$$

by replacing $m\delta(22') + ie\gamma(\bar{\xi}, 22')\,\delta/\delta J(\bar{\xi})$ by the mass operator (*cf.*, Ref. 1) $M^{p}(22')$ and from $F^{e}(11')$ by the sign of the charge, to both sides of equation (8), we shall obtain the following functional equation for G_{ep}:

$$F^{p}(22')\mathcal{F}^{e}(11')G_{ep}(1'2', \xi\xi')$$
$$= - e\gamma(\bar{\xi}, 11')C(1'2)\frac{\delta}{\delta J(\bar{\xi})}D(\xi\xi')$$
$$- ie\gamma(\bar{\xi}, 11')C(1'2)\frac{\delta}{\delta J(\bar{\xi})}\langle A(\xi)\rangle\langle A(\xi')\rangle, \tag{10}$$
$$D(\xi\xi') \equiv \frac{\delta}{\delta J(\xi')}\langle A(\xi)\rangle;$$

where $D(\xi, \xi')$ is the photon Green function[1].

It is convenient to write the first term of the right-hand side of Eq. (10) in another form, making use of the relation

$$- e^{2}\gamma(\bar{\xi}, 11')C(1'2)\frac{\delta}{\delta eJ(\bar{\xi})}D(\xi\xi')$$
$$= ie^{2}\gamma(\bar{\xi}, 13)C(32)D^{0}(\bar{\xi}\bar{\xi}') \tag{11}$$
$$\times C^{-1}(2'3')\gamma(\bar{\xi}', 3'1')G_{ep}(1'2', \xi\xi'),$$

the correctness of which can be easily ascertained expanding both sides of Eq. (11) in a series in e^{2}. D^{0} denotes the zero approximation of the Green function of the photon.

$$\{F^{p}(22')F^{\bar{e}}(11') - I(12, 1'2')\}G_{ep}(1'2', \xi\xi')$$
$$= - ie\gamma(\bar{\xi}, 11')C(1'2)\frac{\delta}{\delta J(\bar{\xi})}\langle A(\xi)\rangle\langle A(\xi')\rangle, \tag{12}$$

where the interaction operator is defined, according to (10) and (11), in the following way:

$$I(12, 1'2')\,G_{ep}(1'2', \xi\xi')$$
$$= ie^{2}\gamma(\bar{\xi}, 13)C(32)D^{0}(\bar{\xi}\bar{\xi}')$$
$$\times C^{-1}(2'3')\gamma(\bar{\xi}', 3'1')G_{ep}(1'2', \xi\xi') \tag{13}$$
$$+ F^{p}(22')\left[m\delta(11') - ie\gamma(\bar{\xi}, 11')\frac{\delta}{\delta J(\bar{\xi})} - M^{e}(11')\right]G_{ep}(1'2', \xi\xi').$$

Eq. (13) may be transformed by means of Eq. (12) into a functional equation for the operator I:

$^{*}p(12) \equiv i\delta(x_1 - x_2)\gamma^{\nu}_{\alpha_1\alpha_2}\partial/\partial x^{\nu}_{2}$.

$$
\begin{aligned}
I(12,1'2')\,G_{ep}(1'2',\xi\xi') &= ie^2\gamma(\bar{\xi},11')\,D(\bar{\xi}\bar{\xi}')\,\Gamma^{p}(\bar{\xi}',22')\,G_{ep}(1'2',\xi\xi') \\
&+ ie^2\gamma(\bar{\xi},13)\,C(32)\,D^0(\bar{\xi}\bar{\xi}')\,C^{-1}(2'3')\,\gamma(\bar{\xi}',3'1')\,G_{ep}(1'2',\xi\xi') - \\
&- e^2\gamma(\bar{\xi},11')\,G^{e}(1'3)\,\gamma(\bar{\xi}',33')\,C(3'2)\,\frac{\delta^2}{\delta J(\bar{\xi})\,\delta J(\bar{\xi}')}\langle A(\xi)\rangle\langle A(\xi')\rangle \\
&- ie^2\gamma(\bar{\xi},11')\,G^{e}(1'3)\,\frac{\delta}{\delta eJ(\bar{\xi})}\left(I(32,3'2')\,G_{ep}(3'2',\xi\xi')\right),
\end{aligned} \tag{14}
$$

