Theory of Cyclotron Resonance in Metals*

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Investigation of a new type of resonance, which takes place in metals located in a high frequency electromagnetic field and a stationary magnetic field parallel to the surface of the metal, the frequency ω of the alternating field being a multiple of the cyclotron frequency $\Omega = eH/mc$.

The shape of the resonance curve depends considerably on the electron dispersion law and permits one to determine from the experimental data the topology of the Fermi surface and find some of its concrete characteristics.

The surface impedance of the metal has been computed for an arbitrary direction of the stationary magnetic field relative to the surface. The analysis has been performed on basis of the most general assumptions of the electron theory of metals (arbitrary law of dispersion and collision integral). It has been proved that it is possible to introduce the mean free path time of the electrons under the conditions of an anomalous skin effect at arbitrary temperatures.

1. DIAMAGNETIC AND CYCLOTRON RESONANCES

IT IS WELL KNOWN that a free electron travelling through a uniform magnetic field moves in a spiral whose axis is in the direction of the magnetic field. The frequency of rotation of the electron in a plane perpendicular to the magnetic field, $\Omega = eH/mc$, is the same for all electrons and is independent of the magnitude or the direction of their velocity. As a consequence, an external high frequency field of frequency $\omega = \Omega$, impressed on a free electron gas, produces resonance.

In metals and semiconductors this resonance is "smeared out" because of electron collisions with phonons, with lattice imperfections and with the surface. For a substantial resonance to occur, it is necessary in every case, that the electron succeed in making a large number of revolutions over the mean free path $l = vt_0$, *i.e.*

$$t_0 \gg 2\pi/\Omega, \; H \gg 2\pi mc/et_0 \tag{1.1}$$

(note that this condition requires that $r \ll l$). In these equations, v is the velocity, t_0 is the mean free path time and e is the electronic charge; $r = mvc/eH$ is the radius of the orbit of the electron in a magnetic field H. Since near resonance, ω is approximately equal to Ω, it follows that

$$\omega \gg 2\pi/t_0 \tag{1.2}$$

(the case of an arbitrary value of ωt_0 and $\Omega \gg 2\pi(\omega + 1/t_0)$ has been previously examined[2]).

The mean free path time of electrons, t_0, is between 10^{-13} and 10^{-14} sec at room temperature and between 10^{-10} and 10^{-11} sec at liquid helium temperatures. Therefore, as the approximations in Eqs. (1.1) and (1.2) show, the resonance should become observable at helium temperatures in centimeter and millimeter wavelength ranges in magnetic fields with H between 10^3 and 10^4 oersteds.

Consider in particular the motion of electrons in the resonant condition with $\omega \approx \Omega \gg 2\pi/t_0$. The skin depth in this case does not depend on t_0 and is equal* to

$$\delta = (c^2 t_0/2\pi\sigma)^{1/2} = (mc^2/2\pi ne^2)^{1/2},$$

where $\sigma = ne^2 t_0/m$ is the dc conductivity and n is the density of electrons.

The ratio of δ to r, the radius of the orbit of the electron in a magnetic field, is:

$$\delta/r = H/(4\pi n\varepsilon)^{1/2}$$

(ε is the energy of the electron). In semiconductors, ε is approximately equal to kT, and since $H \gg 2\pi mc/et_0$ (Eq. 1.1), it follows that

*A preliminary report on this resonance is contained in Ref. 1.

*It should be recalled that the skin depth for the normal skin effect is given by $\delta = (c^2 |1 + \omega t_0| / 2\pi\omega\sigma)^{1/2}$. For rough estimates this formula is always applicable.

$$\delta / r \gg mc/et_0 \sqrt{nkT} \gg 1$$

Thus, if one assumes $m \sim 10^{-10}$g, $t_0 \sim 10^{-10}$ sec, $T \sim 4$°K and $n \sim 10^{14}$ cm^{-3}, one finds that $\delta/r \gg 300$. Consequently, the electron in its orbit in semiconductors finds itself in a practically uniform electric field.

In metals*

$$\delta/r \sim Hm^{1/2}h^{-1}n^{-5/6} \sim 10^{-6}H \text{ Oe} \tag{1.3}$$

($n \sim 10^{22}$ cm^{-3}) so that in all actually attainable magnetic fields, $\delta \ll r$. This means that only those electrons which move in a very thin layer ($z \sim \delta_{\text{eff}} \ll r \ll l/2\pi$) near the surface of the metal find themselves in a non-ignorable electric field.** For this reason it is exceedingly important whether the fixed magnetic field is parallel to the surface or not. In the latter case the electron passes through the layer δ_{eff} only once, the time of passage through if being δ_{eff}/v, which is small compared to the period of the field, and the resonance obviously does not occur*** (in the zero approximation for δ_{eff}/r).

If the magnetic field is parallel to the surface of the metal ($z = 0$) (or more exactly, if the angle between the surface and the field is given by $\Phi \lesssim \delta_{\text{eff}}/l$, then the principal contribution to the current density is made by those electrons moving near the surface (Fig. 1), but not colliding with it, which enter and re-enter the layer $z \sim \delta_{\text{eff}}$ many times ($l/2\pi r \gg 1$). They do not move into the interior of the metal, since the value of the projection of their velocity on the axis, averaged over the period, is

$$\overline{v}_z = -(c/eH)\,\overline{dp_y/dt} = 0.$$

The motion of these electrons is exactly analogous to the motion of charged particles in a cyclotron

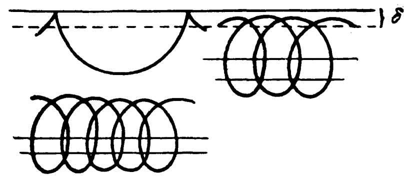

FIG. 1. Schematic form of some possible electron trajectories in a metal.

with only one gap, and at frequencies ω, which are multiples of Ω, resonance can occur.

This resonance in metals differ essentially from resonance in semiconductors. In the latter materials $\delta \gg r$, l, and Ohm's law is obeyed [$\mathbf{j} = \sigma(\mathbf{H})\mathbf{E}$] so that the ordinary skin effect takes place. Resonance evidently appears at a frequency $\omega = \Omega$ and it should of course be called diamagnetic resonance[3].

In metals which are in the resonance condition ($\delta_{\text{eff}} \ll r \ll l/2\pi$), Ohm's law $\mathbf{j} = \sigma\,\mathbf{E}$ is not satisfied, since the electric fields are essentially changed over the mean free path. The resonance arises under conditions of the anomalous skin effect for multiple frequencies with $\omega \approx q\Omega$ ($q = \pm 1, \pm 2, \pm 3, \ldots$). Furthermore in the zero approximation for $\delta_{\text{eff}}/r \ll 1$, resonance will occur only if the magnetic field is parallel to the surface of the metal. Such a resonance, which can be truly called "cyclotron" resonance, has not been previously described in the literature. (In the literature[3], diamagnetic resonance is also called "cyclotron" resonance. Since we have shown that diamagnetic and cyclotron resonances are based on phenomena which are physically distinguishable, the terminology change proposed here would take this distinction into account.)

The purpose of the present work is a calculation of the principal values Z_α of the complete surface impedance tensor of the metal $Z_{\mu\nu} = R_{\mu\nu} + iX_{\mu\nu}$:

$$-\frac{4\pi i\omega}{c^2}E_\mu(0) = \sum_{\nu=1}^{2} Z_{\mu\nu}E'_\nu(0) \qquad (\mu, \nu = x, y),$$

where $E_\mu(0)$ is the value of the component of the electric field at the surface of the metal ($z = 0$).

At resonance R and X pass through their minima (see below). The minima of R correspond to the maximum Q in the resonator and the minima of X occur at the smallest displacement of the resonant frequency.

2. THE ROLE OF THE DISPERSION LAW

Thus far, it was surmised that the "cyclotron" frequency Ω is the same for all electrons. This is true only in those cases, when the dispersion law

*Bismuth may be an exception since in bismuth $n \sim 10^{18}$ cm^{-3}, and the effective masses in certain directions are very small. Under such conditions diamagnetic resonance is possible[3].

**Note that δ is different from δ_{eff}, which is the effective skin depth in the general case

$$\delta_{\text{eff}} = \int_0^\infty j(z)\,dz/j(0).$$

***In this case δ_{eff} is approximately $\sigma\delta_{\text{eff}}/l$, *i.e.*, it assumes its value in the absence of a magnetic field[4]. Consequently, the impedance [$Z = 2\pi\omega\delta_{\text{eff}}(1+i)/c^2$] is independent of the magnetic field.

(*i.e.*, the relation between the energy ε of the quasi-particles, and their quasi-momentum $\mathbf{p}$) is quadratic:

$$\varepsilon(\mathbf{p}) = \sum_{i,k} \mu_{ik} p_i p_k / 2.$$

For the general case of an arbitrary dispersion law it is necessary to consider two possibilities. The trajectory of the quasi-particle in a fixed magnetic field $\mathbf{H}$ is given by the equations:

$$\varepsilon(\mathbf{p}) = \varepsilon, \; p_H = \text{const.} \tag{2.1}$$

(*i.e.*, the energy ε and the projection of the quasi-momentum p_H are integrals of the motion!) If the trajectory is not closed, the motion of the quasi particle in momentum space is infinite, non-periodic and, obviously, resonance is impossible.

