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Ionization Losses of High-Energy Heavy Particles

P. V. VAVILOV

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Analysis and rigorous solution of the problem of ionization losses of heavy particles in "thin absorbers," *i.e.*, absorbers in which the ionization losses are much smaller than the initial energy of the particles

IN PASSING THROUGH MATTER a charged particle loses its energy by collision with atomic electrons. Individual collisions are independent events, so that the energy losses may vary. The kinetic equation for the distribution function is of the form[1]

$$\frac{\partial f(x, \Delta)}{\partial x} = \int_0^b w(\varepsilon) f(x, \Delta-\varepsilon)\, d\varepsilon - f(x, \Delta) \int_0^{\varepsilon_{max}} w(\varepsilon)\, d\varepsilon, \tag{1}$$

and it is assumed that the total loss $\Delta = E_0 - E$ in a path x is small compared with the initial energy E_0, so that the probability $w(E, \varepsilon)$ of energy loss per unit length may be considered independent of the final energy E. Further, it is assumed that $w(\varepsilon) = 0$ for $\varepsilon > \varepsilon_{max}$, where ε_{max} is the maximum energy transferred during a single collision, $b = \Delta$ for $\Delta < \varepsilon_{max}$, and $b = \varepsilon_{max}$ for $\Delta > \varepsilon_{max}$. (We are using Landau's notation.[1])

We shall solve Eq. (1) by Laplace transforms:

$$f(x, \Delta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{p\Delta} \frac{\varphi(x, p)}{p}\, dp,$$

$$\varphi(x, p) = p \int_0^\infty e^{-p\Delta} f(x, \Delta)\, d\Delta,$$

$$w(\varepsilon) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{p\varepsilon} \frac{w(p)}{p}\, dp, \tag{2}$$

$$w(p) = p \int_0^\infty e^{-p\varepsilon} w(\varepsilon)\, d\varepsilon.$$

When $b = \Delta$, we use the well known multiplication theorem for Laplace transforms; inserting (2) into (1), we obtain

$$\frac{\partial \varphi(x, p)}{\partial x} = \varphi(x, p) \left(\frac{w(p)}{p} - \int_0^{\varepsilon_{max}} w(\varepsilon)\, d\varepsilon \right),$$

whence

$$\varphi(x, p) = \varphi(0, p) \exp\left[-x \left(\int_0^{\varepsilon_{max}} w(\varepsilon)\, d\varepsilon - \frac{w(p)}{p} \right) \right], \tag{3}$$

But

$$\frac{w(p)}{p} = \int_0^\infty e^{-\varepsilon p} w(\varepsilon)\, d\varepsilon = \int_0^{\varepsilon_{max}} e^{-\varepsilon p} w(\varepsilon)\, d\varepsilon.$$

further, when $x = 0$ we have $f(0, \Delta) = \delta(\Delta)$, so that

$$\varphi(0, p) = p \int_0^\infty e^{-p\Delta} \delta(\Delta)\, d\Delta = p.$$

Inserting this expression into (3), we arrive at

$$f(x, \Delta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp\left\{ p\Delta - x \int_0^{\varepsilon_{max}} w(\varepsilon)(1 - e^{-\varepsilon p})\, d\varepsilon \right\} dp. \tag{4}$$

The case $b = \varepsilon_{max}$ leads again to the same equation, so that Eq. (4) is the exact solution for arbitrary Δ. The solution obtained differs from Landau's[1] only in that Landau has $\varepsilon_{max} = \infty$.

Let us write the exponent in Eq. (4) in the form

$$I = p(\Delta - \alpha x) - x \int_0^{\varepsilon_{max}} w(\varepsilon)(1 - e^{-\varepsilon p} - \varepsilon p)\, d\varepsilon. \tag{4'}$$

Now distant collisions do not play as important a role in the integral with respect to $d\varepsilon$ as they do in α, and therefore for heavy particles (which are all we are considering in this article) we may use the expression

$$x w(\varepsilon) = \xi \varepsilon^{-2} (1 - \beta^2 \varepsilon / \varepsilon_{max}),$$
$$\varepsilon_{max} = 2 m_e c^2 \beta^2 / (1 - \beta^2), \tag{5}$$
$$\xi = 0{,}300\, x (m_e c^2 / \beta^2) Z / A,$$

where x is given in gm/cm^2, Z is the atomic number of the substance, A is the atomic weight, m_e is the electron mass, βc is the particle velocity, and the initial energy of the particle $E_0 \ll Mc^2(M/m_e)$, where M is the mass of the particle. Inserting (5) into (4'), simple operations lead to

$$I = p(\Delta - \alpha x) - p\xi(1+\beta^2) + \varkappa(1 - e^{-\varepsilon_{\max}p})$$
$$+ (\varkappa\beta^2 + \xi p)\int_0^{\varepsilon_{\max}} \frac{1-e^{-\varepsilon p}}{\varepsilon}\, d\varepsilon,\ \varkappa = \frac{\xi}{\varepsilon_{\max}}.$$

