where

$$L_1 = T^{-3} \ln \frac{T + v(1-\gamma)}{T - v(1-\gamma)}, \quad L_2 = \ln \frac{v_0^2 + v_0 v(1-\gamma) - \gamma}{v_0^2 - v_0 v(1-\gamma) - \gamma}, \quad L_3 = \ln \frac{1+v}{1-v}; \tag{24}$$

$$T^2 = \gamma^2 + v_0^2 - 2\gamma v_0 \cos \vartheta_0, \quad \Delta_0 = 1 - v_0 \cos \vartheta_0. \tag{25}$$

Equations (21)–(23) are valid for values of ε_0 which are not too large ($\varepsilon_0 \ll 137 \, mZ^{1/3}$) and for values of γ which are not too small, for which the screening effect is unimportant.

[1] M. May and G. C. Wick, Phys. Rev. 81, 628 (1951).

[2] M. M. May, Phys. Rev. 84, 265 (1951).

[3] Gluckstern, Hull, and Breit, Phys. Rev. 90, 1026 (1953).

[4] R. L. Gluckstern and M. H. Hull, Jr., Phys. Rev. 90, 1030 (1953).

[5] L. N. Rosentsveig, J. Exptl. Theoret. Phys. (U.S.S.R.) 31, 520 (1956), Soviet Physics JETP 4, 455 (1957).

[6] Ia. B. Zel'dovich, Dokl. Akad. Nauk SSSR 83, 63 (1952).

[7] F. W. Lipps and H. A. Tolhoek, Physica 20, 85 (1954).

[8] W. H. McMaster, Am. J. Phys. 22, 351 (1954).

[9] A. I. Akhiezer and V. B. Berestetskii, *Quantum Electrodynamics*, Moscow, 1953.

[10] A. A. Kresnin and L. N. Rosenzweig, J. Exptl. Theoret. Phys. (U.S.S.R.) 32, 353 (1957), Soviet Phys. JETP 5, 288 (1957).

Translated by W. M. Whitney
225

Theory of Kinetic Phenomena in Liquid He³

A. A. ABRIKOSOV AND I. M. KHALATNIKOV

Institute for Physical Problems, Academy of Sciences, U.S.S.R.

(Submitted to JETP editor May 9, 1956)

J. Exptl. Theoret. Phys. (U.S.S.R.) 32, 1083-1091 (May, 1957)

The kinetic coefficients of liquid He³ have been computed on the basis of the theory of a Fermi liquid by Landau. The temperature dependences of the coefficients and their numerical order of magnitude have also been computed.

IN THE PRESENT WORK, which is based on the theory of a Fermi liquid developed by Landau[1], we shall consider the problem of the viscosity and thermal conductivity of He³. In accord with the Landau theory, the excitation energy in a Fermi liquid is a functional of the distribution function n. At temperatures close to $T = 0$, where the diffuse region of the Fermi function is not large, we can, according to the Landau theory, represent this functional dependence in the form of a decomposition in the deviation of the distribution function from its equilibrium value at $T = 0$. Limiting ourselves to terms up to first order of smallness, we have

$$\varepsilon = \varepsilon(p) + \int f(\mathbf{p}, \mathbf{p}') \nu d\tau',$$
$$d\tau = 2 dp_x dp_y dp_z / (2\pi\hbar)^3, \tag{1}$$

where ν is the difference between the actual distribution function and its value at $T = 0$.

It is most natural to consider that the distribution is the Fermi sphere at $T = 0$. Then at not too high temperatures the excitation energy will be described by the expression

$$\varepsilon(p) = a + p_0 (p - p_0) / m, \tag{2}$$

where p_0 is the limiting momentum and a and m are constants (in the ideal gas case, this expression becomes $\varepsilon = p^2/2m$). By Ref. 2, it follows from the measurement of the density and entropy of He³ that $p_0/\hbar = 0.76 \times 10^8$ cm⁻¹, $m = 1.43 \, m_{\mathrm{He}^3}$. In view of the fact that the energy ε enters into the Fermi distribution in the combination $\varepsilon - \mu$, the constant a

can be added to μ; as we shall see, it is inconsequential in the calculation of the kinetic coefficients.

