Agapkin and G. I. Griskuk for their assistance in the work.

Note added in proof (April 19, 1957). Later measurements with a thinner source force one to believe that the level with 172 kev energy apparently exists. The intensity of the a-decay to this level lies within the stated limits.

In the 4.675—4.850 Mev region, levels with excitation energies 373 kev ($\sim$ 0.002%), 426 kev ($\sim$ 0.005%), 497 kev ($\sim$ 0.0015%) and, apparently, 336 kev (0.0035%) have been discovered. The first two of these levels correspond to states observed in $\gamma - \gamma$ coincidences.

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Reactions Involving Polarized Particles

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(Submitted to JETP editor June 9, 1956)
J. Exptl. Theoret. Phys. (U.S.S.R.) **32**, 1022-1035 (May, 1957)

The statistical tensors of particles produced in reactions of the types $a + b \to c + d$ and $a \to c + d$ have been obtained in the most general case, when the incident beam and the target particles are in definite spin states. New selection rules are deduced which supplement the general selection rules of Simon and Welton[1,2]. The first of these rules can be regarded as a generalization of the rule according to which the polarization vector of the particles produced in the reaction is perpendicular to the plane of the reaction if the incident and target particles are unpolarized. The second rule states that the statistical tensor of a particle produced in the reaction, defined relative to the direction of its momentum, is either purely real or purely imaginary if the incident and target particles are unpolarized. As a particular case, the decay of an unstable particle into particles of spins $\frac{1}{2}$ and 0 is considered, and it is shown that the polarization and angular distribution of these particles depend only on the spin and spin state of the decaying particle.

THE GENERAL THEORY of nuclear reactions has already been broadly developed in the papers of Simon and Welton[1,2]. In the present paper a new method is proposed for obtaining the statistical tensors (for definition see Appendix I) of particles produced in a reaction. It has been given brief-

ly and in a somewhat different form in Ref. 3 (for a method of very similar conception, see Ref. 4). This approach permits a relativistic generalization and also immediate generalization to the case of nuclear reactions with more than two particles in the final state. In a similar fashion one can use the diagonal character of the S matrix with respect to other conserved physical quantities to obtain a number of general properties of processes in quantum-mechanical systems.

In this paper the formalism of Dirac[5] is widely used (cf. Ch. I-IV, especially Sec. 17).

1. We consider reactions of the types $a + b \to c + d$ and $a \to c + d$. a, b, c, d denote either "elementary" particles or nuclei, with spins i_a, i_b, i_c, i_d. The spin is treated in the Pauli approximation, and in this sense the treatment will be nonrelativistic (in particular, particles with rest mass zero are not considered). Knowing the initial state of the system (before the reaction), and assuming as known the elements of the S matrix in the corresponding representation, we can obtain the wave function of the final state*.

$$\psi'_{\eta} \xi_0 = (\eta' \mid S \mid \xi)_\xi \Psi_{\xi_0}. \qquad (1.1)$$

Summation or integration with respect to ξ is understood. The indices ξ and η denote a complete set of quantities characterizing the states of the system (type of particle, etc., see below the complete set for a system of two particles). $\psi'_{\eta} \xi_0$ is the probability amplitude for the quantities η to have the values η' in the final state, if the system was initially in the state ξ_0.[5]

But the initial and final states must be described not by wave functions, but by density matrices, since the states of the system before and after the reaction are, generally speaking, mixed ensembles[7, 13].

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*The relation (1.1) is ordinarily written as applied to reactions of the type $a + b \to c + d$. In this case $\hat{S} = \hat{U}(\infty, -\infty)$, where the operator $\hat{U}(t, t_0)$, for example, satisfies the Schroedinger equation (in the interaction representation)[6]

$$i\hbar\partial \hat{U}(t, t_0)/\partial t = \hat{H}_{\text{int}} \hat{U}(t, t_0).$$

But if the particle a has sufficiently long lifetime so that one can speak approximately about a definite (quasistationary) initial state (with definite energy, etc.), then (1.1) can be written also for the reaction $a \to c + d$, where now $\hat{S} = \hat{U}(\infty, 0)$ (the time is counted from the instant of production of the unstable particle).

For example, an unpolarized beam of particles incident on the target is described by an ensemble in which the probabilities of all possible orientations of the spin are equal. We define the density matrix as

$$(\xi_1 \mid \rho \mid \xi_2) = \sum_\alpha P_\alpha \xi_1 \Psi_\alpha \xi_2 \Psi^*_\alpha, \qquad (1.2)$$

where the P_α are the weights of the pure states $\left(\sum_\alpha P_\alpha = 1\right)$.

By means of Eq. (1.1) we find that the density matrix of the final state is given by:

$$(\eta'_1 \mid \rho' \mid \eta'_2) = (\eta'_1 \mid S \mid \xi_1)(\eta'_2 \mid S \mid \xi_2)^*(\xi_1 \mid \rho \mid \xi_2)$$

$$= (\eta'_1 \mid S\rho S^+ \mid \eta'_2), \qquad (1.3)$$

the diagonal elements $(\eta' \mid \rho' \mid \eta')$ give the probability that after the reaction the system will be in the state η' if the initial state was characterized by the density matrix (1.2).

We shall use henceforth instead of the S matrix the scattering matrix $\hat{R} = \hat{S} - 1$, all elements of which are the same as those of the S matrix with the exception of those for transitions without change of state[6].

It is well known that in an isolated system the total angular momentum J, its projection M, the total momentum $\mathbf{P}$, and the total energy E are conserved. An expression of this fact is the diagonal character of the S (or R) matrix with respect to these quantities. Generally speaking, however, J, M, $\mathbf{P}$, and E cannot belong to the same complete set. Therefore in the general case one has to write down the condition of the diagonal character of the elements of the R matrix with respect to J, M, and E in the representation of a set including J, M, and E, then express in terms of these elements the matrix elements of the R matrix in the representation of another complete set including $\mathbf{P}$, and write the condition that the matrix be diagonal with respect to $\mathbf{P}$. The performance of these operations in practice would require rather cumbersome unitary transformations.

There does exist, however, a system of coordinates in which we can make use of the diagonal character of the R matrix with respect to J, M, E and $\mathbf{P}$ all at once. Namely, if for the description of a system of two particles we have chosen the system of coordinates in which $P^2 = 0$, then in a set including J, M, and E there can also be included: the spins i_c and i_d of the two particles, the to-

tal spin s, the total orbital angular momentum l of the particles relative to the center of mass*, the orbital angular momentum L of the entire system relative to the origin of coordinates (see below), the total orbital angular momentum $\mathcal{L}$ ($\hat{\mathcal{L}} = \hat{L} + \hat{l}$), and the absolute value P of the total momentum of the system (equal to zero)† . Since the operator $\hat{L}$ can be written $[\hat{\mathbf{R}}_c \times \mathbf{P}]$, where $\hat{\mathbf{R}}_c$ is the operator of the center of mass, then in those states in which $\mathbf{P} = 0$ we have also $L = 0$ (and we are indeed considering only states of the system for which $P = 0$ and consequently $\mathbf{P} = 0$). Since P^2 is conserved, then $L = 0$ always, and the total orbital angular momentum of the system is equal to l. For a reaction of the type $a + b \rightarrow c + d$, for example, we have in the the representation of this complete set

$$(i_c i_d s' l' L' \mathcal{L}' P' J' M' E' \alpha' \mid R \mid i_a i_b s l L \mathcal{L} 0 J M E \alpha)$$

$$= (i_c i_d s' l' 0 l' \alpha' \mid R_0^{0JE} \mid i_a i_b s l 0 l \alpha) \qquad (1.4)$$

$$\times \delta(\mathbf{P}' - 0) \delta_{J'J} \delta_{M'M} \delta(E' - E).$$

The upper indices indicate the dependence of the diagonal elements on P, J, and E; it can be shown that they are independent of M. In what follows the indices i_c, i_d, etc., for the spins of the particles will not be written out, and also the indices for the total momentum and the quantities L and $\mathcal{L}$ are dropped, both in the elements of the R matrix and in those of the density matrix (in other words, no indication is shown of the motion of the system as a whole, i.e., of the fact that it is at rest in the chosen system of coordinates).