where

$$\Gamma^{p}(\xi,22') \equiv \delta F^{p}(22')/\delta e\langle A(\xi)\rangle,$$

$$\Gamma^{e}(\xi,11') \equiv -\delta F^{e}(11')/\delta e\langle A(\xi)\rangle$$

denote vertex operators.

Since the term in Eq. (14) containing the functional derivative of (IG_{ep}) is of secondary importance compared with other terms (this term includes the radiative corrections), Eq. (14) can be solved for the operator I by the method of successive approximations. This makes it possible to find the interaction operator in any approximation of e^2. If the iteration process of solution of Eq. (14) is continued ad infinitum, the operator I will be represented as the sum of an infinite number of terms corresponding to all the irreducible diagrams[2] describing both the electron-positron interaction and their two-photon annihilation. The interaction operator found in this way yields, upon substitution in Eq. (12), when $J = 0$, the covariant equation for the determination of the Green function G_{ep}.

It should be noted that, although the interaction operator I is found in the form of a series in e^2, the fact that we obtain the corresponding approximations of the Green function G_{ep} does not mean that this function has been expanded in powers of the charge. The situation we encounter here is similar to the case of the Bethe-Salpeter equation[2]. In fact, if we retain in the operator only the terms proportional to e^2, then it is equivalent in the S-matrix scheme to accounting, in the infinite sum determining G_{ep}, of an infinite number of reducible diagrams of the ladder type* (Fig. 2) beside the irreducible diagram of Fig. (1).

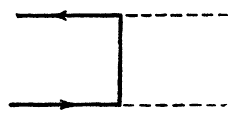

FIG. 1.

In order to obtain finite results (for $J = 0$) it is necessary to carry out a renormalization of the operators $F(11')$, $F(22')$ and $I(12,1'2')$ in Eq. (12) at all degrees of approximation in e^2. As it can be seen from Eq. (14) and the relation $F = G^{-1}$, the renormalization of $F(11')$, $F(22')$ and $I(12,1'2')$ is carried out in the usual way (*cf.*, for example, Ref. 14)

Eq. (12) will take the following form for $J = 0$ and for the operator I corresponding to the first non-vanishing approximation in e^2

$$
\begin{aligned}
&\{F^0(22')\,F^0(11') - ie^2\gamma(\bar{\xi},11')\,D^0(\bar{\xi}\bar{\xi}')\,\gamma(\bar{\xi}',22') \\
&\quad - ie^2\gamma(\bar{\xi},13)\,C(32)\,D^0(\bar{\xi}\bar{\xi}')\,C^{-1}(2'3')\,\gamma(\bar{\xi}',3'1')\}\,G_{ep}(1'2',\xi\xi') \\
&= -e^2\gamma(\bar{\xi},11')\,G^0(1'3)\,\gamma(\bar{\xi}',33')\,C(3'2)\left(D^0(\xi\bar{\xi})\,D^0(\bar{\xi}'\xi') + D^0(\xi\bar{\xi}')\,D^0(\bar{\xi}'\xi)\right),
\end{aligned} \tag{15}
$$

where the index 0 denotes that the functions are taken in the lowest order in e^2. The indices p and e in G and F may be dropped for $J = 0$.

The first term of the operator I in (15) represents the interaction of particles by means of exchange of one virtual quantum, while the second one refers to the interaction of particles due to a single-photon virtual annihilation.