It is easily shown [2] that

$$\int_0^{\varepsilon_{\max}} \frac{1-e^{-\varepsilon p}}{\varepsilon}\, d\varepsilon = C + \ln(\varepsilon_{\max}p) - \mathrm{Ei}(-\varepsilon_{\max}p), \tag{6}$$

where $C = 0.577\ldots$ is Euler's constant, and Ei is the exponential integral function. If we use the expression

$$x\alpha = \xi\left[\ln\frac{2m_e c^2\beta^2\varepsilon_{\max}}{(1-\beta^2)I^2(Z)} - 2\beta^2\right] \tag{7}$$

for the mean energy loss, we obtain (with the replacement $p\varepsilon_{\max} = z$)

$$f(x, \Delta) = \frac{1}{2\pi i \varepsilon_{\max}} e^{\varkappa(1+\beta^2 C)} \int_{c-i\infty}^{c+i\infty} \exp\{z\lambda_1$$
$$+ \varkappa[(z+\beta^2)(\ln z - \mathrm{Ei}(-z)) - e^{-z}]\}\, dz, \tag{8}$$
$$\lambda_1 = (\Delta - \alpha x)/\varepsilon_{\max} - \varkappa(1 + \beta^2 - C).$$

If we set $z\kappa = p$, then $\lambda_1 \to \lambda$, where λ is Landau's parameter. It is easily seen that when $\kappa = 0$ we obtain Landau's solution (with the replacement $z\kappa = p$)

$$f(x, \Delta) = \frac{1}{2\pi i\xi}\int_{c-i\infty}^{c+i\infty} e^{p\lambda + p\ln p}\, dp. \tag{9}$$

Let us consider the case $\kappa \gtrsim 1$. Expanding the exponent in Eq. (4) into a series and introducing the notation

$$\gamma = \int_0^{\varepsilon_{\max}} \varepsilon^2 w(\varepsilon)\, d\varepsilon, \quad \delta = \int_0^{\varepsilon_{\max}} \varepsilon^3 w(\varepsilon)\, d\varepsilon, \tag{10}$$

we obtain

$$\exp\left\{p(\Delta - \alpha x) + \frac{\gamma x}{2!}p^2 - \frac{\delta x}{3!}p^3\right\}.$$

If we restrict ourselves to the term containing γ, we obtain a Gaussian curve, so that the third term gives the asymmetry of the curve. Let us write

$$z = (\delta x/2)^{1/3}(p - \gamma/\delta). \tag{11}$$

After some simple operations we obtain

$$f(x, \Delta) = e^{at - a^3/3}\frac{1}{2\pi i\eta}\int_{c-i\infty}^{c+i\infty} e^{zt - z^3/3}\, dz,$$
$$\frac{1}{\eta} = \left(\frac{2}{x\delta}\right)^{1/3} = \frac{1}{\xi}\left[(2\varkappa)^2 \Big/ \left(1 - \frac{2}{3}\beta^2\right)\right]^{1/3}, \tag{12}$$
$$a = \eta\frac{\gamma}{\delta} = \left(1 - \frac{\beta^2}{2}\right)\left[(2\varkappa) \Big/ \left(1 - \frac{2}{3}\beta^2\right)^2\right]^{1/3},$$
$$t = (\Delta - \alpha x)/\eta + a^2.$$

Integrating along the imaginary axis, we obtain

$$f(x, \Delta) = \frac{1}{\eta\sqrt{\pi}} e^{at - a^3/3} v(t),$$
$$v(t) = \frac{1}{\sqrt{\pi}}\int_0^{\infty}\cos\left(yt + \frac{y^3}{3}\right) dy. \tag{13}$$

where $v(t)$ is Airey's function, which has been tabulated by Fock[3]. Hines[4] has used a similar expansion, but was unable, using Mellin transforms, to obtain an exact solution.

The position of the maximum of the distribution function is found by differentiating Eq. (13) with respect to t,

$$a = -v'(t)/v(t). \tag{14}$$

From Eq. (14) and the tables we find $t_{\max} = (\Delta_{\max} - \alpha x)/\eta + a^2$.