As was shown in Ref. 2, it is possible that the temperature dependence of thermodynamical quantities in He² is better explained if we take as the ground state not the Fermi sphere but a thin spherical shell, such that the excitation spectrum has the form

$$\varepsilon(p) = a + (p - p_0)^2/2m. \tag{3}$$

We shall discuss the results which follow from such a form of the spectrum in Sec. 4.

1. THE KINETIC (BOLTZMANN) EQUATION

Let the motion in the liquid under consideration take place with a slightly inhomogeneous (in the coordinates) velocity u and let there be a small temperature gradient. In this case the distribution function will differ but slightly from its equilibrium value

$$n = n_0 + \delta n, \tag{4}$$

where

$$n_0 = \left[\exp\left\{\frac{\varepsilon - \mathbf{p}\mathbf{u} - \mu}{kT}\right\} + 1\right]^{-1}, \quad |\delta n| \ll n_0. \tag{5}$$

The quantity δn is found from the kinetic or Boltzmann equation

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial \mathbf{r}}\frac{\partial \varepsilon}{\partial \mathbf{p}} - \frac{\partial n}{\partial \mathbf{p}}\frac{\partial \varepsilon}{\partial \mathbf{r}} = I(n). \tag{6}$$

As usual, we must substitute the function n_0 on the left side of the Boltzmann equation. In this case, we shall consider that at the point in the liquid being considered, $u = 0$. Substituting (5) in Eq. (6), we get

$$\frac{\partial n_0}{\partial t} - \frac{1}{3}\frac{\partial n_0}{\partial \varepsilon}\mathbf{p}\frac{\partial \varepsilon}{\partial \mathbf{p}}\operatorname{div}\mathbf{u}$$

$$- \frac{1}{2}\frac{\partial n_0}{\partial \varepsilon}\left(p_i\frac{\partial \varepsilon}{\partial p_k} - \frac{1}{3}p_l\frac{\partial \varepsilon}{\partial p_l}\delta_{ik}\right)$$

$$\times \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial u_l}{\partial x_l}\right) \tag{7}$$

$$- \frac{\partial n_0}{\partial \varepsilon}\left(\frac{\varepsilon - \mu}{T} - s\right)\frac{\partial \varepsilon}{\partial \mathbf{p}}\nabla T = I(n),$$

where s is the entropy per particle, and in all terms except $\partial n_0/\partial t$, we take $\varepsilon = \varepsilon(p)$, since they

contain only small values for the velocity and temperature gradients.

We now transform $\partial n_0/\partial t$ and shall show that this expression no longer depends on the term with f in Eq. (1). In accord with Eq. (5), we can write

$$\delta n_0 = \frac{\partial n_0}{\partial \varepsilon}\left(\delta\varepsilon - \delta\mu - \frac{\varepsilon - \mu}{T}\delta T\right) \tag{8}$$

(in the variation, we consider the rate of n to be fixed). But, as is well known, the derivative $\partial n/\partial \varepsilon$ is different from zero in a small circle about the point $\varepsilon = \mu$, where it is a rapidly changing function. Therefore, we can consider the quantities in the bracket relative to this point [corrections will have the relative order $(kT/\mu)^2$]. In this variation, $\delta\varepsilon$ and $\delta\mu$ are arbitrary and are not equal to each other. We note incidentally that the second term in the energy ε of (1) is of importance only in $\delta\varepsilon$. On the other hand, following the Landau theory, the distribution function is normalized by the relation

$$\int n_0 d\tau = N, \tag{9}$$

where N is the number of atoms per unit volume. Differentiating this relation, we find

$$\delta N = \int \frac{\partial n_0}{\partial \varepsilon}\left(\delta\varepsilon - \delta\mu - \frac{\varepsilon - \mu}{T}\delta T\right)d\tau$$

$$\approx -\left[(\delta\varepsilon - \delta\mu)\frac{d\tau}{d\varepsilon}\right]_{\varepsilon=\mu}, \tag{10}$$

since $\int(\partial n_0/\partial \varepsilon)\,d\varepsilon = -1$. Comparing Eqs. (8) and (10), we get

$$\frac{\partial n_0}{\partial t} = -\frac{\partial N}{\partial t}\frac{\partial n_0}{\partial \varepsilon}\left(\frac{d\varepsilon}{d\tau}\right)_{\varepsilon=\mu}.$$

Thus we have succeeded in eliminating $\delta\varepsilon$, while in the remaining terms, and therefore in the last equation, we can consider $\varepsilon = \varepsilon(p)$.