Hereinafter we shall be interested in the interaction operator I, taken with an accuracy of e^4. For this purpose, it is necessary to calculate in Eq. (14) the variational derivative of $(I^{(1)}G_{ep})$ with respect to the current, where $I^{(1)}$ is the first non-

*For brevity we omitted diagrams corresponding to the single-photon virtual annihilation of particles.

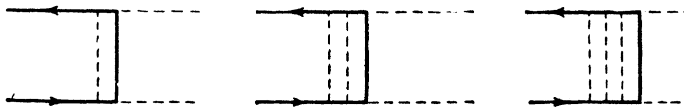

FIG. 2.

vanishing approximation of the operator I. Besides, in all other terms of the right-hand side of Eq. (14) the functions G, D and Γ should be taken in such approximation that the terms up to e^4 inclusively should be taken into account.

For the calculation of the variational derivative it is convenient to represent $(I^{(1)}G_{ep})$, by means of relations (14) and (12), in the following way:

$$\begin{aligned}
&\{(F^p(22')F^e(11') - ie^2\gamma(\bar{\xi}, 11')D^0(\bar{\xi}\bar{\xi}')\gamma(\bar{\xi}', 22') \\
&- ie^2\gamma(\bar{\xi}, 13)C(32)D^0(\bar{\xi}\bar{\xi}')C^{-1}(2'3')\gamma(\bar{\xi}', 3'1'))G^e(1'5)G^p(2'4)\} \\
&\times I^{(1)}(54, 5'4')G_{ep}(5'4', \xi\xi') = \\
&= -e^2\gamma(\bar{\xi}, 11')G^e(1'3)\gamma(\bar{\xi}', 33')C(3'2)\frac{\delta^2}{\delta J(\bar{\xi})\,\delta J(\bar{\xi}')}\langle A(\xi)\rangle\langle A(\xi')\rangle \\
&+ e^3\gamma(\bar{\xi}, 11')D^0(\bar{\xi}\bar{\xi}')\gamma(\bar{\xi}', 22')G^e(1'3)G^p(2'4)\gamma(\bar{\bar{\xi}}, 33')C(3'4) \\
&\times\frac{\delta}{\delta J(\bar{\bar{\xi}})}\langle A(\xi)\rangle\langle A(\xi')\rangle + e^3\gamma(\bar{\xi}, 13)C(32)D^0(\bar{\xi}\bar{\xi}')C^{-1}(2'3') \\
&\times\gamma(\bar{\xi}', 3'1')G^e(1'5)G^p(2'4)\gamma(\bar{\bar{\xi}}, 55')C(5'4)\frac{\delta}{\delta J(\bar{\bar{\xi}})}\langle A(\xi)\rangle\langle A(\xi')\rangle.
\end{aligned} \tag{16}$$

Denoting for the time being the left-hand side of Eq. (16) by $\Lambda(12, 54)I^{(1)}(54, 5'4')G_{ep}(5'4', \xi\xi')$ and the right hand side by $B(12, \xi\xi')$, we shall rewrite Eq. (16) in the form

$$\Lambda(I^{(1)}G_{ep}) = B. \tag{17}$$

From this we obtain the expression for the required derivative

$$\delta(I^{(1)}G_{ep})/\delta J = -\Lambda^{-1}\frac{\delta\Lambda}{\delta J}(I^{(1)}G_{ep}) + \Lambda^{-1}\frac{\delta B}{\delta J}. \tag{18}$$