If, on the other hand, the curve of Eq. (2.1) is closed, the motion in momentum space will be periodic with a frequency Ω, given by[5]

$$\Omega = (2\pi eH/c)\, \partial S/\partial\varepsilon, \tag{2.2}$$

where $S(\varepsilon, p_H)$ is the area of the cross section, bounded by the curves (2.1) (only this case will be considered in what follows). The quantity $(1/2\pi)\,\partial S/\partial\varepsilon$ plays the role of an effective mass. In the case of a non-quadratic dispersion law, Ω depends on ε and p_H, so that the resonance frequency can coincide exactly with one of the frequencies $\Omega = \Omega(\varepsilon; p_0)$. For electrons with p_H nearly equal to p_0 the frequency $\Omega(p_H)$ is equal to:

$$\Omega(p_H) = \Omega(p_0) + \Omega'(p_0)(p_H - p_0) + {}^1\!/_2\Omega''(p_0)(p_H - p_0)^2 + \ldots$$

It is obvious that the number of electrons finding themselves close to resonance [*i.e.*, having a frequency $\Omega \approx \Omega(p_0)$], will be greatest when $\Omega'(p_0) = 0$, *i.e.*, when p_0 corresponds to an extremum of Ω, and consequently an extremum of $\partial S/\partial\varepsilon$. Therefore one naturally anticipates a resonance at a frequency ω, equal to

$$\Omega_{\text{ext}} = (2\pi eH/c)\,(\partial S/\partial\varepsilon)_{\text{ext}}.$$

Thus, cyclotron resonance is very sensitive to the shape of the boundary Fermi surface $\varepsilon(\mathbf{p}) = \varepsilon_0$. The resonance is absent if the direction of the magnetic field corresponds to an open section of Eq. (2.1). Furthermore, the shape of the resonance is different depending on how closely the Fermi surface resembles an ellipsoid. Therefore in the remainder of this paper we shall not introduce any assumptions concerning the form of the dispersion law.

3. THE COMPLETE SYSTEM OF EQUATIONS

The complete set of equations consists of Maxwell's equations:

$$\text{curl } \mathbf{H} = \frac{4\pi}{c}\mathbf{j}; \quad \text{curl } \mathbf{E} = -\frac{i\omega}{c}\mathbf{H}$$

and the kinetic equations for the perturbation $f_1 \exp(i\omega t)$ to the equilibrium Fermi distribution function

$$f_0(\varepsilon) = \left[\exp\left(\frac{\varepsilon - \varepsilon_0}{kT}\right) + 1\right]^{-1}$$

Eliminating $\mathbf{H}$ from Maxwell's equation, we get

$$E''_{x,y}(z) + (4\pi i\omega/c^2)\, j_{x,y}(z); \; j_z = 0,$$
$$j_i = -\frac{2e}{h^3}\int v_i f_1\, d\tau_{\mathbf{p}}, \; \mathbf{v} = \nabla_{\mathbf{p}}\varepsilon(\mathbf{p}) \; (i = x, y, z); \tag{3.1}$$

where $d\tau_{\mathbf{p}} = dp_x\, dp_y\, dp_z$, and the integration is to be performed over all momentum space. The Oz axis is the direction of the inward normal to the surface of the metal, and the Ox axis is in the direction of the projection of the stationary magnetic field onto the metal surface. The kinetic equations for f_1 will be written with variables ε, p_H and $\tau = \Omega_0 t$. Here $\Omega_0 = eH/m_0c$ (m_0 is the characteristic mass of the electron) and t is the time expended by the electron during one revolution in its orbit[5,6], which for closed orbits is given by $t = (c/eH)\,\partial S_t/\partial\varepsilon$ (where S_t is the area of the region of intersection between $\varepsilon(\mathbf{p}) = \varepsilon$, and $p_H = \text{const.}$) For simplicity, it will be assumed that $\partial S/\partial\varepsilon > 0$, *i.e.*, that the quasi particles under consideration are electrons.

The linearized kinetic equations have the form[5]:

$$v_z\frac{\partial f_1}{\partial z} + \Omega_0\frac{\partial f_1}{\partial\tau} + i\omega f_1 + \left(\frac{\partial f_1}{\partial t}\right)_{\text{CT}} = e\,\mathbf{v}\mathbf{E}\,\frac{\partial f_0}{\partial\varepsilon}, \tag{3.2}$$

where $(\partial f_1/\partial t)_{\text{col}}$ is the integral for collisions of electrons with photons, impurities and lattice imperfections. Here it is taken into account that $\dot{\varepsilon} = e(\mathbf{v}\cdot\mathbf{E})$ and $\dot{p}_k = -eE_k$. The boundary conditions for the z coordinate are determined from the nature of the reflection of the electrons at the surface. We shall assume that the electrons are scattered diffusely (this is almost always so[6,7]), *i.e.*, that the reflected electrons have an equilibrium distribution function*

$$f_1|_{z=0,\, v_z>0} = 0. \tag{3.3}$$

At infinity f_1 is necessarily zero. A boundary condition of f_1 arises from the periodicity of f_1 with respect to τ with a period $\theta = m_0^{-1}\,\partial S/\partial\varepsilon$:

$$f_1(\tau+\theta) = f_1(\tau). \tag{3.4}$$

To solve the problem it is convenient to introduce

$$\Psi(z;\,\mathbf{v}) = f_1(z;\,\mathbf{v}) - f_1(z;\,-\mathbf{v}).$$

In order to obtain an equation for Ψ, we write Eq. (3.2) separately for $f_1(z;\,\mathbf{v})$ and for $f_1(z;\,-\mathbf{v})$. When the symmetry of the collision integral relative substitution of $\mathbf{v}$ by $-\mathbf{v}$ is taken into account, one finds the following:

$$\frac{\partial f_1(z;\,\mathbf{v})}{\partial z} + \hat{L}f_1(z;\,\mathbf{v}) = e\,\mathbf{E}\frac{\mathbf{v}}{v_z}\frac{\partial f_0}{\partial\varepsilon}; \tag{3.5}$$

$$\frac{\partial f_1(z;\,-\mathbf{v})}{\partial z} - \hat{L}f_1(z;\,-\mathbf{v}) = e\,\mathbf{E}\frac{\mathbf{v}}{v_z}\frac{\partial f_0}{\partial\varepsilon}; \tag{3.5a}$$

where

$$\hat{L} = \frac{1}{v_z}\left\{\Omega_0\frac{\partial}{\partial\tau} + i\omega + \left(\frac{\partial}{\partial t}\right)_{\text{CT}}\right\}.$$

If one now performs the operation $(\partial/\partial z - \hat{L})$ on (3.5) and the operation $(\partial/\partial z + \hat{L})$ on (3.5a) and substracts the resulting equations, one obtains:

$$\left(\frac{\partial^2}{\partial z^2} - \hat{L}^2\right)\Psi(z;\,\mathbf{v}) = -2e\,\mathbf{E}\,\hat{L}\left(\frac{\mathbf{v}}{v_z}\frac{\partial f_0}{\partial\varepsilon}\right). \tag{3.6}$$

Equation (3.1) is now written in the form:

$$E''_{x,y}(z) = -\frac{4\pi i\omega}{c^2}\frac{em_0}{h^3}\int v_{x,y}\Psi(z;\,\mathbf{v})\,d\tau\,d\varepsilon\,dp_H,$$
$$\int v_z\Psi(z;\,\mathbf{v})\,d\tau\,d\varepsilon\,dp_H = 0 \tag{3.7}$$

Here the relation $\varepsilon(-\mathbf{p}) = \varepsilon(\mathbf{p})$ has been used. Now it is possible to expand the functions $\mathbf{E}(z)$ and $\Psi(z;\,\mathbf{v})$ over the region $z<0$ [$\mathbf{E}(-z) = \mathbf{E}(z)$; $\Psi(-z;\,\mathbf{v}) = \Psi(z;\,\mathbf{v})$] and to introduce Fourier transformations:

$$k^2\mathcal{E}_\mu(k) + 2E'_\mu(0) = \frac{4\pi i\omega}{c^2}\frac{em_0}{h^3}\int v_\mu\psi(k;\,\mathbf{v})\,d\tau\,d\varepsilon\,dp_H; \tag{3.8}$$

$$\int v_z\psi(k;\,\mathbf{v})\,d\tau\,d\varepsilon\,dp_H = 0; \tag{3.8a}$$

$$k^2\psi(k;\,\mathbf{v}) + 2\Psi'(0;\,\mathbf{v}) + \hat{L}^2\psi(k;\,\mathbf{v}) = 2e\vec{\mathcal{E}}(k)\,\hat{L}\left(\frac{\mathbf{v}}{v_z}\frac{\partial f_0}{\partial\varepsilon}\right). \tag{3.9}$$

It is still necessary to determine the boundary conditions for $\Psi(z;\,\mathbf{v})$ based on Eq. (3.3). Subtracting (3.5) from (3.5a) and substituting $z=0$, we find:

$$\Psi'(0;\,\mathbf{v}) = -\hat{L}\{f_1(0;\,\mathbf{v}) + f_1(0;\,-\mathbf{v})\}. \tag{3.10}$$

But from Eq. (3.3):

$f_1(0;\,\mathbf{v}) = -f_1(0;\,\mathbf{v}) = 0$ for $v_z>0$;

$f_1(0;\,-\mathbf{v}) = -f_1(0;\,-\mathbf{v}) = 0$ for $v_z<0$.