The number of atoms N satisfies the continuity equation

$$(\partial N/\partial t) + N\operatorname{div}\mathbf{u} = 0. \tag{11}$$

Thus the term with $\partial n_0/\partial t$ gives the contribution to the term with $\operatorname{div}\mathbf{u}$ and as a result this term takes the form:

$$-\frac{\partial n_0}{\partial \varepsilon}\left(\frac{1}{3}\mathbf{p}\frac{\partial \varepsilon}{\partial \mathbf{p}} - N\left(\frac{d\varepsilon}{d\tau}\right)_{\varepsilon=\mu}\right)\operatorname{div}\mathbf{u}. \tag{12}$$

The collision integral appears on the right side of Eq. (7):

$$I(n) = -\int w[n_1 n_2 (1 - n_1') (1 - n_2')$$

$$- (1 - n_1)(1 - n_2)(n_1' n_2')]$$

$$\times \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1' - \mathbf{p}_2') \qquad (13)$$

$$\times \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_1' - \varepsilon_2')\, d\tau_2 d\tau_1' d\mathbf{p}_2'.$$

Introducing the function

$$\delta n = -n_0(1 - n_0)\,\psi = kT\,(\partial n_0/\partial \varepsilon)\,\psi \qquad (14)$$

and neglecting quadratic terms, we can rewrite it in the form

$$I(n) = \int w n_{01} n_{02}(1 - n_{01}')(1 - n_{02}')$$

$$\times(\psi_1 + \psi_2 - \psi_1' - \psi_2')\,\delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1' - \mathbf{p}_2') \qquad (15)$$

$$\times \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_1' - \varepsilon_2')\, d\tau_2 d\tau_1' d\mathbf{p}_2'.$$

The collision probability w depends, generally speaking, on all four momenta. However, in this temperature region, where Eq. (1) is generally valid, the momenta whose absolute magnitude is close to the bounding value on the Fermi surface are of interest. Therefore, we can consider that w depends only on θ—the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$ and φ—the angle formed by the planes $(\mathbf{p}_1, \mathbf{p}_2)$ and $(\mathbf{p}_1', \mathbf{p}_2')$.

We can put the function ψ in a different form depending on which of the kinetic coefficients is being calculated, i.e., which of the terms on the left side we keep. However, independently of this, we apply only one method of transformation of the integrals. We make use of the fact that the momenta of the particles in the basic region of integration differ slightly in absolute magnitude from the limiting momentum p_0. If we turn the plane of the vectors $(\mathbf{p}_1, \mathbf{p}_2)$ relative to the axis, directed along $\mathbf{p}_1 + \mathbf{p}_2$, through the angle φ, so that this plane coincides with the plane $(\mathbf{p}_1, \mathbf{p}_2)$, then we get the diagram shown in the Figure. It is clear that the vector $\mathbf{f}$ will be small in absolute magnitude, and the
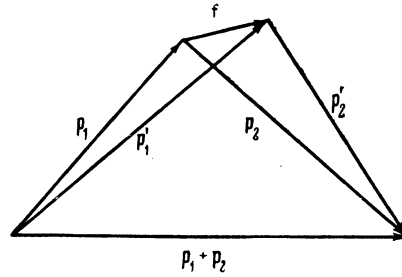
angles between all the momenta and the $\mathbf{p}_1 + \mathbf{p}_2$ axis will be approximately equal to $\theta/2$ or to $-\theta/2$. Then we obtain

$$p_1' \approx p_1 + f_z \cos\frac{\theta}{2} + f_r \sin\frac{\theta}{2},$$

$$p_2' \approx p_2 - f_z \cos\frac{\theta}{2} + f_r \sin\frac{\theta}{2}, \qquad (16)$$

where f_z is the component of $\mathbf{f}$ along the $\mathbf{p}_1 + \mathbf{p}_2$ axis, while f_r is the perpendicular component. Integrating over $d\mathbf{p}_2'$, we solve the δ-function of the momenta, while we replace the integral over $d\tau_1'$ by an integral over $df_r\, df_z\, d\varphi$, introducing cylindrical coordinates with axis along $\mathbf{p}_1 + \mathbf{p}_2$:

$$d\tau_1' = 2p_0 \sin(\theta/2)\, df_r df_z d\varphi\,/(2\pi\hbar)^3. \qquad (17)$$