After a calculation we obtain (for $J = 0$) the following expression for the interaction operator I with accuracy up to e^4:

$$\begin{aligned}
&I(12, 1'2')G_{ep}(1'2', \xi\xi') = I_i(12, 1'2')G_{ep}(1'2', \xi\xi') \\
&- e^2\Gamma(\bar{\xi}, 11')G(1'3)\Gamma(\bar{\xi}', 33')C(3'2)(D(\bar{\xi}\xi)D(\bar{\xi}'\xi') + D(\bar{\xi}\xi')D(\bar{\xi}'\xi)),
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
&I_i(12, 1'2') = ie^2\Gamma(\bar{\xi}, 11')D(\bar{\xi}\bar{\xi}')\Gamma(\bar{\xi}', 22') \\
&+ ie^2\gamma(\bar{\xi}, 13)C(32)D^0(\bar{\xi}\bar{\xi}')C^{-1}(2'3')\gamma(\bar{\xi}', 3'1') \\
&+ (ie^2)^2\gamma(\bar{\bar{\xi}}, 13)G^0(33')\gamma(\bar{\xi}, 3'1')D^0(\bar{\xi}\bar{\xi}')D^0(\bar{\bar{\xi}}\bar{\bar{\xi}}')\gamma(\bar{\xi}', 24)G^0(44')\gamma(\bar{\bar{\xi}}', 4'2') \\
&+ (ie^2)^2\gamma(\bar{\bar{\xi}}, 13)G^0(33')\gamma(\bar{\xi}, 3'5)C(52)D^0(\bar{\xi}\bar{\xi}')D^0(\bar{\bar{\xi}}\bar{\bar{\xi}}') \\
&\times[C^{-1}(1'4)\gamma(\bar{\xi}', 44')G^0(4'6)\gamma(\bar{\bar{\xi}}', 62') \\
&- C^{-1}(2'7')\gamma(\bar{\xi}', 7'7)G^0(75')\gamma(\bar{\bar{\xi}}', 5'1')].
\end{aligned} \tag{19'}$$

As it can be easily seen, the interaction operator I (19) contains two groups of terms. The first group, denoted by I_i, is determined by the effects of exchange of one and of two virtual quanta and the one-and two-photon virtual pair annihilation. This group coincides exactly with the electron-positron interaction operator, found in Ref. 6. The second group of terms in expression (19) is determined by the possibility of the real annihilation of particles.

The required equation for the Green function G_{ep} with the interaction operator I calculated up to e^4 inclusively is of the form:

$$\{F(22')F(11') - I_i(12, 1'2')\}G_{ep}(1'2', \xi\xi')$$
$$= -e^2\Gamma(\xi, 11')G(1'3)\Gamma(\xi', 33')C(3'2)(D(\xi\bar{\xi})$$
$$D(\bar{\xi}'\xi') + D(\bar{\xi}\xi')D(\bar{\xi}'\xi).) \quad (20)$$

where the operators $F(11')$ and $F(22')$ contain also the radiative corrections up to terms proportional to e^4. The latter can be easily found if we note that $F = G^{-1}$ and make use of the corresponding corrections[15] to the single-particle Green function G.

Eq. (20) describes the production (or the annhilation, if transformation (4) is carried out) of free and bound particles. It can be applied as well to the calculation of radiative corrections to the photoproduction and the single-photon annihilation of positronium in an external field. The application of Eq. (20) leads to the following wave equation of the electron and positron with possible annihilation of the particles:

$$\{F(22')F(11') - I_i(12, 1'2')\}\Psi(1'2') = 0, \quad (21)$$

where $\Psi(1'2')$ is the wave function of the electron and positron and the operators F and I_i were defined above. This wave equation for the system electron-positron was obtained by another method in the work of Karplus and Klein[6].

The generalization of the results for the case of the n-photon annihilation of particles is straightforward.

2. ACCOUNT OF THE INTERACTION BETWEEN THE ELECTRON AND POSITRON DURING PAIR PRODUCTION

As an application of equations derived in the preceding paragraph we shall consider the problem of accounting for the electron-positron interaction during pair production (or annihilation), paying attention to radiative corrections.

The interaction between the electron and positron is usually not taken into account in all calculations of pair production. The final state of each particle is considered as free. This is caused by a considerable simplification of calculations involved, since accounting for the interaction between the components of a pair correspond to taking into the account the higher approximations of the scattering matrix.