In the representation of the complete set s, l, J, M, E, a, and with the use of Eq. (1.4), Eq. (1.3) takes the form:

$$(s_1' l_1' J_1 M_1 E_1 \alpha_1' \mid \rho' \mid s_2' l_2' J_2 M_2 E_2 \alpha_2')$$

$$= (s_1' l_1' \alpha_1' \mid R^{J_1 E_1} \mid s_1 l_1 \alpha_1)(s_2' l_2' \alpha_2' \mid R^{J_2 E_2} \mid s_2 l_2 a_2)^*$$

$$\times (s_1 l_1 J_1 M_1 E_1 \alpha_1 \mid \rho \mid s_2 l_2 J_2 M_2 E_2 \alpha_2). \qquad (1.5)$$

---

*In the center-of-mass system

$$\hat{l} = [(\hat{\mathbf{r}}_c - \hat{\mathbf{R}}_c)\hat{\mathbf{p}}_c] + [(\hat{\mathbf{r}}_d - \hat{\mathbf{R}}_c)\hat{\mathbf{p}}_d]$$

$$= [(\hat{\mathbf{r}}_c - \hat{\mathbf{R}}_c)\hat{\mathbf{p}}_c] + [(\hat{\mathbf{r}}_d - \hat{\mathbf{R}}_c)(-\hat{\mathbf{p}}_c)] = [(\hat{\mathbf{r}}_c - \hat{\mathbf{r}}_d)\hat{\mathbf{p}}_c].$$

Here $\hat{\mathbf{p}}_c = -\hat{\mathbf{p}}_d$ are the momentum operators of the particles in the center-of-mass system.

† We remark that, generally speaking, a complete set must also include the masses of the particles. For brevity they are not written out explicitly. The complete set may also include a number of other variables, for example the intrinsic parities of the two particles. All of these are denoted by the letter a.

2. The meaning of Eq. (1.1) or (1.3) consists in the expression of the unknown parameters that determine ψ' or ρ' in the form of elements of the S matrix. The conservation laws reduce the number of these parameters. We must now find out how this manifests itself in the description of the angular distribution and spin state of the products c and d of the reaction in question. The essence of what follows is comprised in transformations (for the most part unitary) from the representation of the final and initial states of the reaction in the quantities s, l, J, M, E, which are not directly measured, into a representation of quantities that are experimentally measured.

To characterize the final state we require the elements of the density matrix that are diagonal with respect to $\mathbf{p}_c$ (the momentum in the center-of-mass system), and also the statistical tensors (briefly: the c-tensors) describing the spin state of the products of the reaction (cf. Appendix I). We denote the set of these quantities by the symbol $\rho'(\mathbf{p}_c; q_c \nu_c q_d \nu_d)$. We note that according to Eqs. (1.1)–(1.3) the indices $\mathbf{p}_c$ can be regarded also as parameters of the c-tensors of particles c and d (these being quantities proportional to the mean values of the corresponding spin operators in the ensemble of particles with momentum $\mathbf{p}_c$).

In a problem of the type $a + b \rightarrow c + d$ the initial state is characterized by a definite value of the quantity $\mathbf{p}_c$ and by c-tensors. The symbolic notation is

$$\rho(\mathbf{p}_a; q_a \nu_a q_b \nu_b)(\mathbf{p}_a' \mid \mathbf{p}_a)(\mathbf{p}_a'' \mid \mathbf{p}_a)$$

(cf. Appendix II).

Hereafter we shall specify momenta $\mathbf{p}$ in terms of their absolute values p and the corresponding unit vectors n with spherical angles ϑ, φ.

Thus we have to express

$$\rho'(\mathbf{n}_c, p_c, \alpha'; q_c \nu_c q_d \nu_d) \text{ in terms of}$$

$$(s_1' l_1' J_1' M_1' E_1' \alpha' \mid \rho' \mid s_2' l_2' J_2' M_2' E_2' \alpha'), [cf. (1.4)] \text{ and}$$

$$(s_1 l_1 J_1 M_1 E_1 \alpha_1 \mid \rho \mid s_2 l_2 J_2 M_2 E_2 \alpha_2)$$

in terms of $(\mathbf{n}_a, p_a; q_a \nu_a q_b \nu_b)(\mathbf{p}_a' \mid \mathbf{p}_a)(\mathbf{p}_a'' \mid \mathbf{p}_a)$.

This is accomplished by means of successive transformations from one representation to another, using the Clebsch-Gordan coefficients $(j_1 j_2 m_1 m_2 \mid j_1 j_2 j m)$ or $C_{j_1 m_1 j_2 m_2}^{jm}$[8,9] and the correctly normalized (cf. Appendix II) transformation functions $(\vartheta \varphi p \mid l \mu E)$:

$$\varrho' \left(\mathbf{n}_c,\, p_c,\, \alpha';\ q_c \nu_c q_d \nu_d \right) = \left[(2i_c + 1)(2i_d + 1) \right]^{1/2}$$

$$\times \sum_{m_c,\, m_c',\, m_d,\, m_d'} (-1)^{i_c - m_c' + i_d - m_d'} (i_c i_c m_c - m_c' \,|\, i_c i_c q_c \nu_c)(i_d i_d m_d - m_d' \,|\, i_d i_d q_d \nu_d)$$

$$\times \sum (i_c i_d m_c m_d \,|\, i_c i_d s_1' m_1')\, (\mathbf{n}_c\, p_c \,|\, l_1'\, \mu_1'\, E_1')\, (s_1'\, l_1'\, m_1'\, \mu_1' \,|\, s_1' l_1' J_1'\, M_1') \tag{2.1}$$

$$\times\, (s_1'\, l_1'\, J_1'\, M_1'\, E_1'\, \alpha' \,|\, \varrho' \,|\, s_2'\, l_2'\, J_2'\, M_2'\, E_2'\, \alpha')$$

$$\times\, (s_2'\, l_2'\, J_2'\, M_2' \,|\, s_2'\, l_2'\, m_2'\, \mu_2')\, (l_2'\, \mu_2'\, E_2' \,|\, \mathbf{n}_c\, p_c)\, (i_c i_d s_2'\, m_2' \,|\, i_c i_d m_c'\, m_d').$$

Σ denotes summation or integration (with respect to E_1' and E_2') over all indices occurring twice. We insert into Eq. (2.1) the expression (II.2) for $(\vartheta\varphi p \,|\, l\mu E)$ and use the formula

$$Y_{l_1 \mu_1}(\mathbf{n})\, Y_{l_2 \mu_2}(\mathbf{n})$$
$$= \sum_{L,\, m_L} \left[\frac{(2l_1 + 1)(2l_2 + 1)}{4\pi (2L + 1)} \right]^{1/2} C_{l_1 0 l_2 0}^{L 0} C_{l_1 \mu_1 l_2 \mu_2}^{L m_L}\, Y_{L m_L}(\mathbf{n}). \tag{2.2}$$

The majority of the sums over magnetic quantum numbers in Eq. (2.1) actually do not contribute, by virtue of the properties of the Clebsch-Gordan coefficients, but it is convenient to keep them for the carrying-out of the summation over these indices. For this we use the following typical formula:

$$\sum_{m_1 m_2 \mu_1 \mu_2} C_{j_1 m_1 l_1 \mu_1}^{s_1 \sigma_1} C_{j_2 m_2 l_2 \mu_2}^{s_2 \sigma_2} C_{j_1 m_1 j_2 - m_2}^{j m j} C_{l_1 \mu_1 l_2 - \mu_2}^{l m l}$$

$$= (-1)^{s_2 - j_2 - l_2} \left[(2s_1 + 1)(2s_2 + 1)(2j + 1)(2l + 1) \right]^{1/2} \tag{2.3}$$

$$\times \sum_{g,\, m_g} C_{s_1 \sigma_1 s_2 - \sigma_2}^{g m_g} C_{j m_j l m_l}^{g m_g} X(j_1 j j_2;\ s_1 g s_2:\ l_1 l l_2).$$