We now introduce the following notation:

$$x = (\varepsilon_1' - \mu)/kT, \quad y = (\varepsilon_2' - \mu)/kT,$$

$$t = (\varepsilon - \mu)/kT, \quad \varkappa = \mu/kt \qquad (18)$$

with ε from (2). Transformation from the variables f_r, f_z, to x and y is easily accomplished with the help of Eq. (16). We then obtain

$$d\tau_1' \int d\tau_2 \delta(\varepsilon) = \frac{m^3 (kT)^2\, dx\, dy\,(d\Omega/2\pi)\,(d\varphi_2/2\pi)}{8\pi^4 \hbar^6 \cos(\theta/2)}, \qquad (19)$$

where $d\Omega$ denotes the solid angle differential $\sin\theta d\theta d\varphi$ on which $w(\theta, \varphi)$ depends. We note that, in view of the indistinguishability of particles, the angle φ changes only within the limits 0 to π.

2. COEFFICIENT OF VISCOSITY

We begin with the calculation of the viscosity coefficient. In this case, the tensor term on the left side of Eq. (7) is dominant. From symmetry considerations, it is evident that ψ must have the form

$$\psi = \frac{1}{2}q(t)\left(p_i \frac{\partial \varepsilon}{\partial p_k} - \frac{1}{3} p_l \frac{\partial \varepsilon}{\partial p_l}\delta_{ik}\right)\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial u_m}{\partial x_m}\right). \qquad (20)$$

Upon substitution of this expression in the collision integral, the second factor can be transformed by the addition theorem of spherical functions. After integration over the angle $d\varphi_2$, only the first terms are left, i.e.,

$$P_2(\theta_1 + \theta_2) \to P_2(\theta_1) P_2(\theta_2), \text{ where } P_2(\theta) = {}^3/_2 \cos^2\theta - 1.$$

As a result, we get the following equation for q from Eq. (7):

$$n_0(t)[1 - n_0(t)] = \frac{(mkT)^3}{8\pi^4\hbar^6\cos(\theta/2)} \int \frac{d\Omega}{2\pi} \int_{-\varkappa}^{\infty} dx \int_{\substack{-\varkappa \\ x+y>t-\varkappa}}^{\infty} dy\, w(\theta,\varphi) \tag{21}$$

$$\times n_0(t) n_0(x+y-t)[1-n_0(x)][1-n_0(y)][q(t) + q(x+y-t) P_2(\theta) - q(x) P_2(\theta_1') - q(y) P_2(\theta_2')].$$

In view of the fact that, in the temperature region we have considered, only the values of x, y, and t much smaller than $\varkappa$ are important, we can consider the lower limit in Eq. (21) to be $-\infty$. Under these conditions, and assuming that q is a symmetric function (this is verified by the result), we are easily convinced that all terms with different q can be put in the same form in the sense of its dependence on x and y. Thus the bracketed expression can be put in the form:

$$q(t) + q(x)[P_2(\theta) - P_2(\theta_1') - P_2(\theta_2')].$$

Expressions for the spherical functions in terms of the angles θ and φ are easily obtained with the aid of the Figure. Integrating over the variable y, on which q no longer depends, we get the following expression after some transformations:

$$\frac{8\pi^4\hbar^6}{(mkT)^3} = \overline{\frac{w(\theta,\varphi)}{\cos(\theta/2)}\left[\frac{3}{4}(1-\cos\theta)^2\sin^2\varphi - 1\right]}$$

$$\times \left\{\int_t^{\infty}\frac{dx\,x[q(x+t) + q(x-t)]}{e^x - 1} + 2\int_0^{\infty}\frac{dx\,xq(x)}{e^x + 1}\right. \tag{22}$$

$$\left. + \int_0^t dx\,xq(x-t)\right\} + \overline{\frac{w(\theta,\varphi)}{\cos(\theta/2)}}\,q(t)\,\frac{\pi^2 + t^2}{2},$$

where the bar denotes averaging over the solid angle.