It was established by Sakharov[10] that for the case of a small relative velocity of produced particles, the account of the Coulomb interaction reduces to the multiplication of the differential cross-section $d\sigma_f$ of the free-particle production by the factor $|\psi(0)|^2/|\psi_f(0)|^2$, where $\psi(x)$ is the non-relativistic wave function of interacting particles in the relative system of coordinates and $\psi_f(x)$ is the wave function of free particles,

$$d\sigma = (|\psi(0)|^2/|\psi_f(0)|^2)\,d\sigma_f \quad (22)$$

where $d\sigma$ is the differential cross-section with the interaction between the produced particles taken into account.

We shall show in which way relation (22) should be generalized for the case of arbitrary relative velocity of produced particles and an arbitrality high degree of approximation.* For the sake of generality we shall deal first with interacting particles which are not necessarily in a bound state, while for simplicity we shall study in detail the case of pair production by two quanta. For the case of photoproduction in an external field it is necessary to add to the matrix element considered below a certain number of matrix elements corresponding to the single, triple, *etc.*, scattering of the produced particle by the external field (higher Born approximations). These matrix elements can be found by means of the Green functions $G_{ep}(12, \xi\xi'\xi'')$, $G_{ep}(12, \xi\xi'\xi''\xi''')$, *etc.*, describing pair production by two quanta with the complementary emission of one, two, *etc.*, quanta. The given proof is afterwards extended for the case of pair production in an external field.

a) *First non-vanishing approximation*

The relation (22) can be easily obtained if we make use of the expression (15) for the Green function describing the photoproduction of interacting particles. In fact, the solution of Eq. (15) is of the form

*The problem whether Eq. (22) remains correct for higher approximations was considered by Sakharov[10]. In that work, however, the meaning of higher approximations remains unclear, since non-covariant perturbation theory is used.

$$G_{ep}(12,\xi\xi')=-e^2K(12,1'2')\gamma(\xi,1'3)G^0(33')\gamma(\xi',3'5)C(52')(D^0(\bar{\xi}\xi)D^0(\bar{\xi}'\xi')+D^0(\bar{\xi}\xi')D^0(\bar{\xi}'\xi)),\tag{23}$$

where $K(12, 1'2')$ is the Green function of the interacting electron and positron. Making use of Eq. (23) we can write the amplitude $\mathcal{A}$ of photoproduction of particles as

$$\begin{aligned}\mathcal{A} &= -e^2\bar{\Psi}(12)\gamma(\xi,11')\\ &\times G^0(1'3)\gamma(\xi',33')C(3'2)\Phi_{kk'}(\xi\xi')\\ &=-2\frac{-e^2}{\sqrt{k_0k_0'}}\int\bar{\Psi}(x_1x_2)(\hat{l}^rG^0(x_1x_2)\hat{l}^{r'}Ce^{i(kx_1+k'x_2)}\\ &+\hat{l}^{r'}G^0(x_1x_2)\hat{l}^rCe^{i(kx_2+k'x_1)})d^4x_1d^4x_2,\end{aligned}\tag{24}$$

where the wave function of produced particles $\Psi(12)$ fulfills Eq. (21), with the operator l_i containing only terms proportional to e^2.

$$\Phi_{kk'}(\xi\xi')=\frac{1}{2}(k_0k_0')^{-1/2}[l^r_{\nu\xi}\,l^{r'}_{\nu\xi'}\exp i(k\xi+k'\xi')+l^r_{\nu\xi'}\,l^{r'}_{\nu\xi}\exp i(k\xi'+k'\xi)]$$

is the symmetrized function of photons with momenta k and k' and polarization l^r and $l^{r'}$, while $\hat{l}=\gamma^\nu l_\nu$.

The calculation of the matrix element (24) with the exact function $\Psi(12)$ of interacting particles is difficult. We shall, therefore, as in the non-relativistic case (22), find the relation between the photoproduction amplitude of interacting particles and the amplitude of photoproduction of particles which are free in the final state, this task being much more simple.