The relation (2.3) can be obtained by means of Eq. (1), (18), and (19) of Ref. 9, and also from Eq. (3) in Ref. 10, and the coefficients X are defined in Ref. 10 and 1. By means of Eq. (2.3) one at once carries out in Eq. (2.1) the sum over m_c, m_c', m_d, m_d' of the product of the first, second, third, and last Clebsch-Gordan coefficients (we note that $-m_c' - m_d' = -m_2'$). After this one can take the sum over m_1', m_2', μ_1', μ_2' [again a sum of the type (2.3)], and one finally gets from Eq. (2.1):

$$\varrho' \left(\mathbf{n}_c,\, p_c,\, \alpha';\ q_c \nu_c q_d \nu_d \right) = N_1' N_2' (4\pi)^{-1/2} \left[(2i_c + 1)(2i_d + 1) \right]^{1/2}$$

$$\times \sum (\mathbf{n}_c \,|\, L'\, m_L')\, C_{q' \nu' L' m_L'}^{J' M'}\, C_{q_c \nu_c\, q_d \nu_d}^{q' \nu'}\, C_{l_1'\, 0 l_2'\, 0}^{L' 0} (-1)^{J_2' - M_2'} C_{J_1' M_1' J_2' - M_2'}^{J' M'}$$

$$\times \left[(2s_1' + 1)(2s_2' + 1)(2q_c + 1)(2q_d + 1) \right]^{1/2} X(i_c q_c i_c;\ s_1' q' s_2';\ i_d q_d i_d) \tag{2.4}$$

$$\times \left[(2l_1' + 1)(2l_2' + 1)(2J_1' + 1)(2J_2' + 1)(2q' + 1) \right]^{1/2}$$

$$\times X(s_1' q' s_2';\ J_1' J' J_2';\ l_1' L' l_2')\, i^{-l_1' - l_2'}\, (s_1' l_1' J_1' M_1' E_1' \alpha' \,|\, \varrho' \,|\, s_2' l_2' J_2' M_2' E_2' \alpha').$$

The sum Σ is taken over s_1', s_2', l_1', l_2'; J_1', J_2', q', L', J' and over ν', M_L', M_1', M_2', M'; and

$$N_{1,2}' = 2\sqrt{2}\,\pi h\,[R/V]^{1/2} p_c^{-1}\,(p_c \mid E_{1,2}')$$

(cf. Appendix II).

Equations (2.1) and (2.4) are written in the center-of-mass coordinate system, the z, y, and x directions being so far arbitrary. For the flux of particles the most obvious distinguished direction in space is the direction $\mathbf{n}$ of this flux. We introduce the c-tensors $\rho'(\mathbf{n}_c, p_c, \alpha'; q_c\tau_c q_d\tau_d)$ with indices τ referred to $\mathbf{n}_c$ as axis of quantization. The expression of these tensors in terms of the old c-tensors has the form:

$$\rho'(\mathbf{n}_c, p_c,\alpha'; q_c\tau_c q_d\tau_d) = \sum_{\nu_c'\nu_d} D^{q_c}_{\tau_c\nu_c}(-\pi, \vartheta_c, \pi - \varphi_c)$$

$$\times\, D^{q_d}_{\tau_d\nu_d}(-\pi, \vartheta_c, \pi - \varphi_c)\, \rho'(\mathbf{n}_c, p_c, \alpha'; q_c\nu_c q_d\nu_d),$$

$$(2.5)$$

$\pi - \varphi_c$, ϑ_c, and $-\pi$ are the Eulerian angles* of the

rotation g_c of the axes zyx that makes the z axis coincide with the direction of $\mathbf{n}_c$ and the y axis perpendicular to the old direction of the z axis and to $\mathbf{n}_c$; the rotation inverse to $g_c = \{-\pi, \vartheta_c, \pi - \varphi_c\}$, is written $g_c^{-1} = \{\varphi_c,\ \vartheta_c,\ 0\}$. We note that the spherical angles are reckoned as before from the old axes zyx; one has only introduced a new axis of quantization for the spin indices.

Substituting Eq. (2.4) into Eq. (2.5), we can first carry out the summation over ν_c and ν_d:

$$\sum_{\nu_c,\nu_d} D^{q_c}_{\tau_c\nu_c}(g)\, D^{q_d}_{\tau_d\nu_d}(g)\, C^{q'\nu'}_{q_c\nu_c q_d\nu_d}$$

$$= D^{q'}_{\tau_c+\tau_d,\nu'}(g)\, C^{q'\tau_c+\tau_d}_{q_c\tau_c q_d\tau_d}.$$

$$(2.6)$$

We introduce the notation $\tau' = \tau_c + \tau_d$. Using this same formula (2.6)[1], we can carry out the sum over ν' and m_L', if we use the relation

$$Y_{ln}(\vartheta, \pi - \varphi_1) = [(2l+1)/4\pi]^{1/2} D^l_{0,n}(\varphi_2, \vartheta, \varphi_1).$$

The final result is:

$$\rho'(\mathbf{n}_c, p_c; \alpha';\ q_c\tau_c q_d\tau_d) = N_1' N_2'\, (4\pi)^{-1} [(2i_c+1)(2i_d+1)]^{1/2} \times$$

$$\times \sum D^{J'}_{\tau',M'}(-\pi, \vartheta_c, \pi-\varphi_c)\, C^{q'\tau'}_{q_c\tau_c q_d\tau_d}(-1)^{q'+\tau'}\,(-1)^{J_2'-M_2'} C^{J'M'}_{J_1'M_1'J_2'-M_2'}$$

$$(2.7)$$

$$\times [(2q_c+1)(2q_d+1)]^{1/2} X(i_c q_c i_c;\ s_1' q' s_2';\ i_d q_d i_d)\,\times$$

$$\times G^*_{\tau'}(J_1' l_1' s_1';\ J'q';\ J_2' l_2' s_2')\,(s_1' l_1' J_1' M_1' E_1' \alpha' \mid \rho' \mid s_2' l_2' J_2' M_2' E_2' \alpha'),$$

*The indices ν (or τ) of the c-tensors are "contravariant" (indices of the representation). Therefore $\rho(q\nu)$ transforms on three-dimensional rotation like $Y^*_{q\nu}(\theta, \Phi) = (q\nu \mid \theta\Phi)$. Equation (2.5) is obtained by simply taking the complex conjugate of the formula for transforming the spherical functions, which we write in the form

$$Y_{qn}(\mathbf{n}') = \sum_m Y_{qm}(\mathbf{n})\, D^q_{m,n}(g^{-1}),$$

if the transformation of a unit vector by the rotation g is written $\mathbf{n}' = \hat{g}\mathbf{n}$. If the rotation g is treated as a rotation of the coordinate system (right-handed), it can be specified by the Eulerian angles φ_1 (rotation around the z axis), ϑ (rotation around the y' axis), and φ_2 (rotation around the z' axis). All rotations are clockwise. We emphasize that the Eulerian angle ϑ is ordinarily defined as a rotation around the x' axis, but only with our definition will the formula written above be correct.

$$D^l_{m,n}(\varphi_2\vartheta_1\varphi_1) = \exp(-im\varphi_2)\, i^{n-m} P^l_{mn}(\cos\vartheta)\exp(-in\varphi_1),$$

with the functions P^l_{mn} as in Ref. 11. See this reference also for the definition of the spherical functions.

where the coefficient $G_{\tau'}$ is defined in Ref. 2, Eq. (3.3); the sum is taken over s_1', s_2', l_1', l_2', q', J_1', J_2', J' and over M_1', M_2', M'.