This expression is very complicated. However, analysis shows that for arbitrary assumptions concerning the form of $w(\theta,\varphi)$, the error in the viscosity coefficient will be less than 10% if we simply take it into account that the values we need are $t^2 \ll \pi^2$. In this case, q turns out to be constant, equal to

$$q = \frac{64\pi^2\hbar^6}{3\,(mkT)^3}\left[\overline{\frac{w(\theta,\varphi)}{\cos(\theta/2)}(1-\cos\theta)^2\sin^2\varphi}\right]^{-1}. \tag{23}$$

In accordance with Landau's theory[1], the momentum flux is equal to

$$\Pi_{ik} = \int p_i \frac{\partial\varepsilon}{\partial p_k} n\,d\tau + \delta_{ik}\left(\int n\varepsilon\,d\tau - E\right). \tag{24}$$

However, it is necessary to recall that the energy is also a functional of the distributing function. Substituting Eqs. (4) and (11) with $\nu = \delta n$, we get

$$\Pi_{ik} = \int p_i \frac{\partial\varepsilon(p)}{\partial p_k}\delta n\,d\tau$$

$$- \int \delta n f(\mathbf{p},\ \mathbf{p}') \frac{\partial n'}{\partial\varepsilon'} p_i' \frac{\partial\varepsilon(p')}{\partial p_k'}\,d\tau\,d\tau'. \tag{25}$$

Substituting δn in accordance with Eqs. (14), (20) and (23), we find the value of the viscosity coefficient, defining it as the coefficient of proportionality between Π_{ik} and $(\delta u_i/\delta x_k) + (\delta u_k/\delta x_i) - {}^2/_3\,\delta u_l/\delta x_l$ with opposite sign:

$$\eta = \frac{64}{45}(kT)^{-2}\frac{\hbar^3 p_0^5}{m_4}\left[1 + \overline{f(\theta)P_2(\theta)}\frac{p_0 m}{\pi^2\hbar^3}\right]$$

$$\times \left[\overline{\frac{w(\theta,\varphi)}{\cos(\theta/2)}(1-\cos\theta)^2\sin^2\varphi}\right]^{-1}. \tag{26}$$

Thus, it is shown that $\eta \sim T^{-2}$. This dependence was predicted earlier by Pomeranchuk[3] on the basis of qualitative considerations. So far as the numerical value of the viscosity is concerned, this depends on the definite form of averaging $f(\theta)$ and $w(\theta,\varphi)$, and therefore cannot be determined precisely. However, with the help of Eq. (26), we can estimate the order of magnitude. We shall assume that $w(\theta,\varphi)$ does not depend upon the angle φ, and make use of the fact that, from Landau's theory,

$$w(\theta, 0) = 2\pi f^2(\theta)/\hbar. \tag{27}$$

If we consider the quantity f to be approximately given by $f_0 + f_1 \cos \theta$, we can find the coefficients f_0 and f_1 from experimental data on the compressibility (4) and the relation between m and m_{He^3}. This gives

$$(mp_0 / \pi^2 h^3) f = 5,4 + 1,3 \cos \theta. \qquad (28)$$

We then obtain the following expression for the order of magnitude of the viscosity:

$$\eta \sim 1 \cdot 10^{-6} T^{-2} \text{ poise}. \qquad (29)$$

Simple analysis shows that the second viscosity is a quantity of much higher order of smallness than η. Actually, taking into account that the collision integral for both cases is approximately identical, we can compare both coefficients, taking the corresponding integrals from the tensor term on the left side of Eq. (7), and from Eq. (12). Recalling that $p_0^3 / 3\pi^2 \hbar^3 = N$, we find that the terms of zeroth order

for the second viscosity vanish. Thus $\zeta/\eta \sim (kT/\mu)^2$.