Therefore Eq. (3.10) must written as

$$\Psi'(0;\,\mathbf{v}) = \hat{L}\{\operatorname{sgn} v_z\cdot\Psi(0;\,\mathbf{v})\};\ \operatorname{sgn} x = \begin{cases}+1 & (x>0)\\ -1 & (x<0)\end{cases}. \tag{3.11}$$

Substituting Eq. (3.11) into Eq. (3.9), we get

$$\{k^2+\hat{L}^2\}\psi(k;\,\mathbf{v}) = 2\hat{L}\left\{\frac{g(k;\,\mathbf{v})}{v_z}\right\}, \tag{3.12}$$

where

$$g(k;\,\mathbf{v}) = e\vec{\mathcal{E}}(k)\,\mathbf{v}\,\partial f_0/\partial\varepsilon - |v_z|\,\Psi(0;\,\mathbf{v}).$$

Hence

$$\psi(k;\,\mathbf{v}) = (\hat{L}+ik)^{-1}g/v_z + (\hat{L}-ik)^{-1}g/v_z. \tag{3.13}$$

The condition that ψ is periodic with respect to τ has obviously been conserved. Thus the problem reduces to finding $(\hat{L}\pm ik)^{-1}g/v_z = \zeta$, *i.e.*, to finding a periodic solution to the equation

$$(\Omega_0\partial\zeta/\partial\tau) + i\omega\zeta + (\partial\zeta/\partial t)_{\text{col}} \pm ikv_z\zeta = g \tag{3.14}$$

and solving Eqs. (3.8) and (3.8a).

*Incidentally, the nature of the reflection from the surface does not materially affect the results.

4. THE FEASIBILITY OF INTRODUCING THE RELAXATION TIME INTO THE ANOMALOUS SKIN EFFECT

The formulae of the previous section are true with the same general conditions for the case of the normal, as well as for the anomalous skin effect, in both metals and semiconductors. Let us now examine the highly anomalous skin effect ($\delta_{eff} \ll r, l$). In this case the principal contribution to the current density is made by those electrons which are moving nearly parallel to the surface of the metal ($|v_z| \ll v$), *i.e.*, the electrons near to the zone $v_z = 0$ on the Fermi surface (Fig. 2, *a*). This is evident from the fact that, from Eq. (3.14), $kv_z \sim \Omega_0$, $|v_z|/v \sim \Omega_0/kv$, and, from Eq. (3.8), $k \sim 1/\delta_{eff}$, so that

$$|v_z|/v \sim \Omega_0 \delta_{eff}/v \sim \delta_{eff}/r \ll 1.$$

Accurate calculations lead to the same conclusion.

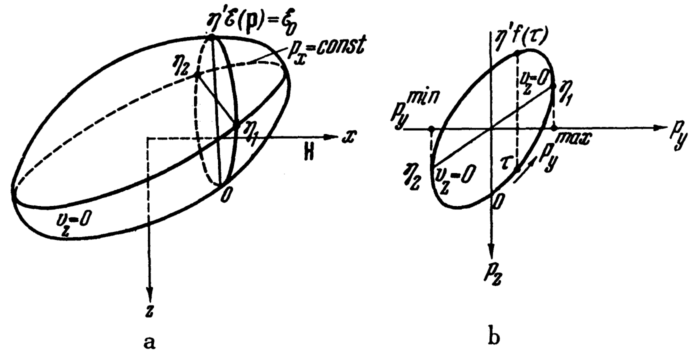

FIG. 2. a – Fermi surface $\varepsilon(\mathbf{p}) = \varepsilon_0$ b – intersection between the Fermi surface and p_x = const.

Therefore, the term $v_z\mathcal{E}_z$ in Eq. (3.12) can be neglected compared with $v_x\mathcal{E}_x + v_y\mathcal{E}_y$ and Eq. (3.8a), which determines $\mathcal{E}_z$, need not be considered further. For this reason the function ζ is large only when $|v_z| \ll v$.

Now let us return to the collision integral which is composed of two terms:

$$(\partial\zeta/\partial t)_{col} = \zeta(\mathbf{p}) \int A(\mathbf{p}; \mathbf{p}')\, d\tau_{\mathbf{p}'} - \int B(\mathbf{p}; \mathbf{p}')\, \zeta(\mathbf{p}')\, d\tau_{\mathbf{p}'},$$

where the magnitudes of A and B are related to the transition probabilities (their actual form depends on temperature[8]).

Because of the fact that $\zeta(\mathbf{p})$ has a sharp maximum for $v_z \sim 0$, in this region the first term will be significantly larger than the second (where $\zeta(\mathbf{p})$ is averaged), and the second term can be ignored in the approximation to the zeroth order of δ_{eff}/r. Consequently near the limit of the anomalous skin effect (when $\delta_{eff} \ll r, l$) we have at any temperature (consistent with this condition):

$$(\partial\zeta/\partial t)_{col} \approx \zeta(\mathbf{p})/t_0(\mathbf{p}); \quad 1/t_0(\mathbf{p}) = \int A(\mathbf{p}; \mathbf{p}')\, d\tau_{\mathbf{p}'}, \tag{4.1}$$

i.e., it is always possible to introduce the mean free path time of the electrons, $t_0(\mathbf{p})$.* The physical reason for this is that the populations of the non-equilibrium states with $|v_z| \ll v$ in the anomalous skin effect are significantly greater than the populations of states with $|v_z| \sim v$. Therefore, through collision, transitions from states with $|v_z| \ll v$ are more probable. Thus, Eq. (3.14) may finally be written in the form:

$$\frac{\partial\zeta}{\partial\tau} + \gamma_0\zeta \pm i\frac{kv_z}{\Omega_0}\zeta = \frac{1}{\Omega_0}g;$$

$$g = e\frac{\partial f_0}{\partial\varepsilon}\sum_{\mu=1}^{2}\mathcal{E}_\mu(k)\, v_\mu - |v_z|\,\Psi(0; \mathbf{v}). \tag{4.2}$$

$$\gamma_0(\mathbf{p}) = i\frac{\omega}{\Omega_0} + \frac{1}{\Omega_0 t_0(\mathbf{p})}.$$

5. SOLUTION OF THE EQUATIONS

Equation (4.2) is readily solved. Its periodic solutions have the form:

$$\zeta(k; \tau) = \frac{1}{\Omega_0}\int_{-\infty}^{\tau} g(\tau_1)\exp\left\{\int_{\tau}^{\tau_1}\left(\gamma_0 \pm i\frac{kv_z}{\Omega_0}\right)d\tau_2\right\}d\tau_1.$$

*This conclusion permits a considerably simpler representation of the results of Ref. 9.

Hence according to Eq. (3.13)

$$\psi(k;\tau) = \frac{2}{\Omega_0}\int_{-\infty}^{\tau} g(\tau_1)\exp\left(\int_{\tau}^{\tau_1}\gamma_0 d\tau_2\right)\cos\left(\frac{k}{\Omega_0}\int_{\tau}^{\tau_1} v_z d\tau_2\right)d\tau_1. \quad (5.1)$$

First let us consider the case of a magnetic field $\mathbf{H}(H, 0, 0)$, parallel to the surface of the metal. In the zeroth approximation relative to the electric field*

$$\frac{d\mathbf{p}}{dt} = -\frac{e}{c}[\mathbf{vH}],\quad v_z = -\frac{1}{m_0}\frac{dp_y}{d\tau}.$$

Consequently,

$$\int_0^{\theta} v_z d\tau = 0$$

Using this relation and the periodicity of $g(\tau)$ with respect to τ, we get from (5.1):

$$\psi(k;\tau) = \frac{2}{\Omega_0\{e^{2\pi\gamma}-1\}}\int_{\tau}^{\tau+\theta}\left\{e\frac{\partial f_0}{\partial\varepsilon}\sum_{\mu=1}^{2}\mathscr{E}_\mu(k)v_\mu - |v_z|\Psi(0;\mathbf{v})\right\}\times\exp\left(\int_{\tau}^{\tau_1}\gamma_0 d\tau_2\right)\cos\left(\frac{k}{\Omega_0}\int_{\tau}^{\tau_1}v_z d\tau_2\right)d\tau_1, \quad (5.2)$$

where

$$\gamma = i\frac{\omega}{\Omega} + \overline{\left(\frac{1}{\Omega t_0}\right)};\quad \Omega = \frac{2\pi eH}{c\partial S/\partial\varepsilon};\quad \overline{\frac{1}{t_0}} = \frac{1}{\theta}\int_0^{\theta}\frac{d\tau}{t_0};\quad \theta = \frac{1}{m_0}\frac{\partial S}{\partial\varepsilon}.$$

Now let us determine

$$\Psi(0;\mathbf{v}) = \frac{1}{\pi}\int_0^{\infty}\psi(k;\mathbf{v})\,dk = \frac{1}{\pi}\lim_{N\to\infty}\int_0^{N}\psi(k;\mathbf{v})\,dk.$$

For this purpose, note that

$$\lim_{M\to\infty}\int_{\tau}^{\tau+\theta}\varphi(\tau_1)\frac{\sin M[p_y(\tau_1)-p_y(\tau)]}{p_y(\tau_1)-p_y(\tau)}d\tau_1 = \pi\left\{\frac{\varphi(\tau)+\varphi(\tau+\theta)}{2|p_y'(\tau)|} + \frac{\varphi(f(\tau))}{|p_y'(f(\tau))|}\right\}, \quad (5.3)$$

where $f(\tau)$ is a point symmetrical about τ (see Fig. 2, b). This point is determined from the relations

$$p_y(\tau) = p_y(f(\tau));\quad \tau < f(\tau) < \tau+\theta. \quad (5.4)$$

Here for simplicity it has been assumed that curve $\varepsilon(\mathbf{p}) = \varepsilon_0$, $p_x = \text{const.}$ is convex and that the point $f(\tau)$ is unique. However, these last formulae are are also satisfied in the general case of non-convex curves, since only the value of $f(\tau)$ in the vicinity of $v_z = 0$ needs to be evaluated. If at the same time on the surface $\varepsilon(\mathbf{p}) = \varepsilon_0$ there are several curves with $v_z = 0$, then the resonance for given values of p_x will occur at the point $v_z = 0$, $p_x = \text{const}$, and $p_y = p_{y\,\max}$. This is connected with the fact that

$$z = \Omega_0\int v_z d\tau = -(\Omega_0/m_0)\,p_y(\tau),$$

and the recurring entrances of the electron into the layer $z \sim \delta_{\text{eff}}$ can occur only when the highest point of the trajectory with $z = z_{\min}$ occurs within this layer.