Similarly one gets the expressions of $(s_1 l_1 J_1 M_1 E_1 \alpha_1 \mid \rho \mid s_2 l_2 M_2 E_2 \alpha_2)$ in terms of

$$\rho_{\alpha_1\alpha_2}(\mathbf{n}_a, p_a;\ q_a\nu_a q_b\nu_b)$$

or of $\rho_{\alpha_1\alpha_2}(\mathbf{n}_a, p_a;\ q_a\tau_a q_b\tau_b)$;

[we refer to the corresponding formulas as Eqs. (2.8) and (2.9)].

Refraining from a change in notation, we remark that the corresponding transformation functions are obtained by simply taking complex conjugates of the transformation functions in Eqs. (2.4) and (2.7) (i.e., taking complex conjugates of all the coefficients of $(s_1' l_1' J_1' M_1' E_1' \alpha' \mid \rho' \mid s_2' l_2' J_2' M_2' E_2' \alpha')$ in Eqs. (2.4) and (2.7) and replacing $[(2i_c+1)(2i_d+1)]^{1/2}$ by $[(2i_a+1)(2i_b+1)]^{-1/2}$ [cf. Eqs. (1.3) and (1.4)].

Combination of Eqs. (2.4), (1.5), and (2.8) gives the first final formula [we shall call it Eq. (2.10)];

all spin indices and spherical angles are referred to an arbitrarily chosen system of axes zyx. Since $J_1' = J_1$, $J_2' = J_2$, $M_1' = M_1$, and $M_2' = M_2$, the sum over M_1 and M_2 is easily carried out.

$$\sum_{M_1 M_2} (J_1 J_2 M_1 - M_2 \mid J_1 J_2 J' M')$$

$$\times (J_1 J_2 M_1 - M_2 \mid J_1 J_2 J M) = \delta_{J'J}\delta_{M'M}$$

and thus we also have $J' = J$, $M' = M$. Moreover,

$$(p_c \mid E_1)(E_1 \mid p_a) = (p_c \mid E_2)(E_2 \mid p_a) = (p_c; p_a),$$

where the symbol $(p_c; p_a)$ expresses the law of conservation of energy in the momentum representation. All the transformations are exhausted by this, and Eq. (2.10) can easily be written out in final form, but it is very cumbersome.

Combination of Eqs. (2.7), (1.5) and (2.8) gives the second final formula, Eq. (2.11). If in Eq. (8)* of Ref. 3 we replace $\lambda_c^2/4$ by the normalizing factor $N/(4\pi)^2$ (cf. Appendix II) and $D_{\varkappa,\varkappa'}^J(\varphi_\Lambda, \vartheta_\Lambda, 0)$ by

$$\sum_M D_{\tau'M}^J(-\pi, \vartheta_c, \pi - \varphi_c) D_{\tau M}^{J*}(-\pi, \vartheta_a, \pi - \varphi_a)$$

$$= D_{\tau'\tau}^J(g_c g_a^{-1}) \qquad (2.12)$$

[this is required by the change of definition of the c-tensors, cf. Eq. (5) in Ref. 3 and Eq. (II.3)], then the combination of Eqs. (7), (8′), and (9) in Ref. 3 will also be Eq. (2.11).

It can be shown that it follows from Eq. (2.12) that the c-tensors of the final state depend essentially on the parameters of the rotation $g_c g_a^{-1}$ of the system of axes $z_a y_a x_a$ distinguished by the initial state (the axis $z_a \parallel \mathbf{n}_a$, and the direction of the axis y_a can be given by the spin state; for example, it can be directed along the component of the polarization vector perpendicular to $\mathbf{n}_a$) into the system of axes $z_c y_c x_c$ (the axis $z_c \parallel \mathbf{n}_c$, and axis $y_c \parallel [\mathbf{n}_a \times \mathbf{n}_c]$). Thus all the quantities in Eq. (2.11) can be defined relatively to a number of physical directions of the reaction $a + b \to c + d$, so that for Eq. (2.11) one can dispense with the introduction of any auxiliary coordinate system. In the coordinate system ordinarily used, coinciding with the system of axes $z_a y_a x_a$, we have $g_c g_a^{-1} = \{-\pi, \vartheta, \pi - \varphi\}$, where ϑ, φ are the spherical

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* There is a mistake in Eq. (8) of Ref. 3: the exponent on the square brackets

$[(2s_1 + 1)(2s_2 + 1)(2s_1' + 1)(2s_2' + 1)]$ must be $-\frac{1}{2}$, not $+\frac{1}{2}$.

angles of the direction $\mathbf{n}_c$ in such a system of coordinates.

If the spin state of the incident beam a and the target particles b is completely unpolarized or has axial symmetry

$$\rho(q_a \tau_a q_b \tau_b) = \rho(q_a 0 q_b 0)\,\delta_{\tau_a 0}\delta_{\tau_b 0}.,$$

then the sum over τ in Eq. (2.11) there remains only (cf. note * on page 839)

$$D_{\tau'0}^J(-\pi, \vartheta, \pi - \varphi) = P_{\tau'0}^J(\cos\vartheta)$$

and the c-tensors of the final state [including the angular distribution $\rho(\mathbf{n}_c, p_c, a'; 0000)$] do not depend on φ. This is natural, since in this case the choice of the axis y_a is completely arbitrary and nothing can depend physically on such a choice.

By a procedure analogous to that explained for Eq. (2.11), the general formula for a reaction of the type $a \to c + d$ can be obtained:

$$\rho'(\mathbf{n}_c, p_c, \alpha'; q_c \tau_c q_d \tau_d)$$

$$= N_0^2 (4\pi)^{-1} [(2i_c + 1)(2i_d + 1)]^{1/2} (2s + 1)^{-1/2}$$

$$\times \sum (-1)^{q'+\tau'} C_{q_c \tau_c q_d \tau_d}^{q'\tau'} [(2q_c + 1)$$

$$\times (2q_d + 1)]^{1/2} X(i_c q_c i_c; s_1' q' s_2'; i_d q_d i_d)$$

$$\times G_{\tau\tau'}^*(s l_1' s_1'; q\ q';\ s l_2' s_2') \qquad (2.13)$$

$$\times (s_1 l_1 \alpha' \mid R^{sE} \mid \alpha_1)(s_2 l_2' \alpha' \mid R^{sE} \mid \alpha_2)^*$$

$$\times D_{\tau'\nu}^q(-\pi, \vartheta_c, \pi - \varphi_c)\rho_{\alpha_1, \alpha_2}(q\nu).$$

The sum is taken over q', τ', s_1', s_2', l_1', l_2', q, ν; and

$$N_0 = 2\pi h\, [R/V]^{1/2} p_c^{-1}(p_c \mid E);$$

$\rho(q\nu)$ are the c-tensors of particle a, s is its spin, and E is its total energy. The symbol $(p_c \mid E)$ is equal to unity if p_c is a root of the equation

$$\sqrt{p_c^2 c^2 + \varkappa_c^2 c^4} + \sqrt{p_c^2 c^2 + \varkappa_d^2 c^4} = E = \varkappa_a c^2,$$

and zero if p_c does not satisfy this equation (cf. Appendix II).