We shall now show that for $\kappa \gg 1$ Eq. (13) becomes a Gaussian curve. Indeed, $a \approx (2\kappa)^{1/3}$ and as $\kappa \to \infty$ we have $a \to \infty$, so that the maximum of the curve moves towards large t. Using the asymptotic expression for Airey's function[3], we obtain

$$f(x, \Delta) \approx \frac{1}{2\eta\sqrt{\pi}t^{1/4}}\exp\left\{-\frac{a^3}{3} + at - \frac{2}{3}t^{3/2}\right\}.$$

Further

$$t^{1/4} \approx \sqrt{a},\ at - {}^2/_3\, t^{3/2} \approx (a^3/3) - z^2/4a$$

($z = (\Delta - \alpha x)/\eta \ll a$) and by expressing a, η, and z in terms of γ and δ, we obtain

$$f(x, \Delta) \approx (2\pi\gamma x)^{-1/2}\exp\{-(\Delta$$
$$-\alpha x)^2/2\gamma x\},\ \varkappa \gg 1. \tag{15}$$

Performing the integration in Eq. (8) along the imaginary axis, we obtain the following expression for the distribution function:

$$f(x, \Delta) = \frac{1}{\pi\xi} \varkappa e^{\varkappa(1+\beta^2 C)} \int_0^\infty e^{\varkappa f_1} \cos(y\lambda_1 + \varkappa f_2)\, dy,$$

$$f_1 = \beta^2(\ln y - \mathrm{Ci}(y)) - \cos y - y \,\mathrm{Si}(y), \quad (16)$$

$$f_2 = y(\ln y - \mathrm{Ci}(y)) + \sin y + \beta^2 \mathrm{Si}(y),$$

where Si and Ci are the sine and cosine integral functions, respectively. All the numerical calculations were performed in the Digital Control Systems Laboratory of the Academy of Sciences, USSR. Fig. 1 gives graphs of the functions

$$\varphi(\lambda_1) = \xi f(x, \Delta)$$

and $$(\lambda_1) = \int_\Delta^\infty f(x, \Delta)\, d\Delta = \frac{1}{\varkappa} \int_{\lambda_1}^\infty \varphi(x)\, dx$$

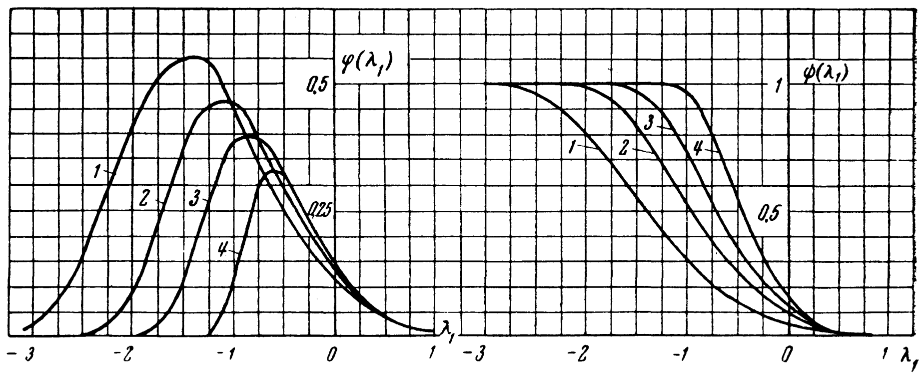

FIG. 1. For curve 1, $\varkappa$ = 1.0; for curve 2, $\varkappa$ = 0.7; for curve 3, $\varkappa$ = 0.5; for curve 4, $\varkappa$ = 0.3.

for various values of the parameter κ and for $\beta^2 = 0.9$.

Fig. 2 gives the curves for $\kappa = 0.1$ (curve 1) and $\kappa = 0.01$ (curve 2). For comparison, we also give Landau's function (curve L). The abscissa gives Landau's parameter:

$$\lambda = (\lambda_1/\varkappa) - \ln\varkappa = (\Delta - \alpha x)/\xi - 1 - \beta^2 + C - \ln\varkappa.$$

It is seen from Fig. 2 that when $\kappa = 0.01$ the exact function is practically the same as Landau's.

Thus Landau's approximation is valid for $\kappa \lesssim 0.01$, the exact solution (16) must be used in the interval $0.01 \lesssim \kappa \lesssim 1$, and the approximation of Eq. (13) may be used in the region $\kappa \gtrsim 1$.

In conclusion, I consider it my duty to express my gratitude to Iu. F. Orlov for valuable discussions and remarks. The author also expresses his gratitude to the staff of the Calculating Division, who performed the numerical calculations.

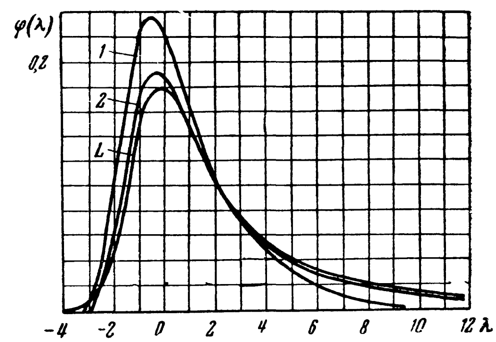

FIG. 2

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[2] N. N. Lebedev, *Special Functions and Their Applications*, M., Gostekhizdat, 1953.

[3] V. A. Fock, *Table of Airey Functions*, M., 1946.

[4] K. C. Hines, Phys. Rev. 97, 1725 (1955).