With the help of Eq. (5.3), it is easy to show from from Eq. (5.2), that

$$\Psi(0;\tau) + \Psi(0;f(\tau))\exp\left(\int_{\tau}^{f(\tau)}\gamma_0 d\tau_2 - 2\pi\gamma\right) = \frac{1}{\Omega_0}e^{-2\pi\gamma}F(\tau),$$

$$F(\tau) = \frac{e}{\pi}\frac{\partial f_0}{\partial\varepsilon}\sum_{\mu=1}^{2}\int_{\tau}^{\tau+\theta}v_\mu(\tau_1)\exp\left(\int_{\tau}^{\tau_1}\gamma_0 d\tau_2\right)d\tau_1\times\int_0^{\infty}\mathscr{E}_\mu(k')\cos\left(\frac{k'}{\Omega_0}\int_{\tau}^{\tau_1}v_z d\tau_2\right)dk'. \quad (5.5)$$

*Remember that this equation defines the variable t.

In Eq. (5.5), let us substitute $f(\tau)$ for τ. Using the definition of $f(\tau)$, it is readily shown that $f(f(\tau)) = \tau + \theta$. Therefore, after the substitution, one has:

$$\Psi(0; f(\tau)) + \Psi(0, \tau)\exp\left(\int_{f(\tau)}^{\tau+\theta}\gamma_0 d\tau_2 - 2\pi\gamma\right) = \frac{1}{\Omega_0}e^{-2\pi\gamma}F(f(\tau)). \tag{5.6}$$

From Eq. (5.5) and (5.6)

$$\Psi(0; \tau) = \frac{e}{\pi\Omega_0}\frac{\partial f_0/\partial\varepsilon}{e^{2\pi\gamma}-1}\sum_{\mu=1}^{2}\int_0^\infty \mathcal{E}_\mu(k')\,dk'\left\{\int_\tau^{\tau+\theta} v_\mu(\tau_1)\exp\left(\int_\tau^{\tau_1}\gamma_0 d\tau_2\right)\right.$$
$$\left.\times\cos\left(\frac{k'}{\Omega_0}\int_\tau^{\tau_1} v_z d\tau_2\right)d\tau_1 - \int_{f(\tau)-\theta}^{f(\tau)}\cdots\right\} = \tag{5.7}$$
$$= \frac{e}{\pi\Omega_0}\frac{\partial f_0}{\partial\varepsilon}\sum_{\mu=1}^{2}\int_0^\infty \mathcal{E}_\mu(k')\,dk'\int_{f(\tau)-\theta}^{\tau} v_\mu(\tau_1)\exp\left(\int_\tau^{\tau_1}\gamma_0 d\tau_2\right)\cos\left(\frac{k'}{\Omega_0}\int_\tau^{\tau_1} v_z d\tau_2\right)d\tau_1.$$

We have also made use of the fact that

$$\int_{f(\tau)}^{\tau_1} v_z d\tau_2 = \int_\tau^{\tau_1} v_z d\tau_2$$

and that in the case of a periodic function $\Phi(\tau)$

$$\{e^{2\pi\gamma}-1\}^{-1}\int_\alpha^{\alpha+\theta}\Phi(\tau_1)\exp\left(\int_\tau^{\tau_1}\gamma_0 d\tau_2\right)d\tau_1 = \int_{-\infty}^{\alpha}\Phi(\tau_1)\exp\left(\int_\tau^{\tau_1}\gamma_0 d\tau_2\right)d\tau_1.$$

Substituting the simple expression of Eq. (5.7) for $\Psi(0, \tau)$ into Eq. (5.2) and calculating the current density as

$$j_\mu(k) = -\frac{em_0}{h^3}\int v_\mu\psi\, d\varepsilon\, dp_x\, d\tau,$$

we find

$$j_\mu(k) = \sum_{\nu=1}^{2}\left\{K_{\mu\nu}(k)\,\mathcal{E}_\nu(k) - \int_0^\infty Q_{\mu\nu}(k; k')\,\mathcal{E}_\nu(k')\,dk'\right\}, \tag{5.8}$$

where

$$K_{\mu\nu}(k) = \frac{2e^2 m_0}{h^3\Omega_0}\int_{\varepsilon=\varepsilon_0}[e^{2\pi\gamma}-1]^{-1}dp_x\int_0^\theta v_\mu(\tau)\,d\tau\int_\tau^{\tau+\theta} v_\nu(\tau_1)\exp\left(\int_\tau^{\tau_1}\gamma_0 d\tau_2\right)\cos\left(\frac{k}{\Omega_0}\int_\tau^{\tau_1} v_z d\tau_2\right)d\tau_1; \tag{5.9}$$

$$Q_{\mu\nu}(k; k') = \frac{2e^2 m_0}{\pi h^3\Omega_0^2}\int_{\varepsilon=\varepsilon_0}[e^{2\pi\gamma}-1]^{-1}dp_x\int_0^\theta v_\mu(\tau)\,d\tau\int_\tau^{\tau+\theta}|v_z(\tau_1)|\cos\left(\frac{k}{\Omega_0}\int_\tau^{\tau_1} v_z d\tau_2\right)d\tau_1\cdot\int_{f(\tau_1)-\theta}^{\tau_1} v_\nu(\xi)$$
$$\times\exp\left(\int_\tau^{\xi}\gamma_0 d\tau_2\right)\cos\left(\frac{k'}{\Omega_0}\int_{\tau_1}^{\xi} v_z d\tau_2\right)d\xi. \tag{5.10}$$

Now we take into account that $kv/\Omega_0 \gg 1$. Using the method of steepest descent and noting that $df/d\tau = -1$*, in the vicinity of points of steepest descent $\eta_1(p_x)$ and $\eta_2(p_x)$, where $v_z(\varepsilon_0; p_x; \eta_\alpha) = 0$,

*In the vicinity of η_α we have $f(\tau) = \begin{cases} 2\eta_\alpha - \tau + \theta & (\text{for } 0\eta,\ \eta'\eta_2), \\ 2\eta_\alpha - \tau & (\text{for } \eta_1\eta',\ \eta_2 0). \end{cases}$

(α = 1, 2) (see Fig. (2, b).) after lengthy calculation we arrive at the expression:

$$j_\mu(k)=\frac{2\pi e^2 m_0}{h^3}\sum_{\nu,\alpha=1}^{2}\left\{\frac{\mathcal{E}_\nu(k)}{k}\int\frac{v_\mu(\eta_\alpha)v_\nu(\eta_\alpha)}{|v_z'(\eta_\alpha)|}\frac{dp_x}{1-e^{-2\pi\gamma}}-\right.$$
$$-\frac{1}{4\pi}\int_0^\infty\frac{\mathcal{E}_\nu(k')\,dk'}{(k+k')\sqrt{kk'}}\int\frac{v_\mu(\eta_\alpha)v_\nu(\eta_\alpha)}{|v_z'(\eta_\alpha)|}\frac{(1+e^{-2\pi\gamma})^2}{1-e^{-2\pi\gamma}}dp_x- \tag{5.11}$$
$$\left.-\frac{1}{2\pi^2}\int_0^\infty\frac{\ln(k/k')}{k^2-k'^2}\mathcal{E}_\nu(k')\,dk'\int\frac{v_\mu(\eta_\alpha)v_\nu(\eta_\alpha)}{|v_z'(\eta_\alpha)|}(3+e^{-2\pi\gamma})\,dp_x\right\}.$$