Let us analyze the case in which particles a and b are identical (and/or particles c and d). Let the variables of the complete set ξ_1 (and ξ_2) in the definition of the density matrix (1.2) be the momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ of the particles and their spin components m_1 and m_2. Then it is necessary that

$$(\xi_1 \mid \rho \mid \mathbf{p}_1, \mathbf{p}_2, m_1, m_2)$$
$$= (-1)^{2i} (\xi_1 \mid \rho \mid \mathbf{p}_2, \mathbf{p}_1, m_2, m_1), \qquad (2.14)$$

i.e., the elements of the density matrix with fixed ξ_1 must either change sign, if the identical particles have half-integral spin, or remain unchanged, for Bose particles, depending on whether we as-

cribe to the "first" particle a the momentum $\mathbf{p}_1$ and the spin component m_1 and to the "second" the momentum $\mathbf{p}_2$ and the component m_2, or ascribe $\mathbf{p}_2$ and m_2 to the "first" and $\mathbf{p}_1$ and m_1 to the "second". The same must hold independently for the indices ξ_1 with fixed ξ_2. Introducing instead of $\mathbf{p}_1$ and $\mathbf{p}_2$ the total momentum $\mathbf{P}$ and the momentum in the center-of-mass system $\mathbf{p}$, we can write Eq. (2.14) in the form:

$$(\xi_1 \mid \rho \mid \mathbf{P}, \mathbf{p}, m_1, m_2) = (-1)^{2i} (\xi_1 \mid \rho \mid \mathbf{P}, -\mathbf{p}, m_2, m_1).$$

Taking into account the fact that

$$(\xi_1 \mid \rho \mid -\mathbf{p}, i, i, m_2, m_1)$$
$$= (\xi_1 \mid \rho \mid l\mu piism)(l\mu p \mid \pi - \vartheta, \varphi + \pi, p)(iism \mid iim_2 m_1)$$
$$= (\xi_1 \mid \rho \mid l\mu piism)(-1)^l (l\mu p \mid \vartheta, \varphi, p)(-1)^{s-2i}(iism \mid iim_1 m_2)$$

(index $\mathbf{P}$ omitted), we find that the element of the density matrix symmetrized with respect to the right-hand indices [satisfying Eq. (2.14)] has the form

$$(\xi_1 \mid \rho \mid \mathbf{p}, m_1, m_2)\text{sym} = 2^{-1/2}[(\xi_1 \mid \rho \mid \mathbf{p}\, m_1 m_2) + (-1)^{2i}(\xi_1 \mid \rho \mid -\mathbf{p}, m_2 m_1)]$$
$$= 2^{-1/2}(\xi_1 \mid \rho \mid l\mu psm)[1 + (-1)^{l+s}](l\mu p \mid \vartheta \varphi p)(iism \mid iim_1 m_2). \qquad (2.15)$$

Therefore, if particles a and b are identical, then in addition to making the corresponding indices equal ($i_a = i_b = i$, etc.) in Eqs. (2.10) and (2.11) we must insert under the sign of summation a factor

$$\tfrac{1}{2}[1 + (-1)^{l_1+s_1}][1 + (-1)^{l_2+s_2}]$$

A similar factor (with primes on l and s) is inserted in Eqs. (2.4), (2.7), (2.11), etc., if c and d are identical.

3. The final state of a reaction of the type $a + b \rightarrow c + d$ is described, generally speaking, by $(2i_c + 1)^2 + (2i_d + 1)^2$ c-tensors (*cf.* Appendix I). But it turns out that in the case of a completely unpolarized initial state a happy choice of the axis of quantization reduces the number of the c-tensors of the final state, for the calculation of which a knowledge of the matrix elements (1.4) is necessary.

Ordinarily the z axis of the coordinate system in which a formula of the type of Eq. (2.10) is written is directed along $\mathbf{n}_a$. Let us direct it perpendicular to the plane of the reaction, *i.e.*, along the vector

$[\mathbf{n}_a \times \mathbf{n}_c]$. Thus for each case of the reaction its own z axis is chosen. We then direct the x axis along $\mathbf{n}_a$, so that the direction of the incident beam is always described in the same way: $\vartheta_a = \pi/2$, $\varphi_a = 0*$. The angle φ_c now is the angle between the directions of the momentum $\mathbf{p}_c$ and of the incident beam. For the following considerations we need the following factor in the general term of the sum Σ of Eq. (2.10), in the chosen coordinate system:

$$C^{JM}_{q'v'L'm'_L} C^{JM}_{qvLm_L} C^{L'0}_{l'_1 0 l'_2 0} C^{L0}_{l_1 0 l_2 0} \cdot Y_{L', m'_L}$$
$$\times (\pi/2, \varphi_c) Y^*_{L, m_L}(\pi/2, 0). \qquad (3.1)$$

From the properties of the coefficients $(l'_1 l'_2 00 \mid l'_1 l'_2 L'0)$ and $(l_1 l_2 00 \mid l_1 l_2 L0)$ and from the

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*But the c-tensors of the spin states of the beam and the target will, generally speaking, be different for different z axes. They can be expressed in terms of c-tensors referred, for example, to $\mathbf{n}_a$ as axis of quantization, by the formulas of Sec. 2 [of the type of Eq. (2.5)].

law of conservation of spatial parity of the system it follows that $l_1' + l_2' + L'$, $l_1 + l_2 + L$, and $l_1' + l_2' + l_1 + l_2$ must be even numbers. Therefore $L' + L$ is also even. This fact is a sort of selection rule that does not depend on the choice of the coordinate system.

In virtue of the properties of associated Legendre polynomials of the first kind, the functions $Y_{l\mu}(\pi/2, \varphi)$ are nonvanishing only for $l + \mu$ even. Therefore the only terms in Eq. (2.10) not equal to zero are those with even $L' + m_L'$, $L + m_L$ and consequently even $L' + L + m_L' + m_L$, and finally even $m_L' + m_L$. From the properties of the first two Clebsch-Gordan coefficients in (3.1) we have $\nu' + m_L' = \nu + m_L$, from which we have $\nu' - \nu = m_L - m_L'$. Since ν', ν, m_L, and m_L' are whole numbers, the selection rule obtained can be expressed as follows: $\nu' + \nu$ must be an even number if the axis of quantization is chosen perpendicular to the plane of the reaction. In the important special case of completely unpolarized incident beam and target[in all coordinate systems all $\rho(\mathbf{n}_a, p_a; q_a \nu_a q_b \nu_b)$ other than $\rho(\mathbf{n}_a, p_a; 0000)$ are equal to zero] this rule states that only the $\rho(\mathbf{n}_c, p_c; q_c \nu_c q_d \nu_d)$ with even $\nu_c + \nu_d$ are nonvanishing.

In particular, the vanishing of $\rho'(\mathbf{n}_c, p_c; 1, \pm 1, 0, 0)$ and $\rho'(\mathbf{n}_c, p_c; 0,0,1, \pm 1)$ means that the polarization vector for each of the particles c and d must be perpendicular to the plane of the reaction if the incident beam and the target are unpolarized[1].

If in Eq. (2.11) we assume the initial state completely unpolarized and choose the usual coordinate system (z axis $\parallel \mathbf{n}_a$), then we can obtain one more selection rule. In this case $\rho'(\mathbf{n}, p; q_c \tau_c 00)$ [or $\rho'(\mathbf{n}, p; 00 q_d \tau_d)$] depends on τ_c owing to the following factors in the general term of the sum (cf. Eq. (8) in Ref. 3, with $D_{\varkappa\varkappa}^J$, $(\varphi_\Lambda, \vartheta_\Lambda, 0)$ replaced by $D_{\tau_c, 0}^J (-\pi, \vartheta, \pi - \varphi)$—cf. Sec. 2):

$$(-1)^{\tau_c} G_{\tau_c} (J_1 l_1' s_1'; J\ q_c;$$

$$J_2 l_2' s_2') D_{\tau_c, 0}^J (-\pi, \vartheta, \pi - \varphi),$$

or, taking into account the expression (3.3) of Ref. 2 for G_{τ_c} and the formula

$$D_{\tau, 0}^J (\varphi_2, \vartheta, \varphi_1)$$

$$= [4\pi / (2J + 1)]^{1/2} (-1)^\tau Y_{J, \tau} (\vartheta, \pi - \varphi_2),$$