The wave function $\Psi^{E\sigma}(x_1x_2)$ of interacting particles, entering into Eq. (24), is an eigenfunction of the total energy E and also of all other constants of motion σ, forming the given full set of physical values. We shall denote by $\Psi_f^{E\sigma}(x_1x_2)$ the wave function of free particles which is an eigenfunction of the full set E and σ. Since, however, the equations of motion (determining the eigenvalues of E) of the wave functions $\Psi^{E\sigma}$ and $\Psi_f^{E\sigma}$ are different, the following expansion is true:

$$\Psi^{E\sigma}(x_1x_2)=\int C(E,E')\Psi_f^{E'\sigma}(x_1x_2)\,dE'.\tag{25}$$

The coefficient $C(E, E')$ is a δ-like function with a sharp maximum at the point $E'=E$, while interaction disappears, $e^2\to 0$, the coefficient $C(E, E')$ tends to $\delta(E-E')$. This means that the essential domain of integration over E' is, for the coefficient $C(E, E')$, the region close to E. We shall, therefore, bring the smooth (with respect to E') function $\Psi_f^{E'\delta}$ outside the integral sign at the maximum point of the coefficient $(E=E')$, and obtain as an approximation

$$\Psi^{E\sigma}(x_1x_2)=\left(\int C(E,E')\,dE'\right)\Psi_f^{E\sigma}(x_1x_2)\equiv N\,\Psi_f^{E\sigma}(x_1x_2),\tag{26}$$

where the coefficient N is defined by the relation

$$\psi_f^{E\sigma}(0)=N\psi_f^{E\sigma}(0).\tag{27}$$

where $\psi^{E\sigma}(0)$ and $\psi_f^{E\sigma}(0)$ are the same wave functions as in Eq. (26) in relative coordinates, taken at the point $x=0$. For the square of the absolute value of N we have

$$|N|^2=\bar{\psi}_{\alpha_2\alpha_1}(0)\psi_{\alpha_1\alpha_2}(0)/\bar{\psi}_{f\,\beta_2\beta_1}(0)\psi_{f\,\beta_1\beta_2}(0),\tag{28}$$

(identical indices denote summation and the indices E and σ are omitted).

The amplitude (24) of the photoproduction of interacting particles can be now written, making use of (26), as follows:

$$\begin{aligned}\mathcal{A}&=N^*[-e^2\bar{\Psi}_f(12)\gamma(\xi,11')\\ &G^0(1'3)\gamma(\xi',33')C(3'2)\Phi_{kk'}(\xi\xi')].\end{aligned}\tag{29}$$

The coefficient of N^* in (29) represents the amplitude of photoproduction of free particles.

Consequently, if we denote the differential cross-section for the photoproduction of free particles by $d\sigma_f$, the differential cross-section $d\sigma$ for the photoproduction of a pair of interacting particles can be written in the form

$$d\sigma=[(\bar{\psi}(0)\psi(0))/(\bar{\psi}_f(0)\psi_f(0))]\,d\sigma_f,\tag{30}$$

where $(\bar{\psi}(0)\psi(0))=\mathrm{Sp}[\bar{\psi}(0)\psi(0)]$, and the wave functions (in relative coordinates) ψ and ψ_f of interacting and free particles respectively are eigenfunction of the same set of values.

In contrast to Eq. (22), Eq. (30) is true for arbitrary relative velocities of produced particles, including relativistic values. For small relative velocities of the produced particles the coefficient $(\overline{\psi}(0)\,\psi(0))$ is equal to its non-relativistic value and relation (30) coincides with formula (22) given by Sakharov[10].

If pair production takes place in the external field of a nucleus of charge Ze, then, in order that Eq. (30) be applicable, it is necessary that the produced particles move with relativistic velocities (namely $Ze^2/\hbar v_1 \ll 1$, $Ze^2/\hbar v_2 \ll 1$), since in derivation of Eq. (30) the external field is regarded as a perturbation, while the relative velocity of these particles can be arbitrarily small.