(Thus, we have verified the assumptions made in Section 4, and cited repeatedly, namely that only values $v_z \approx 0$ plays an important role.) Noting, that

$$J=m_0\sum_{\alpha=1}^{2}\int\left[\frac{v_\mu v_\nu}{|v_z'(\tau)|}\right]_{\tau=\eta_\alpha}\Phi(p_x)\,dp_x$$
$$=\int v_\mu v_\nu\Phi\delta(v_z)\,\delta(\varepsilon-\varepsilon_0)\,d\tau_{\mathbf{p}},$$

and introducing spherical coordinates $\mathbf{v} = v\mathbf{n} = (v\sin\vartheta\cos\varphi,\ v\sin\vartheta\sin\varphi,\ v\cos\vartheta)$ we find

$$J=\int_0^{2\pi}n_\mu(\varphi)\,n_\nu(\varphi)\,\Phi(\varphi)\,d\varphi/K(\varphi). \tag{5.11a}$$

where K is the Gaussian curvature of the Fermi surface. All the quantities are determined at $\vartheta = \pi/2$. Using Eqs. (5.11) and (5.11a), we can write Eq. (3.8) in the form

$$-k^2\mathcal{E}_\mu(k)-2E_\mu'(0)=i\frac{3\pi^2\omega}{c^2}\sum_{\nu=1}^{2}\left\{\frac{A_{\mu\nu}}{k}\mathcal{E}_\nu(k)\right.$$
$$-\frac{2}{\pi^2}C_{\mu\nu}\int_0^\infty\frac{\ln(k/k')}{k^2-k'^2}\mathcal{E}_\nu(k')\,dk' \tag{5.12}$$
$$\left.-\frac{1}{\pi}\left(A_{\mu\nu}-C_{\mu\nu}\right)\int_0^\infty\frac{\mathcal{E}_\nu(k')\,dk'}{(k+k')\sqrt{kk'}}\right\},$$

where*

$$A_{\mu\nu}=\frac{8e^2}{3h^3}\int_0^{2\pi}\frac{n_\mu n_\nu}{K}\frac{d\varphi}{1-\exp\{-2\pi i(\omega/\Omega)-(2\pi/\Omega t_0)\}};$$

$$B_{\mu\nu}=\frac{8e^2}{3h^3}\int_0^{2\pi}\frac{n_\mu n_\nu}{K}\,d\varphi;$$

$$C_{\mu\nu}=B_{\mu\nu}-\frac{2e^2}{3h^3}\int_0^{2\pi}\frac{n_\mu n_\nu}{K}\left[1-\exp\left\{-2\pi i\frac{\omega}{\Omega}-\left(\frac{2\pi}{\Omega t_0}\right)\right\}\right]. \tag{5.13}$$

Eqs. (5.12) and (5.13) allow $\mathcal{E}_\mu(k)$ to be determined in principle:

$$\mathcal{E}_\mu(k)=\sum_{\nu=1}^{2}W_{\mu\nu}(k)\,E_\nu'(0), \tag{5.14}$$

where $W_{\mu\nu}(k)$ are certain known functions.

Hence

$$E_\mu(0)=\sum_{\nu=1}^{2}E_\nu'(0)\frac{1}{\pi}\int_0^\infty W_{\mu\nu}(k)\,dk.$$

But, by definition, the surface impedance tensor, $Z_{\mu\nu}$, is given by:

$$-(4\pi i\omega/c^2)\,E_\mu(0)=\sum_{\nu=1}^{2}Z_{\mu\nu}E_\nu'(0)$$

Therefore,

$$Z_{\mu\nu}=R_{\mu\nu}+iX_{\mu\nu}=-(4i\omega/c^2)\int_0^\infty W_{\mu\nu}(k)\,dk. \tag{5.15}$$

The following section will be devoted to a calculation of $W_{\mu\nu}(k)$.

Note, that in the general case it is not possible to reduce the complex tensor $Z_{\mu\nu}$ to its principal

*Only the mean value of $t_0^{-1}(\varepsilon_0;\ p_x(\varphi))$ appears in the answer. This is not unexpected since the integrals of the motion (ε_0 and p_x) determine the system of quantum-mechanical states, and a degeneracy occurs with respect to τ and leads to an averaging of the collision frequency $\nu(p_x) = 1/t_0$.

axes. This means, that there are no directions, along which an electromagnetic wave is reflected without a rotation of the plane of polarization.

6. CALCULATION OF THE SURFACE IMPEDANCE. ANALYSIS OF THE RESULTING EQUATIONS.

First we will show that the surface impedance has a resonant character. $Z_{\mu\nu}$ can be calculated, along with $A_{\mu\nu}$ and $B_{\mu\nu}$ from Eqs. (5.12) through (5.15). For $\omega = q\Omega$ ($q = \pm 1, \pm 2, \ldots$), the denominator of the expression under the integral sign in the case of $A_{\mu\nu}$ is equal to $\overline{2\pi/\Omega t_0}$, *i.e.*, it is very small. Thus, there can exist two essentially different cases according to their dependence on the dispersion law.

For a quadratic dispersion law, or with

$$\Big| \varepsilon(\mathbf{p}) - \frac{1}{2}\sum_{i,k} \mu_{ik} p_i p_k \Big| / \varepsilon_0 \ll \overline{(2\pi/\Omega t_0)},$$

the cyclotron frequency Ω is independent of φ;

$$\Omega = (eH/c)\,(\mu_{yy}\mu_{zz} - \mu_{yz}^2)^{1/2}$$

and resonance occurs for all electrons at the Fermi surface for $\omega \approx q\Omega$.

For an arbitrary dispersion law $\varepsilon(\mathbf{p}) = \varepsilon$ the magnitude of Ω is a function of φ and resonance occurs only at a frequency $\omega \approx q\Omega_{\text{ext}}$, which corresponds to the extreme value of $\partial S/\partial\varepsilon$ with respect to φ.

Outside the resonance region* one can equate $\overline{2\pi/\Omega t_0}$ to zero in Eq. (5.13) for either dispersion law. (In the case of a non-quadratic dispersion law and $\omega/q = \Omega \neq \Omega_{\text{ext}}$, the integral $A_{\mu\nu}$ should be understood to be a principal value.)

Let us present the results of calculations in the vicinity of resonance in the case of a non-quadratic and of a quadratic dispersion law.

1. In the case of a non-quadratic dispersion law, in the vicinity of resonance ($\omega \approx q\Omega_{\text{ext}}$) the main contribution to $A_{\mu\nu}$ is given by the points $\varphi = \varphi_i$ ($i = 1, 2, \ldots, \beta$) where Ω has an extremum. Therefore,

$$A_{\mu\nu} \approx \frac{16e^2}{3h^3}\Big\{\sum_{i=1}^{\beta}\Big(\frac{n_\mu n_\nu}{\beta K}\Big)_{\varphi=\varphi_i}\int_0^{\pi}\frac{d\varphi}{1-\exp(-2\pi i\omega/\Omega - \overline{2\pi/\Omega t_0})} + d_{\mu\nu}\Big\};$$
$$d_{\mu\nu} = \text{const}, \quad |d_{\mu\nu}| \sim K^{-1}; \quad \Omega^{-1}(\varphi) = \Omega^{-1}(\varphi_i) + \frac{1}{2}\frac{\partial^2\Omega^{-1}(\varphi_i)}{\partial\varphi^2}(\varphi-\varphi_i)^2. \tag{6.2}$$

It is easy to see that in this case, reducing these sums to principle axes and assuming, for simplicity, that $\overline{t_0^{-1}(\varphi_i)} = \overline{t_0^{-1}(\varphi_1)}$ we find that:

$$Z_{\alpha\alpha} = \frac{32}{9\sqrt{3}}\Big(\frac{\sqrt{3}\pi\omega^2}{c^4 A_\alpha^0}\Big)^{1/3}(q^2\varkappa)^{1/6}\exp\Big\{\frac{i}{3}\Big(\pi + s\tan^{-1}\frac{\sqrt{\varkappa + s\varkappa_1}}{\sqrt{\varkappa - s\varkappa_1} + a_\alpha q\varkappa}\Big)\Big\};$$
$$0 < a_\alpha \sim 1;$$
$$|Z_{xy}| \ll |Z_{\alpha\alpha}|; \tag{6.3}$$
$$\varkappa_1 = (\omega - q\Omega_{\text{ext}})/q\Omega_{\text{ext}}; \quad \varkappa = [\varkappa_1^2 + (\omega t_0^*)^{-2}]^{1/2},$$
$$s = \operatorname{sgn}\frac{\partial^3 S(\varphi_1)}{\partial\varepsilon\,\partial\varphi^2}; \quad \frac{1}{t_0^*} = \overline{t_0^{-1}}(\varphi_1);$$

A_α^0 are the principal values of the tensor

$$A_{\mu\nu}^0 = \frac{16e^2}{3h^3}\Big\{\sum_{i=1}^{\beta}\frac{n_\mu(\varphi_i)\,n_\nu(\varphi_i)}{K(\varphi_i)}\Big[\frac{1}{2}\frac{\partial S(\varphi_1)}{\partial\varepsilon}\Big/\Big|\frac{\partial^3 S(\varphi_1)}{\partial\varepsilon\,\partial\varphi^2}\Big|\Big]^{1/2}\Big\}$$
$$\varepsilon = \varepsilon_0, \quad v_z = 0, \quad p_y > 0.$$

(We have not written out the actual form of $d_{\mu\nu}$ and a_α in view of their unwieldiness).

Clearly (in consequence of the fact that $\varepsilon(-\mathbf{p}) = \varepsilon(\mathbf{p})$), $\beta \geqslant 2$ for all non-central sections and for p_x = const. In the case of the "topmost section" and of central sections which do not coincide

*In the case of strong magnetic fields with either diswith the base plane, $\beta = 1$, one of the principal values of A_y^0 is zero, and resonance takes place [and Eq. (6.3) holds] only if the incident wave is polarized along the x-axis. The direction of the electric field also gives the direction of the velocity at the point $\varphi = \varphi_1$; $v_z = 0$, and $\varepsilon = \varepsilon_0$.

The relative depth of the resonance for $R_{\alpha\alpha}$ and $X_{\alpha\alpha}$ are basically different for the cases of minimum and maximum values of $\partial S/\partial\varepsilon$ (*i.e.*, for $s > 0$ and $s < 0$).