the factors in question are

$$(q_c J \tau_c - \tau_c \mid q_c J L'0)\ Y_{J, \tau_c} (\vartheta,\ 0).$$

Similarly, $\rho'(\mathbf{n}_c, p_c; q_c -\tau_c\ 00)$ depends on $-\tau_c$ owing to the factors

$$(q_c J - \tau_c \tau_c \mid q_c J L'0)\ Y_{J, -\tau_c} (\vartheta,\ 0).$$

From the law of conservation of the spatial parity of the system, and owing to the occurrence of the coefficients $(l_1' l_2' 00 \mid l_1' l_2' L'0)$, $(l_1 l_2 00 \mid l_1 l_2 L0)$, and $(0J00 \mid 0JL0)$ in G_{τ_c} and G_0^*, we have: $l_1 + l_2 + l_1' + l_2'$, $l_1' + l_2' + L'$, and $l_1 + l_2 + J$ are even numbers, from which it follows that $J + L'$ (and, of course, $J - L'$) must also be an even number. Therefore

$$(q_c J - \tau_c \tau_c \mid q_c J L'0)$$

$$= (-1)^{q_c + J - L'} (q_c J \tau_c - \tau_c \mid q_c J L'\ 0)$$

$$= (-1)^{q_c} (q_c J \tau_c - \tau_c \mid q_c J L'0).$$

Taking into account that

$$Y_{J, -\tau_c} (\vartheta, 0) = (-1)^{\tau_c} Y_{J, \tau_c} (\vartheta, 0),$$

we get

$$\rho'(\mathbf{n}, p; q_c, -\tau_c, 0,0)$$

$$= (-1)^{q_c + \tau_c} \rho'(\mathbf{n}, p; q_c, \tau_c, 0, 0).$$

Taking into account now the Hermitian property of the c-tensors*, we conclude finally that if the initial state is completely unpolarized all the $\rho'(\mathbf{n}, p; q_c \tau_c 00)$ with even q_c are real, and those with odd q_c are purely imaginary[†]. This fact makes almost as great a simplification in the problem of finding the spin state of the system $c + d$ as the preceding selection rule. In particular, if the particle c is unstable, the selection rules that have been presented make it possible to simplify somewhat the description of the initial state of the reaction of its decay, without any sort of hypotheses about its production.

Similar selection rules can also be found for the

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*One easily convinces oneself from (I.3) that to the Hermitian property of the density matrix, $(m_1 \mid p \mid m_2)^* = (m_2 \mid \rho \mid m_1)$ there corresponds the following property of the c-tensors: $\rho^*(q, \nu) = (-1)^\nu (q, -\nu)$.

†There exists a certain generalization of this selection rule: the contribution to $\rho'(\mathbf{n}, p; q\tau00)$ from each c-tensor $\rho(\mathbf{n}_a, p_a; q_a 0 q_b 0)$ is real if $q + q_a + q_b$ is even, and purely imaginary if $q + q_a + q_b$ is odd.

reaction of the type $a \to c + d$ (directing the z axis respectively perpendicular and parallel to $\mathbf{n}_c$ and regarding particle a as completely unpolarized).

4. We now obtain some consequences that follow from Eq. (2.13) for the reaction $a \to c + d$ in the particular case $i_c = \tfrac{1}{2}$, $i_d = 0$, of which an important example is provided by the decay of the Λ^0 particle: $\Lambda^0 \to p + \pi^-$. First, for $i_d = 0$

$$X (i_c q_c i_c;\ s_1' q' s_2';\ 0 q_d 0)$$

$$= (2i_c + 1)^{-1} (2q_c + 1)^{-1/2} \delta_{q_d 0} \delta_{q_c q'} \delta_{s_1' i_c} \delta_{s_2' i_c}.$$

Second, for $i_c = \tfrac{1}{2}$, from the triads $G_{\tau'}(sl_1' \tfrac{1}{2};\ q q_c;\ sl_2' \tfrac{1}{2})$ in Eq. (2.13) we have $l_1' = s \pm \tfrac{1}{2}$, $l_2' = s \pm \tfrac{1}{2}$, i.e., l_1' and l_2' can differ only by unity. If the elements of the R matrix in Eq. (2.13) are nonvanishing only for transitions that conserve spatial parity, then l_1' and l_2' must have the same parity. We get the rule that $l_1' = l_2' = l' = s + \tfrac{1}{2}$ or else $l_1' = l_2' = l' = s - \tfrac{1}{2}$ depending on the unknown parity of particle a and the parities of c and d. In what follows it is assumed that the ensemble of particles a is a pure state with respect to the variable a [i.e., $\rho_{a_1, a_2}(q\nu) = \rho (a;\ q\nu) \delta_{a a_1} \delta_{a a_2}$].

In Eq. (2.13) there then remains just one nonvanishing element of the R matrix, and if there are no alternative modes of decay of a, then from the unitary property of the S matrix it follows that

$$({}^1\!/_2\, 0\, {}^1\!/_2\, l'\alpha' \mid R^{sE} \mid \alpha) ({}^1\!/_2\, 0\, {}^1\!/_2\, l'\alpha' \mid R^{sE} \mid \alpha)^* = 1. \tag{4.1}$$

If there are other decay schemes $(a \to c' + d')$, then on the right side of Eq. (4.1) we must put in place of unity the total probability w of the decay of a by the scheme $a \to c + d$ ($0 < w < 1$).

If now, furthermore, we integrate $\rho'(\mathbf{n}_c, p_c, a';\ q_c \tau_c 00)$ (with the weight factor $V/(2\pi h)^3$—cf. Appendix II) over the range of momenta $(\mathbf{p}_c, \mathbf{p}_c + \Delta\mathbf{p})$ containing momenta $\mathbf{p}_c$ that make $(p_c \mid E)$ equal to unity, and denote the resulting quantities by $'T_{\tau_c}^{q_c}(\vartheta, \varphi)\,\Delta\Omega$ (index a' omitted), then from Eq. (2.13) we get

$$T_{\tau_c}^{q_c}(\vartheta, \varphi) = \frac{w\,(2s+1)^{-1/2}}{4V\sqrt{2}\,\pi}$$

$$\sum_{q\nu} (-1)^{q_c + \tau_c} G_{\tau_c}(sl'\,{}^1\!/_2, q\,q_c;\ sl'\,{}^1\!/_2) \tag{4.2}$$

$$\times D_{\tau_c, \nu}^q (-\pi, \vartheta, \pi - \varphi)\, \rho\,(\alpha;\ q\nu).$$

On interchange of the upper and lower sets of arguments of the coefficient $G_{\tau_c}(sl'\tfrac{1}{2};\ qq_c;\ sl'\tfrac{1}{2})$, there must appear before G_{τ_c} the factor $(-1)^{2s+1+q+q_c}$ (cf. Ref. 2, page 38). But since these sets are identical, G_{τ_c} is not changed by this, and $2s + 1 + q + q_c$ must be an even number. Since the spin of particle a is half-integral, $2s + 1$ is an even number, and therefore $q + q_c$ must be even. The angular distribution of the decay products—$T_0^0(\vartheta, \varphi)$—is determined by only the even tensors of the initial state of particle a and the polarization of the decay protons (i.e., the c-tensors $T_0^1, T_{\pm 1}^1$ are determined only by the odd ones. By measuring the angular distribution and polarization of particle c we obtain direct information about $\rho(q, \nu)$, the spin and parity of particle a, and, on the other hand, if these latter are known, we can predict all the c-tensors $T_{\tau_c}^{q_c}(\vartheta, \varphi)$.

By integrating the angular distribution $T_0^0(\vartheta, \varphi)$ with respect to ϑ or φ, we can obtain general formulas for distributions that are directly observed experimentally (for example, by integrating over ϑ we obtain the distribution in the angle between the plane of production of the Λ^0 and the plane of its disintegration)[16].