3. COEFFICIENT OF THERMAL CONDUCTIVITY

We now proceed to the thermal conductivity. The function ψ has in this case the form

$$\psi = q \left(\frac{\varepsilon(p) - \mu}{kT} \right) \left(\frac{\partial \varepsilon}{\partial \mathbf{p}} \nabla T \right). \qquad (30)$$

In place of Eq. 12, we get here the same equation but with $\cos \theta$ in place of $P_2(\theta)$ in the collision integral, and the factor $(\epsilon - \mu)/T - s$ on the left side. The presence of such a factor shows that the desired function ought to contain both symmetric and anti-symmetric parts, namely,

$$q(t) = q_s(t) + q_a(t). \qquad (31)$$

Substitution in the collision integral gives two equations

$$\frac{8\pi^4 \hbar^6}{(mkT)^3} s = \overline{\frac{w(\theta, \varphi)}{\cos(\theta/2)}} \left[q_s(t) \frac{\pi^2 + t^2}{2} - \int_0^\infty dx \frac{x[q_s(x+t) + q_s(x-t)]}{e^x - 1} - 2 \int_0^\infty \frac{dx \, x q_s(x)}{e^x + 1} - \int_0^\infty dx \, x q_s(x-t) \right], \qquad (32)$$

$$\frac{8\pi^4 \hbar^6 k}{(mkT)^3} t = \overline{\frac{w(\theta, \varphi)(1 + 2\cos\theta)}{\cos(\theta/2)}} \left[\int_0^\infty dx \frac{x[q_a(x-t) - q_a(x+t)]}{e^x - 1} \right.$$

$$\left. + 2t \int_0^\infty \frac{dx \, q_a(x)}{e^x + 1} + \int_0^t dx \cdot x q_a(x-t) \right] + \overline{\frac{w(\theta, \varphi)}{\cos(\theta/2)}} q_a(t) \frac{\pi^2 + t^2}{2}. \qquad (33)$$

However, in the given case, the Boltzmann equation does not define the complete solution of the problem. To it must be added an additional condition which expresses the conservation of the total momentum of the system, which reduces to the equality $\int \mathbf{p} \delta n \, d\tau = 0$ or, in other words,

$$\int \frac{\partial n}{\partial \varepsilon} \left(\mathbf{p} \, \frac{\partial \varepsilon}{\partial \mathbf{p}} \right) q(t) \, d\tau = 0. \qquad (34)$$

To find the odd part $q_a(t)$, it suffices to solve Eq. (33), in the same way as was done in considering the viscosity. So far as the even part $q_s(t)$ is concerned, the situation is quite different. In the first place we must note that $q_s = \text{const}$ makes the right side of Eq. (32) identically equal to zero. Therefore we must determine the constant term $q_s(0)$ not from this equation, but from Eq. (34). Here it is easy to show that the additional terms in

$q_s(t)$, namely, $a_2 t^2 + a_4 t^4 + \ldots$, generally do not contribute to the thermal conductivity. Actually, the appearance of such a term, for example, $a_m t^{2m}$, should change the constant term in $q_s(t)$ by an amount a_m^0 so that

$$\int \frac{\partial n}{\partial \varepsilon} \mathbf{p} \, \frac{\partial \varepsilon}{\partial \mathbf{p}} [a_m t^{2m} + a_m^0] \, d\tau = 0$$

From this condition, we get

$$a_m^0 = -a_m (2m)! \, R_m, \qquad R_m = \int_0^\infty \frac{z^{2m-1}}{e^z + 1} dz.$$

In the calculation of the energy flow, we must compute integrals of the type

$$\int \frac{\partial n}{\partial \varepsilon} F(\varepsilon) (a_m t^{2m} + a_m^0) \, d\tau$$

$$= \left(F \frac{d\tau}{d\varepsilon} \right)_{\varepsilon = \mu} [a_m (2m)! \, R_m + a_m^0] = 0.$$

We thus come to the conclusion that, to find the thermal conductivity, we need only solve Eq. (33), and select the constant term in q_s in order to satisfy the condition (34). In practice, as in the viscosity case, it suffices to find a solution under the assumption $t^2 \ll \pi^2$. Here we obtain

$$q = \frac{24\pi^2\hbar^6 k}{(mkT)^3}\left(t - \frac{\pi^2 kTm}{p_0^2}\right)\left[\frac{w(\theta,\varphi)(1-\cos\theta)}{\cos(\theta/2)}\right]^{-1} \tag{35}$$