The final forms of these equations are as follows:

a) $s > 0$, $\partial S/\partial\varepsilon$ has a minimum at $\varphi = \varphi_1$:

$$R_{\alpha\alpha}^{\text{res}} = \frac{4}{3\sqrt{3}}\left(\frac{\sqrt{3}\pi\omega^2}{c^4A_\alpha^0}\right)^{1/3}\left(\frac{q^2}{\omega t_0^*}\right)^{4/9}\left(\frac{25}{32a_\alpha^2}\right)^{-5/18};\quad \Omega_{\text{res}} = \frac{\omega}{q}\left[1-\left(\frac{25}{32a_\alpha^2}\right)^{1/3}\left(\frac{1}{q\omega t_0^*}\right)^{2/3}\right];$$

$$X_{\alpha\alpha}^{\text{res}} = \frac{32}{9\sqrt{3}}\left(\frac{\sqrt{3}\pi\omega^2}{c^4A_\alpha^0}\right)^{1/3}\left(\frac{q^2}{\omega t_0^*}\right)^{1/6}\sin^{5/6}\frac{2\pi}{5};\quad \Omega_{\text{res}} = \frac{\omega}{q}\left[1+\frac{1}{\omega t_0^*}\tan\frac{\pi}{10}\right]; \tag{6.4}$$

$$\left(\frac{X_{\alpha\alpha}}{R_{\alpha\alpha}}\right)^{\text{res}} = \left(\frac{16\,\omega t_0^*}{q^2a_\alpha^2}\right)^{1/3};\quad \Omega_{\text{res}} = \frac{\omega}{q}\left[1-\left(\frac{\sqrt{2}}{a_\alpha q\omega t_0^*}\right)^{2/3}\right];$$

b) $s < 0$, $\partial S/\partial\varepsilon$ has a maximum at $\varphi = \varphi_1$:

$$R_{\alpha\alpha}^{\text{res}} = X_{\alpha\alpha}^{\text{res}} = \frac{32}{9\sqrt{3}}\left(\frac{\sqrt{3}\pi\omega^2}{c^4A_\alpha^0}\right)^{1/3}\left(\frac{q^2}{\omega t_0^*}\right)^{1/6}\sin^{5/6}\frac{\pi}{5};$$

$$\Omega_{\text{res}}^{R,X} = \frac{\omega}{q}\left[1\mp\frac{1}{\omega t_0^*}\cot\frac{\pi}{5}\right]; \tag{6.5}$$

(the minus sign is associated with R, the plus sign with X). Thus, in all cases R and X have minimum values at resonance. In all equations

$$\Omega_{\text{res}} = 2\pi eH_{\text{res}}/c\,(\partial S/\partial\varepsilon)_{\text{ext}};$$

The relative breadth of the resonance curve is

$$\Delta H/H \sim |\omega - q\Omega_{\text{res}}|/\omega. \tag{6.6}$$

At the same time* $|q| = 1, 2, 3, \ldots \ll (r/\delta)^{2/3}$, $(\omega t_0^*/2\pi)$. Eqs. (6.4) and (6.5) show that resonance occurs at frequencies Ω_{res} which are slightly shifted from the ratio ω/q. The amount of this shift is different for X_α and for R_α and its origin is different in the two cases. The frequency shift of X_α occurs because a small increase in the magnetic field, while it does not change the resonant condition, leads to an "advantageous" increase in the revolutions of the electron between collisions. The frequency shift of R_α is associated with the changing phase of the electric field as a function of distance from the surface. For a maximum acceleration of the electrons in the skin-layer, it is necessary that the thickness δ_1 at which the phase of the field is significantly changed should be large compared to the effective attenuation depth, δ_{eff}. For this reason it is necessary that $X \gg R$. But X/R depends on the frequency and on the magnetic field. Therefore when relatively small changes of the magnetic field can lead to $X \gg R$, such changes are found to be "advantageous", even if at the same time after $|\omega - q\Omega_{\text{res}}|t_0^*$ revolutions the electron finds itself near the surface with its phase changes by π from that of the electric field. This is possible when $\partial S/\partial\varepsilon$ has a minimum and impossible when $\partial S/\partial\varepsilon$ has a maximum. Precisely for this reason the essentially different depth of the resonance of R_α is explained by its dependence on the sign of $\partial^3S(\varphi_1)/\partial\varepsilon\,\partial\varphi^2$.

2. Let us turn to the quadratic dispersion law. For simplicity consider the case where $\overline{1/t_0}$ is independent of φ. The surface impedance tensor can then be diagonalized, and the resultant formula holds for all frequencies and magnetic fields. In this case we get from (5.12)

$$-k^2\mathcal{E}_\alpha(k) - 2E'_\alpha(0) = i\frac{3\pi^2\omega}{c^2}\frac{B_\alpha}{1-e^{-2\pi\gamma}}\left\{\frac{\mathcal{E}_\alpha(k)}{k} - \frac{(1-e^{-2\pi\gamma})(3+e^{-2\pi\gamma})}{2\pi^2}\int_0^\infty\frac{\ln(k/k')}{k^2-k'^2}\mathcal{E}_\alpha(k')\,dk' - \frac{(1+e^{-2\pi\gamma})^2}{4\pi}\int_0^\infty\frac{\mathcal{E}_\alpha(k')\,dk'}{(k+k')\sqrt{kk'}}\right\}, \tag{6.7}$$

persion law, when $2\pi|i\omega + (\overline{1/t_0})| \ll \Omega$, with $(\Omega/\pi\omega^2)\,\overline{1/t_0} \ll 1$ $[(\overline{1/\omega t_0}) \ll 1]$

$R_\alpha(H) \sim H^{-4/3}$, $X_\alpha(H) \sim H^{-1/3}$,

and for $(\Omega/\pi\omega^2)\,(\overline{1/t_0}) \gg 1$ $[(\overline{1/\omega t_0}) \gtrsim 1]$, we have $Z_\alpha(H) \sim H^{-1/3}$.

*$q \ll (r/\delta)^{2/3}$ at $\Omega \gg \overline{1/t_0}$ corresponds to the condition of the anomalous skin-effect $\delta_{\text{eff}} \ll v/\omega$ whenever cyclotron resonance occurs $[(\delta = mc^2/2\pi e^2)^{1/3}]$.

where B_α are the principal values of the two dimensional tensor $B_{\mu\nu}$. Taking it into account that the only singularities of the solution of (6.7) in the region of the complex variable k, except for $k = 0$, are simple poles which are the roots of the equation

$$k^3 + i3\pi^2\omega B_\alpha/c^2(1 - e^{-2\pi\gamma}) = 0,$$

it is easy to show that

$$Z_\alpha(H) = R_\alpha + iX_\alpha = -\frac{4i\omega}{c^2}\int_0^\infty \mathcal{E}_\alpha(R)\,dk/E'_\alpha(0)$$
$$= \frac{8}{9} IZ_\alpha(0)\left[1 - \exp\left(-2\pi i\frac{\omega}{\Omega} - \frac{2\pi}{\Omega t_0^*}\right)\right]^{1/3};$$
$$Z_\alpha(0) = (\sqrt{3}\pi\omega^2/c^4 B_\alpha)^{1/3}(1 + i\sqrt{3}); \quad (6.8)$$

Here $Z_\alpha(0)$ is the value of the impedance when $H = 0$ (Ref. 10);

$$I = \int_0^\infty G(x)\,dx \sim 1;$$

and $G(x)$ is a solution of the integral equation

$$G(x) = \frac{x}{2\pi(x^3+1)}\left\{3\sqrt{3} + \frac{[1+e^{-2\pi\gamma}]^2}{2}\int_0^\infty \frac{G(x')\,dx'}{(x+x')\sqrt{xx'}} + \frac{(3+e^{-2\pi\gamma})(1-e^{-2\pi\gamma})}{\pi}\int_0^\infty \frac{\ln(x/x')}{x^2 - x'^2}G(x')\,dx'\right\}. \quad (6.9)$$

The integral I, which enters into (6.8), changes slowly with the magnetic field and it can be easily evaluated by the method of successive approximations:

$$G(x) = \sum_{n=0}^\infty G_n(x),$$

$$G_0(x) = 3\sqrt{3}x/2\pi(x^3+1).$$

At resonance, the values of R_α, X_α, X_α/R_α and corresponding resonance frequencies are given by:

$$R_\alpha^{\text{res}} = \frac{16}{9\sqrt{3}}\left(\frac{2\pi q}{\omega t_0^*}\right)^{2/3} R_\alpha(0) = \frac{16}{9\sqrt{3}}\left(\frac{\sqrt{3}\pi}{c^4 B_\alpha t_0^{*2}}\right)^{1/3}(2\pi q)^{2/3};$$
$$\Omega_{\text{res}} = \frac{\omega}{q}[1 - (2\pi q\omega t_0^*)^{-1/3}];$$
$$X_\alpha^{\text{res}} = \frac{32}{27}\left(\frac{\pi q}{\omega t_0^*}\right)^{1/3} X_\alpha(0) = \frac{32}{9\sqrt{3}}\left(\frac{\sqrt{3}\pi\omega^2}{c^4 B_\alpha}\right)^{1/3}\left(\frac{\pi q}{\omega t_0^*}\right)^{1/3};$$
$$\Omega_{\text{res}} = \frac{\omega}{q}\left[1 + \frac{1}{\omega t_0^*}\right]; \quad (6.10)$$
$$\left(\frac{X_\alpha}{R_\alpha}\right)^{\text{res}} = \frac{3}{2}\left(\frac{\omega t_0^*}{\pi q}\right)^{1/2}; \quad \Omega_{\text{res}} = \frac{\omega}{q}[1 - (\pi q\omega t_0^*)^{-1/2}]$$
$$1/t_0^* = \overline{1/t_0}.$$

By way of an example, Fig. 3 shows the functions $R(H)/R(0)$, $X(H)/X(0)$, $X(H)/R(H)\sqrt{3}$ plotted for the case of an isotropic quadratic dispersion law ($\varepsilon(p) = p^2/2m$) for ωt_0^* equal to 1, 10 and 50. The graphs were constructed with the help of (6.8).