We shall prove that Eq. (30) remains in force when we take into the account radiative corrections of any order. $(\overline{\psi}(0)\,\psi(0))$ should then be calculated in Eq. (30) with the same accuracy that is chosen for $d\sigma_f$.

b) *Radiative corrections*

The amplitude of the photoproduction of the interacting electron and positron can be written using Eq. (20) as an approximation with first order radiative corrections:

$$
\begin{aligned}
\mathcal{A} = & -e^2\overline{\Psi}(12)\,\gamma(\xi,11')\,G^0(1',3)\,\gamma(\xi',33')\,C(3'2)\,\Phi_{kk'}(\xi\xi') + \\
& + ie^4\overline{\Psi}(12)\,\gamma(\xi,11')\,G^0(1'3)\,\gamma(\overline{\xi},33')\,G^0(3'5)\,D^0(\overline{\xi}\,\overline{\xi}')\,\gamma(\overline{\xi}',55')\,G^0(5'7)\,\gamma(\xi',77') \\
& \times C(7'2)\,\Phi_{kk'}(\xi\xi') + ie^4\overline{\Psi}(12)\,\gamma(\overline{\xi},11')\,G^0(1'3)\,\gamma(\xi,33')\,G^0(3'5)\,D^0(\overline{\xi}\,\overline{\xi}')\,\gamma(\overline{\xi}',55') \\
& \times G^0(5'7)\,\gamma(\xi',77')\,C(7'2)\,\Phi_{kk'}(\xi\xi') + ie^4\overline{\Psi}(12)\,\gamma(\xi,11')\,G^0(1'3)\,\gamma(\overline{\xi},33') \\
& \times G^0(3'5)\,D^0(\overline{\xi}\,\overline{\xi}')\,\gamma(\xi',55')\,G^0(5'7)\,\gamma(\overline{\xi}',77')\,C(7'2)\,\Phi_{kk'}(\xi\xi').
\end{aligned}
\tag{31}
$$

Calculating in (31) the matrix coefficients proportional to the highest (fourth) power of e we can make use of relation (26), remaining within the limits of the given accuracy. Such an approximation for the wave function $\Psi(12)$ of interacting particles is, however, not satisfactory for the calculation of those matrix elements in Eq. (31) which are proportional to e^2. It is necessary to introduce a correction of the order e^2 into the approximation (26) of the wave function $\Psi(12)$. Owing to the weakness of the electromagnetic binding, this can be done by successive approximations, using for this purpose the wave equation of interacting particles (21) in the integral form

$$
\Psi(12) = \Psi^{(0)}(12) + G(13')\,G(24')\,I_i(3'4',1'2')\,\Psi(1'2'), \tag{32}
$$

where $\Psi^{(0)}$ fulfills the equation for free particles $F(11')F(22')\,\Psi^{(0)}(1'2') = 0$, and the particle interaction operator I_i is assumed to be given [*cf.*, (14) and (19)] in the form of a series in e^2. In fact, the exchange of the wave function of the interacting particles for the wave function of free particles multiplied by N means that we neglect in some way the interaction between the particles.* The proposed method of successive approximations make a correction for this interaction.

We shall obtain the first correction to the zero-order approximation (26) if we exchange the wave function $\Psi(1'2')$ on the right-hand side of Eq. (32) by its zero-order approximation and in the operator I_i we shall leave only the terms proportional to e^2 (denoted below by $I_i^{(1)}$. We have then

$$
\Psi^{(1)}(12) = (N\Psi_f(12)) + G^0(13')\,G^0(24')\,I_i^{(1)}(3'4',1'2')\,(N\Psi_f(1'2')). \tag{33}
$$