In conclusion it must be pointed out that a considerable part of the material presented here was developed by the writer in collaboration with A. M. Baldin (cf. Ref. 3), who is also to be thanked for discussions of some other points in the work. The writer also thanks Prof. M. A. Markov for his constant interest in the work and L. G. Zastavenko for discussion of a number of questions related to the theory of representations of the rotation group.

APPENDIX I

A quantum-mechanical state can be described not only by a wave function or a density matrix, but also by specifying the average values of a certain set of operators[12]. In particular, we shall describe in this way the spin state of a particle (and of a system of particles). We remark that also what is measured experimentally is not the probability of one or another value of the projection of a spin, but, for example, the average value of the spin vector operator (the so-called polarization of the particle).

Let $\hat{A}$ be an operator relating to the spin variables of a particle with spin i; the state of the particle is described by the density matrix $(m_1 \xi_1 \mid \rho \mid m_2 \xi_2)$, where m_1 and m_2 are magnetic

quantum numbers and ξ are all the other variables of the representation. We can find the average value of $\hat{A}$ in this state, if the matrix elements of $\hat{A}$ are known in the representation in which the density matrix is written

$$\overline{A} = \sum_{m_1 m_2 \xi_1 \xi_2} (m_2\xi_2 \mid \hat{A} \mid m_1\xi_1)$$
$$\times (m_1\xi_1 \mid \rho \mid m_2\xi_2) \equiv \mathrm{Sp}\,(A\rho). \qquad (\mathrm{I}.1)$$

We form from the components $\hat{\sigma}_x$, $\hat{\sigma}_y$, $\hat{\sigma}_z$ of the spin vector operator a tensor of rank q, transform-ing under three-dimensional rotations according to an irreducible representation of weight q, and de-note its components by $\hat{A}^{\,q\nu}$. We remark that by its construction this operator is a contravariant tensor, like $\hat{\sigma}$; i.e., it transforms under rotations like $Y^*_{q\nu}(\mathbf{n})$, and not like $Y_{q\nu}(\mathbf{n})$. For example, the $\hat{A}^{\,1\nu}$ are simply the cyclic (or canonical[11]) components of $\hat{\sigma}$; and transform like the vector $\mathbf{n}$, i.e., like $Y^*_{1\nu}(\mathbf{n}) = (1\nu \mid \mathbf{n})$. The dependence of the matrix elements of $\hat{A}^{\,q\nu}$ on m_2 and m_1, according to the suitably modified (ν is a " contravariant" index!) Wigner-Eckart theorem[8], is given by:

$$(m_2\xi_2 \mid \hat{A}^{q\nu} \mid m_1\xi_1) = \sqrt{2i+1}\,(i\xi_1 \| \hat{A}^q \| i\xi_1)\,(\xi_1 \mid \xi_2)\,(-1)^{i-m_2}(iim_1 - m_2 \mid iiq\nu) \qquad (\mathrm{I}.2)$$

(if the spin variables and the variables ξ are separable, then $(i\xi_1 \mid \hat{A}^q \mid i\xi_1)$ does not depend on ξ_1). Substitut-ing (I.2) into (I.1), we see that to find $\overline{A^{q\nu}}$ one need not know the density matrix directly, but the quantity

$$\rho_{\xi_1,\,\xi_2}(q,\nu) = \sqrt{2i+1}\sum_{m_1,\,m_2}(-1)^{i-m_2}(iim_1 - m_2 \mid iiq\nu)\,(m_1\xi_1 \mid \rho \mid m_2\xi_2), \qquad (\mathrm{I}.3)$$

proportional to $\overline{A^{q\nu}}$ (cf. also Ref. 12). From Eq. (I.3) it follows that the quantities $\rho(q,\nu)$, $q = 0, 1, \ldots, 2i$, $\nu = -q, -q+1, \ldots, q$ can char-acterize the spin state of the particle just as ade-quately as the density matrix. The inverse trans-formation from the q, ν representation to the m_1, m_2 representation has the form:

$$(m_1\xi_1 \mid \rho \mid m_2\xi_2) = (2i+1)^{-1/2}\sum_{q\nu}(-1)^{-i+m_2}$$
$$\times (iim_1 - m_2 \mid iiq\nu)\,\rho_{\xi_1,\,\xi_2}(q,\nu). \qquad (\mathrm{I}.4)$$

In the variables ξ, $\rho_{\xi_1,\xi_2}(q,\nu)$ is just the same sort of density matrix as is $(m_1\xi_2 \mid \rho \mid m_2\xi_2)$. From its origin, we can call $\rho_{\xi_1,\xi_2}(q\nu)$ the means or sta-tistical means of the irreducible spin operators. Therefore we give them the name statistical ten-sors (abbreviated c-tensors) following Ref. 13 (page 735).

If we are interested only in the distribution of probabilities for the particle to have definite values of the variables ξ (for example, the angular distri-bution of the particles) in the state characterized by $(m_1\xi_1 \mid \rho \mid m_2 \xi_2)$, then this will be described by the diagonal elements of the density matrix $\left[\sum_m (m\xi_1 \mid \rho \mid m\xi_2)\right]$. The transformation (I.3) is so chosen that

$$\sum_m (m\xi_1 \mid \rho \mid m\xi_2) = \rho_{\xi_1,\,\xi_2}(0,0),$$

and a beam of unpolarized particles would be de-scribed by the c-tensors $\rho_{\xi_1,\xi_2}(0,0)\,\delta_{q0}\delta_{\nu0}$.

The c-tensors of the initial state, $\rho(q_a\nu_aq_b\nu_b)$, used in Sec. 2 are simply products of pairs of c-tensors $\rho(q_a\nu_a)$ and $\rho(q_b\nu_b)$ characterizing the separate spin states of particles a and b.

After the reaction, particles c and d are spa-tially separated, and consequently do not interact. It can be shown that just as the wave function of a system of two noninteracting particles separates as a product of the wave functions of the two parti-cles, the density matrix and the c-tensors have a similar property:

$$\rho(q_c\nu_cq_d\nu_d) = \rho(q_c\nu_c\,00)\,\rho(00\,q_d\nu_d)\,/\,\rho(0,0,0,0),$$
$$(\mathrm{I}.5)$$

where $\rho(q_c\nu_c00)$ are the c-tensors of particle c and $\rho(00q_d\nu_d)$ are those of particle d. These c-tensors suffice to characterize completely the final spin of the system after the reaction, if c and d do not form a bound system.

We note that the "tensor moments" introduced in Ref. 3,

$$T^q_\nu = \sqrt{2i+1}\,\Sigma\,(-1)^{-i-m_1}$$
$$\times (ii - m_1m_2 \mid iiq\nu)\,(m_1 \mid \rho \mid m_2)$$

(there is a misprint in Eq. (5) of Ref. 3) are related to the $\rho(q, \nu)$ defined by Eq. (I.3) in the following way:

$$T_\nu^q = (-1)^{q+\nu} \rho (q, -\nu) = (-1)^q \rho^* (q, \nu).$$

APPENDIX II

We shall normalize all the wave functions (and the density matrix) of our problem in such a way that the system in the state η is located in the volume V (a three-dimensional sphere of radius R) at all times, with probability 1; in the x representation this condition is written

$$\int_V (x \mid \eta)^* (x \mid \eta) \, d^3 x = 1. \qquad \text{(II.1)}$$

The normalization of the wave function of the state η in other representations must correspond to this normalization.