The small maxima of R and X at $\omega = (q + \frac{1}{2})\Omega$ are not due to resonance, since for $\Omega t_0^* \to \infty$ the magnitude of the impedance at these points approaches a constant value. Deviations from a quadratic dispersion law will reduce the height if the maxima in R and X to an even greater degree.

It can be shown by direct calculation that the equations derived hold not only for electrons ($\partial S/\partial\varepsilon > 0$) but also for "holes". To use the formulae for holes, it is only necessary to change $\partial S/\partial\varepsilon$ in all formulae into $|\partial S/\partial\varepsilon|$.

Similar formulae are found also in the case of several bands. It must be noted, that even if the number of holes is equal to the number of electrons and if the corresponding resonances coincide, the formulae retain their form. (This is because one can neglect the influence of the Hall field which would accompany the distortion of the trajectory of the electrons in the region $z \sim \delta_{\text{eff}} \ll r$).

It is interesting to note that the character of the dispersion law changes the form of the resonance curves in a qualitative way. In particular,

$$R_\alpha^{\text{res}}/R_\alpha(0) \sim (\Omega t_0^*/2\pi)^{-2/3}$$

for a quadratic dispersion law;

$$R_\alpha^{\text{res}}/R_\alpha(0) \sim (\Omega t_0^*/2\pi)^{-4/5}$$

for a non-quadratic dispersion law when $\partial S/\partial\varepsilon$ has a minimum, and

$$R_\alpha^{\text{res}}/R_\alpha(0) \sim (\Omega t_0^*/2\pi)^{-1/6}$$

for a non-quadratic dispersion law when $\partial S/\partial\varepsilon$ has a maximum.

7. DETERMINATION OF SOME OF THE MICROSCOPIC CHARACTERISTICS OF THE ELECTRON GAS IN METALS

By measuring the surface impedance in a magnetic field under the condition of the anomalous skin-effect, one can in principle determine the shape of the boundary Fermi surface $\varepsilon(\mathbf{p}) = \varepsilon_0$, the velocity of the electrons $\mathbf{v} = \mathbf{v}(\mathbf{p})$ on this surface and the mean free path time $t_0(\mathbf{p})$.

It has been shown previously[10] that the principal values of the surface impedance Z_α in the presence of the anomalous skin-effect, but in the absence of a magnetic field, are given by

$$Z_\alpha = R_\alpha + iX_\alpha = (\sqrt{3}\pi\omega^2/c^4B_\alpha)^{1/3}(1 + i\sqrt{3}), \tag{7.1}$$

where B_α are the principal values of the tensor $B_{\mu\nu}$

$$B_{\mu\nu} = \frac{8e^2}{3h^3}\int_0^{2\pi}\frac{n_\mu n_\nu}{K}\,d\varphi. \tag{7.1a}$$

(notation the same as in preceding sections).

Consequently, a measurement of R_α makes it possible to find the mean value of $1/K$ on the equator $v_z = 0$:

$$\int_0^{2\pi}\frac{d\varphi}{K(\varphi)} = (B_x + B_y)\frac{3h^3}{8e^2}. \tag{7.2}$$

In Ref. 11 there was derived an equation by means of which it is possible to determine a function by evaluating its mean value along all equators. Therefore, by measuring the dependence of R_α on the angle between crystallographic axes and the normal to the metal surface, it is possible to compute the Gaussian curvature K for any point on the surface $\varepsilon(\mathbf{p}) = \varepsilon$.* If this surface turns out to be convex,

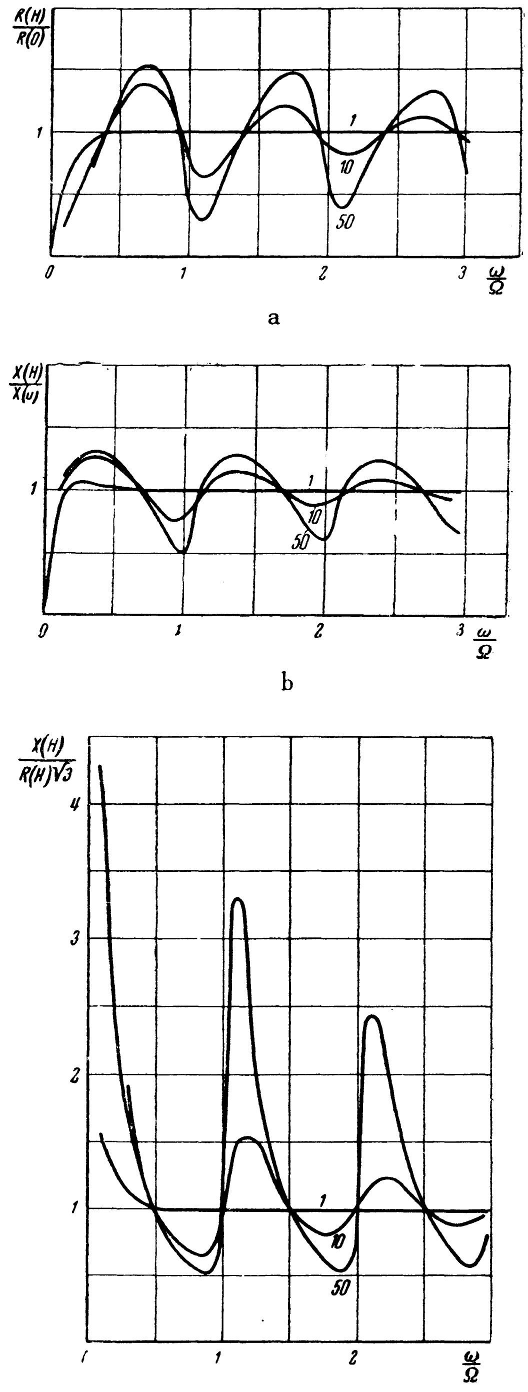

FIG. 3. The numbers alongside the curves are the values of ωt_0; $\omega/\Omega = (m\omega c/e)/H$.

*For this reason, it is necessary to perform experiments, similar to those of Pippard[12].

the computed Gaussian curvature will have its shape determined uniquely. When the equation $\varepsilon(\mathbf{p}) = \varepsilon_0$ defines several surfaces, the problem becomes substantially more complicated.

From Eqs. (6.4), (6.5) and (6.10) it can be seen that, if one knows the resonant frequency for R_α and X_α

$$\Omega_{\text{res}} = 2\pi e H_{\text{res}} / c\,(\partial S / \partial \varepsilon)_{\text{ext}} = (\omega / q)\,(1 + \Delta); \quad |\Delta| \ll 1, \tag{7.3}$$

(where H_{res} and Δ are different for R_α and X_α in addition to which Δ depends only on t_0^*), then one can determine t_0^* and the extreme value $(\partial S/\partial \varepsilon)_{\text{ext}}$

If one knows $(\partial S/\partial \varepsilon)_{\text{ext}}$ and the shape of the surface $\varepsilon(\mathbf{p}) = \varepsilon_0$ it becomes possible, in accordance with Ref. 11, to evaluate the velocity $\mathbf{v} = \mathbf{v}(\mathbf{p})$ on the boundary Fermi surface.

From the quantity,

$$1 / t_0^* = \frac{1}{\theta} \int_0^\theta \frac{d\tau}{t_0(\mathbf{p})}$$

one can find[11]

$$\frac{1}{t_0(\mathbf{p})} = \int A(\mathbf{p};\ \mathbf{p}')\, d\tau \mathbf{p}'$$

[see Eq. (4.1)]. When this integral equation has been solved, one can find in principle $A(\mathbf{p};\ \mathbf{p}')$ and consequently the transition probabilities, $w(\mathbf{p};\ \mathbf{p}')$.

Let us recall that values of $1/t_0(\mathbf{p})$ and $w(\mathbf{p};\ \mathbf{p}')$, determined from similar experiments, are applicable only to the narrow layer $z = \delta_{\text{eff}} \ll r,\ l$ in the vicinity of the surface of the metal. They can, therefore, speaking generally, be distingushed from the values of $1/t_0(\mathbf{p})$ and $w(\mathbf{p};\ \mathbf{p}')$ in a large sample where they can be altered by surface treatment.

8. MAGNETIC FIELDS, INTERSECTING THE METAL SURFACE AT AN ANGLE; INTRODUCTION OF EFFECTIVE VALUES

In a manner similar to that which was followed for a magnetic field parallel to the surface of a metal, calculations can be made for an arbitrary orientation of the magnetic field, when $\sin\Phi \gg (r/l)(\delta/r)^{2/3}$ (here Φ is the angle between the direction of the magnetic field and the surface). In this case the impedance becomes equal to

$$Z_\alpha = (\sqrt{3}\,\pi\omega^2 / c^4 B_\alpha)^{1/3} (1 + i\sqrt{3}). \tag{8.1}$$

The equations which were derived above with the help of the kinetic equation can be understood from elementary considerations of the motion of electrons in stationary magnetic field with a variable electromagnetic field, parallel to the surface of the metal.