For the amplitude (31) we obtain now, using Eq. (26) and (33)

$$
\begin{aligned}
\mathcal{A} = N^*[& -e^2\overline{\Psi}_f(12)\,\gamma(\xi,11')\,G^0(1'3')\,\gamma(\xi',3'5')\,C(5'2)\,\Phi_{kk'}(\xi\xi') \\
& + ie^4\overline{\Psi}_f(12)\,\gamma(\overline{\xi},22')\,D^0(\overline{\xi}\,\overline{\xi}')\,\gamma(\overline{\xi}',11')\,G^0(1'3)\,G^0(2'4)\,\gamma(\xi,33')\,G^0(3'5) \\
& \times\gamma(\xi',55')\,C(5'4)\,\Phi_{kk'}(\xi\xi') + ie^4\overline{\Psi}_f(12)\,\gamma(\xi,11')\,G^0(1'3)\,\gamma(\overline{\xi},33')\,G^0(3'5) \\
& \times D^0(\overline{\xi}\,\overline{\xi}')\,\gamma(\overline{\xi}',55')\,G^0(5',7)\,\gamma(\xi',77')\,C(7'2)\,\Phi_{kk'}(\xi\xi') + ie^4\overline{\Psi}_f(12)\,\gamma(\overline{\xi},11') \\
& \times G^0(1'3)\,\gamma(\xi,33')\,G^0(3'5)\,D^0(\overline{\xi}\,\overline{\xi}')\,\gamma(\overline{\xi}',55')\,G^0(5'7)\,\gamma(\xi',77')\,C(7'2)\,\Phi_{kk'}(\xi\xi') \\
& + ie^4\overline{\Psi}_f(12)\,\gamma(\xi,11')\,G^0(1'3)\,\gamma(\overline{\xi},33')\,G^0(3'5)\,D^0(\overline{\xi}\,\overline{\xi}')\,\gamma(\xi',55')\,G^0(5'7)\,\gamma(\overline{\xi}',77')\,C(7'2)\,\Phi_{kk'}(\xi\xi')].
\end{aligned}
\tag{34}
$$

*It should be noted that the interaction between the particles is already accounted for to a large extent in the coefficient N of the wave function of free particles (26) since the coefficient is proportional to $\psi(0)$ – the exact wave function of interacting particles for $x=0$.

The photoproduction amplitude of interacting particles with the account of the first two radiative corrections differs therefore from the photoproduction amplitude of free particles also by the factor N^* and, consequently, Eq. (30) remains in force and is even more exact since $(\overline{\psi}(0)\,\psi(0))$ is now calculated with the same degree of acurracy as $d\sigma_f$. Correctness of Eq. (30) with radiative corrections of the n-th order in e^2 is proved in an analogous way. It is only necessary to bear in mind that using the method of successive approximations for finding the $(e^2)^n$-th order correction to the corresponding approximation of the wave function, Eq. (32) should be used with the operator I_i written with corresponding accuracy, *i.e.*, with terms up to the order $(e^2)^n$ inclusive only.

It can be easily seen [*cf.*, Eq. (4)] that in the case of annihilation of an electron-positron pair we again obtain Eq. (30).

c) *A Note about the Bound State*

If the produced (or annihilated) particles are in a bound state, the total energy of these particles can be written

$$E = \mathcal{E} - \varepsilon, \tag{35}$$

where $\varepsilon > 0$ is the binding energy $(\varepsilon/m \sim e^4)$. Eq. (26) can then be written

$$\Psi^{E\sigma}(x_1x_2) = \left(\int c(E, E')\, dE'\right) \tag{36}$$

$$\Psi_f^{\mathcal{E}\sigma}(x_1x_2) \equiv N\Psi_f^{\mathcal{E}\sigma}(x_1x_2)$$

where $\Psi_f^{\mathcal{E}\sigma}(x_1x_2)$ is the wave function of free particles with the total energy $\mathcal{E}$ and zero relative velocity. This function enters Eq. (30) in the case of production of particles in a bound state.

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