The wave function $(\theta \Phi r \mid lm \mid \mathbf{k} \mid)$ ($\mathbf{k}$ is the wave vector) normalized in accordance with Eq. (II.1) has the form $g_{lk}(r) Y_{lm}(\theta, \Phi)$, where $g_{lk}(r)$ is given by Eq. (9) in Ref. 14. Furthermore, in accordance with the relation $(\vartheta\varphi k \mid l\mu k) = (\vartheta\varphi k \mid \theta \Phi r)(\theta \Phi r \mid l\mu k)$ (integration over θ, Φ, ϑ, and r is understood; ϑ, φ are the spherical angles of $\mathbf{k}$), we can obtain the correctly normalized wave function or transformation function $(\vartheta\varphi k \mid l\mu k)$. (We note that $(\vartheta\varphi k \mid \theta \Phi r)$ is equal to the expression (13) in Ref. 14, multiplied by $V^{-\frac{1}{2}}$). From these considerations we get:

$$(\vartheta\varphi p \mid l\mu E) = \frac{2\pi h \sqrt{2R}}{V\sqrt{V}} \frac{i^{-l}}{p} Y_{l\mu} (\vartheta, \varphi) (p \mid E), \text{ (II.2)}$$

where $(p \mid E)$ is defined analogously to $(\mathbf{p} \mid \mathbf{p}')$ (see below). The presence of the factor i^{-l} assures the invariance of the action of the time-reversal operator on wave functions with definite l and μ, in relation to the composition of angular momenta[15] (i.e., the result of the action of this operator has the same form for $\psi_{l\mu}$ as for $\psi_{l_1\mu_1}$ and $\psi_{l_2\mu_2}$, if $l = l_1 + l_2$). We note that

$$(l\mu E \mid \vartheta\varphi p) = (\vartheta\varphi p \mid l\mu E)^*$$
$$= (-1)^{-l+\mu} (\vartheta\varphi p \mid l, -\mu, E). \qquad \text{(II.3)}$$

Similarly it can be shown that $(\mathbf{p} \mid \mathbf{p}')$ has the form of Eq. (17) in Ref. 14 (i.e., for $\mathbf{p} = \mathbf{p}'$, $(\mathbf{p} \mid \mathbf{p}') = 1$, and for $\mathbf{p} \neq \mathbf{p}'$, $(\mathbf{p} \mid \mathbf{p}') = 0$). If we require that, in accordance with the formalism of Dirac[5], $(\mathbf{p}'' \mid \mathbf{p})(\mathbf{p} \mid \mathbf{p}')$ be equal to $(\mathbf{p}'' \mid \mathbf{p}')$, then it turns out that the normalization (II.1) requires that the inte-

gration over $\mathbf{p}$ understood in $(\mathbf{p}'' \mid \mathbf{p})(\mathbf{p} \mid \mathbf{p}')$ be carried out with the weight factor $V/(2\pi h)^3$ (which is equivalent to a certain summation). According to Dirac (cf. Ref. 5, Sec. 24), this factor must be introduced into every formula in which integration over a momentum occurs.

In connection with the normalization (II.1) there arises the problem of obtaining from $\rho'(\mathbf{n}_c, p_c; q_c\nu_c q_d\nu_d)$ quantities that can be directly compared with the experimental results for the reaction $a + b \to c + d$. What is the meaning of $\rho'(p_c; q_c\nu_c q_d\nu_d)$? There is a physical system consisting initially (at the time $-T$, where $T = R/v$, and v is the absolute value of the relative velocity of particles a and b: $v = |\mathbf{v}_a| + |\mathbf{v}_b|)$ of the particles a and b existing in the volume V; its state is described by the quantities

$$((N \cdot \rho (\mathbf{p}_a; q_a \nu_a q_b \nu_b) (\mathbf{p}'_a \mid \mathbf{p}_a) (\mathbf{p}''_a \mid \mathbf{p}_a).$$

Then $\rho'(\mathbf{p}_c; 0, 0, 0, 0)$ is the probability of the appearance in the volume V at time $+T$ of particles c and d with momentum $\mathbf{p}_c$. The remaining ρ' with q_c, q_d, ν_c, $\nu_d \neq 0$ are quantities proportional to the mean values of the corresponding spin operators (cf. Appendix I) in the ensemble of particles with that momentum.

For the comparison with experiment we need first of all to know how many of particles c and d will appear per second with momenta in the range $(\mathbf{p}_c, \mathbf{p}_c + \Delta\mathbf{p})$, if the constant incident current and the particle density of the target are prescribed (and not the number of particles a and b in a certain volume). To get this we must first divide $\rho'(\mathbf{p}_c; 0, 0, 0, 0)$ by $2T$ and integrate (with weight factor $V/(2\pi h)^3$) over the range $(\mathbf{p}_c, \mathbf{p}_c + \Delta\mathbf{p})$. If we further multiply the quantity obtained by V/v (normalization to unit current), we get the differential cross-section of the reaction, $\Delta\sigma(\mathbf{n}_c)$:

$$\Delta\sigma (\mathbf{n}_c) = \frac{h^2}{4p_a^2} \frac{(4\pi)^2}{N} \rho' (\mathbf{n}_c, p_c'; 0, 0, 0, 0) \Delta\Omega, \qquad \text{(II.4)}$$
$$N = (2\pi h)^4 (p_c; p_a)^2 (2R)^2 [V^2 p_a^2 p_c^2]^{-1};$$

p_c' is the value of the modulus of the momentum $\mathbf{p}_c$ that is required by the law of conservation of energy. It is assumed that momenta with the absolute value p_c' exist in the range $(\mathbf{p}_c, \mathbf{p}_c + \Delta\mathbf{p})$; $\Delta\Omega$ is the solid angle of this range.

One could obtain quantities $\rho'_{\text{cur}}(\mathbf{p}_c; q_d\nu_c q_d \nu_d)$ normalized to unit current, and related to

$\rho'(\mathbf{p}_c; q_c \nu_c q_d \nu_d)$ in just the same way that $\Delta\sigma(\mathbf{n}_c)$ is related to $\rho'(\mathbf{n}_c, \mathbf{p}_c; 0,0,0,0)$. They would give the mean values of certain spin operators in an ensemble of particles with momenta in the range $(\mathbf{p}_c, \mathbf{p}_c + \Delta\mathbf{p})$. But the mean values so obtained would depend not only on the nature of the spin state, but also on the number of particles in the ensemble. Therefore the quantity used to characterize just the spin state is the mean value of the operator $\hat{A}^{q\nu}$ (for particle c, for example), calculated for one particle:

$$\overline{A^{q\nu}}(\mathbf{p}_c) = (i_c\|\hat{A}^q\|i_c)\,\rho_{\Pi\mathrm{T}}(\mathbf{n}_c,\,p_c';\,q,\,\nu,\,0,\,0)\,/\,\Delta\sigma(\mathbf{n}_c)$$
$$= (i_c\|\hat{A}^q\|i_c)\,\rho(\mathbf{n}_c,\,p_c;\,q\nu\,00)\,/\,\rho(\mathbf{n}_c,\,p_c;\,0,\,0,\,0,\,0).$$
$$\text{(II.5)}$$

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Electric Monopole Transitions of Atomic Nuclei

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Processes arising in $E0$ transitions in nuclei are considered. Equations are derived for the probabilities of shell electron conversion, pair production, two-photon transitions, and electron scattering cross section involving excitation of the $E0$ nuclear transition. The calculations are carried out with Coulomb functions of the electron with allowance for the finite dimensions of the nucleus. The conversion probability in an $E0$ nuclear transition is compared with the competing $E2$ and $M1$ nuclear transitions. Some estimates are given for the $E0$ nuclear transition matrix element for various single-particle and collective nuclear models.

SINGLE-PHOTON NUCLEAR TRANSITIONS between states of zero spin are forbidden by the law of conservation of angular momentum. In this case the radiation transition occurs by emission of two (or more) quanta. A distinction is made between two types of transitions: the $M0$-transition, with a change in parity ($0^{\pm} \to 0^{\mp}$), and the $E0$ transition, in which parity remains unchanged ($0^{\pm} \to 0^{\pm}$). In the second case, shell electron conversion or pair production is possible in addition to the two-photon transition. The $E0$ conversion differs substantially from other multipole conversion processes in that the monopole potential is localized in the region of the nucleus, while in other conversion processes