Following Pippard[4] and Ginsburg[13] let us assume that for electrons which are moving nearly parallel to the surface (and which comprise the main contribution to the electric field).

$$\mathbf{j} = \sigma_{\text{eff}} \mathbf{E}. \tag{8.2}$$

We shall assume that the field $\mathbf{E}(z;\ t)$ inside the metal is parallel to the stationary magnetic field and directed along the x-axis. Then

$$E(z;\ t) = E_0 \exp(-z/\delta_{\text{eff}}) \cos(\omega t + \chi); \qquad \delta_{\text{eff}} = (c^2/2\pi\omega\sigma_{\text{eff}})^{1/2}. \tag{8.3}$$

Now we calculate the mean energy Δw, which the n electrons acquire along their path length, $l = vt_0$, under the action of the high frequency field;

$$\overline{\Delta w} = \frac{ne^2}{2m} E_0^2 \overline{\left[\int_0^{t_0} \exp(-z(t)/\delta_{\text{eff}}) \cos(\omega t + \chi)\, dt\right]^2},$$

$$z(t) = z_0 + (v/\Omega)(1 - \cos\Omega t); \qquad \Omega = eH/mc, \tag{8.4}$$

where v is the mean velocity of the electron.

The average must be taken over all χ (χ is the phase of the field, which is encountered by the electron at the surface) and over all initial coordinates of the electron, z_0. From $v/\Omega\,\delta_{\text{eff}} = r/\delta_{\text{eff}} \gg 1$ we find:

$$\overline{\Delta w} = \frac{\pi\sigma}{2\Omega} \frac{\delta_{\text{eff}}^2}{l} E_0^2 \frac{\sin^2(N\pi\omega/\Omega)}{\sin^2(\pi\omega/\Omega)} \cos^2(\pi\omega/\Omega). \tag{8.5}$$

Here $N\ (= l/2\pi r \gg 1)$ is the number of revolutions made by the electron between collisions and $\sigma = ne^2 t_0/m$ is the dc conductivity.

From Eq. (8.5), it is seen that with $\omega = q\Omega$, $\overline{\Delta w}$ attains a maximum, *i.e.*, resonance takes place. During resonance,

$$\overline{\Delta w} = (\pi\sigma / 2\Omega)\,(\delta_{\text{eff}}^2 / l)\, E_0^2 N^2.$$

On the other hand, the quantity of heat, Q, which is released in the metal during the time $t_0 \gg 1/\omega$ is equal to

$$Q = \int_0^{t_0} dt \int_0^{\infty} \sigma_{\text{eff}} E^2 dz = \frac{\pi\sigma_{\text{eff}}}{2\Omega} N\delta_{\text{eff}} E_0^2, \qquad (8.6)$$

where σ_{eff} is the effective conductivity of the metal.

Equating Q and $\overline{\Delta w}$ we find that

$$\sigma_{\text{eff}} = \sigma\,\delta_{\text{eff}}/2\pi r. \qquad (8.7)$$

But, from Maxwell's equations $\delta_{\text{eff}} = (c^2/2\pi\omega\sigma_{\text{eff}})^{1/2}$, so that

$$\sigma_{\text{eff}} = \frac{1}{2\pi}\left(\frac{c^2\sigma^2}{r^2\omega}\right)^{1/3}, \qquad \delta_{\text{eff}} = \left(\frac{rc^2}{\omega\sigma}\right)^{1/3}. \qquad (8.8)$$

Hence, during resonance

$$R(H) = \left(\frac{2\pi\omega}{c^2\sigma_{\text{eff}}}\right)^{1/2} = \left(\frac{4\pi}{\sqrt{3}}\right)^{1/3}\left(\frac{\sqrt{3}\,\pi\omega^2 l}{c^4\sigma}\right)^{1/3}\left(\frac{2\pi q}{\omega t_0}\right)^{1/3}, \qquad (8.9)$$

where only the numerical factor differs from Eq. (6.8) for $\omega = q\Omega$. Such a difference should have been anticipated, because the complex character of δ_{eff} had not been taken into account. A similar method cannot be determined with Eq. (6.10), since it does not permit one to take into account the variation of the phase of the field with depth of penetration. Besides this, it can be applied only to the case of a quadratic dispersion law. A precise calculation shows that in reality the effective depth for the attenuation of the field, which is given by

$$\delta_{\text{eff}}^{(E)} = \int_0^{\infty} E(z)\,dz\,/\,E(0) \sim \delta\,(\delta/r)^{1/3} \ll \delta_{\text{eff}}$$

is considerably smaller than the effective depth for attenuation of the current, given by

$$\delta_{\text{eff}}^{(j)} = \int_0^{\infty} j(z)\,dz\,/\,j(0) \sim \delta\,(r/\delta)^{1/3} \sim \delta_{\text{eff}}.$$

(similar relations are valid even in the absence of the magnetic field; it is only necessary to replace r by l).

9. CONCLUSIONS

We have shown that in metals, at high frequencies and low temperatures, a new kind of resonance, namely cyclotron resonance, should take place. This resonance has not as yet been observed experimentally. It is readily distinguishable from other resonances, since 1) it occurs at a number of frequencies, rather than at a single frequency as is the case for diamagnetic or paramagnetic resonance (see Fig. 3); 2) it is possible only in stationary magnetic fields, which must be very nearly parallel to the surface of the metal, and 3) it persists through a reversal of the magnetic field.

An experimental investigation of cyclotron resonance is beset with a series of difficulties, such as the following:

(1) The frequency ω, is given by

$$2\pi/t_0 \ll \omega \ll (v/c)\,(2\pi ne^2/m)^{1/2} \qquad (9.1)$$

(where t_0 is the characteristic relaxation time of the electrons), corresponds to centimeter and millimeter waves* for pure metals at very low temperatures;

(2) The stationary magnetic field must be

$$H \sim mc\omega/e \gg 2\pi mc/et_0, \qquad (9.2)$$

i.e., fields of thousands of oersteds** are needed. (For the case of almost empty bands with small electron effective masses, the required value of H drops to tens or hundreds of oersteds).

(3) The stationary magnetic field must be very nearly parallel to the surface. The angle Φ between the field and the surface has to be such that

$$\Phi \lesssim \delta_{\text{eff}}/l \sim (r/l)\,(\delta/r)^{2/3}, \quad \delta = (mc^2/2\pi e^2 n)^{1/2} \qquad (9.3)$$

(here δ_{eff} is the effective depth of the skin layer), *i.e.*, if $l \sim 10^{-2}$ cm, $\delta \sim 10^{-5}$ cm, the angle Φ may not exceed several tens of minutes. This also applies to the angular dimensions of surface inhomogeneities. Otherwise, the free-path-length of the electrons, which contribute to the resonance, is determined by collisions with surface irregularities. Therefore, the finish of the surface becomes especially significant.

By using the anisotropy of the surface impedance in a magnetic field, one can in principle determine the shape of the boundary Fermi surface, the ve-

*The inequality on the right occurs because at very high frequencies (in the infrared region) and at corresponding magnetic fields the normal skin effect is encountered again, since r turns out to be much smaller than δ_{eff}. In this case cyclotron resonance does not occur.

**It would be very interesting to investigate cyclotron resonance at very intense, pulsed magnetic fields with a pulse duration considerably greater than $t_0 \sim 10^{-11}$ sec. Of course in this case, the inequality on the right side of (9.1) should be satisfied.

locity of the electrons on it and the transition probability for passing from one state to the other. (It must be understood that this is valid only for closed cross sections of the Fermi surface. For open cross sections, an investigation of the anisotropy permits one to establish only that such cross sections are present and how they are oriented, which one learns from the absence of resonance in corresponding directions.)

An interpretation of the experimental curves in the case of partially filled bands is simpler than in the de-Haas, van-Alphen effect, because of the resonant nature of the curves.

By examining the dependence of the resonance minimum in R on the magnetic field, one can determine how much the dispersion law for the electrons differs from a quadratic law; *i.e.*, to what extent the electron bands are filled.

It should be emphasized that electrons in the fundamental band (and not just those in very slightly filled bands) participate in cyclotron resonance and that these electrons are the ones which make the main contribution to the electrical conductivity of the metal.

Quantum effects will lead to small oscillations superimposed on the fundamental periodic curves. These oscillations are completely non-essential to the phenomenon discussed here.

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Note added in proof: Recently E. Fawcett[17] reported the experimental observation of cyclotron resonance in tin and of a decrease in the surface resistivity of tin and copper in a high intensity magnetic field, parallel to the surface of the metal. As was shown theoretically (see footnote in Sec. 6), the minuteness of the decrease in the surface resistance in the high intensity magnetic field is associated with the fact, that it is clearly not enough for the magnetic field be parallel to the surface within 1°, as was the case in Fawcett's experiments. As we have shown Φ should satisfy the conditions $\Phi < (r/l)\,(\delta/r)^{2/3} \ll 1°$. The smoother variation of $R(H)$ may also be attributed to an insufficiently smooth surface. A detailed consideration of the experimental results will be presented in a separate article.

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