The energy flow is equal to (see Ref. 1):

$$\mathbf{Q} = \int \varepsilon\,\frac{\partial\varepsilon}{\partial\mathbf{p}}\,n d\tau. \tag{36}$$

Substituting (4), we have

$$\mathbf{Q} = \int \varepsilon(p)\,\frac{\partial\varepsilon(p)}{\partial\mathbf{p}}\,\delta n d\tau$$
$$- \int \delta n f(\mathbf{p},\mathbf{p}')\,\frac{\partial n'}{\partial\varepsilon'}\,\varepsilon(\mathbf{p}')\,\frac{\partial\varepsilon(\mathbf{p}')}{\partial\mathbf{p}'}\,d\tau d\tau'. \tag{37}$$

Substituting δn in accord with (14), (30) and (35), we determine the value of the coefficient of thermal conductivity:

$$\varkappa = \frac{8}{3}\frac{\pi^2\hbar^3 p_0^3}{m^4 T}\left(1 + \frac{p_0^3 m}{2\pi^2\hbar^3}\frac{\partial}{\partial p}\left[\frac{f(p,p_0)\cos\theta}{p}\right]_{p=p_0}\right)\left[\frac{w(\theta,\varphi)(1-\cos\theta)}{\cos(\theta/2)}\right]^{-1}. \tag{38}$$

Thus the temperature dependence of the coefficient of thermal conductivity is expressed by a T^{-1} law, which also coincides with the qualitative prediction of Pomeranchuk[3].

Estimating the numerical value of $\varkappa$ in a fashion similar to what was done in the viscosity case, we obtain the relation*

$$\varkappa \sim (40/T)\ \text{erg/cm-sec-deg}. \tag{39}$$

4. LIMITS OF APPLICABILITY

The resulting expressions for the coefficients of viscosity and thermal conductivity in each case cease to be valid in the temperature region $kT \sim \mu$. However, in addition there exists a limitation which moves back the region of applicability of the theory in the direction of much lower temperatures. It consists in the fact that the energy interval of excitation of order kT of interest to us must be much greater than the quantum uncertainty in the energy arising from the collisions, i.e.,

$$\tau \gg \hbar/kT, \tag{40}$$

where τ is the time between collisions. We note that fulfillment of condition (40) is required not

only for the calculation of the kinetic coefficients, as was the case earlier, but also for the validity of the entire theory of the Fermi liquid[1].

We can determine the collision time for He³ from the Boltzmann equation if we write it in the form

$$Dn = -\delta n/\tau,$$

where Dn denotes the left side of the equation. Comparing this expression with Eqs. (7), (14), (20), (30), we get

$$\tau \sim qkT. \tag{41}$$

For the different processes which we have considered, the functions q are different, but for an estimate, we can make use of any one q, for example, the one obtained in the determination of the viscosity. In this case, condition (40), with consideration of the numerical values of the coefficients, gives:

$$T < 0,05°. \tag{42}$$

It is possible that such a small value is connected with the inaccuracy of our estimate, in the first degree arising from the replacing of $w(\theta,\varphi)$ by $w(\theta,0)$. However, it is sufficiently clear that even in the best case one can hardly expect that the theory would be suitable at temperatures higher than 0.1–0.2°. This completely corresponds to the fact that the theoretical curve computed in Ref. 2 for the entropy or thermal capacity ceases to correspond to the experimental data above 0.3° K.

Now let us consider briefly what would be obtained in the case of a spectrum of type (3). It is not difficult to see that in this case the tempera-

---

*In the estimate, we pick out the term with f in Eq. (35). An argument in support of this is the fact that if we take into account

$$\frac{\partial}{\partial p}\left[\frac{f(p,p_0)}{p}\right]_{p_0} \sim \frac{f}{p_0^2},$$

then the term with f adds about 0.2 to the expression in round brackets.

ture dependences of the kinetic coefficients would remain the same as for a spectrum of type (2). With regard to the numerical values of the coefficients, it turns out that in the given case they depend on a large number of unknown quantities, such as df/dp and the higher terms of the expansion of the energy in $p - p_0$. In view of this, calculation of these coefficients is of no interest. The estimates of the order of magnitude that we have made show that in all probability, the coefficients will be of an order larger than in the case of a spectrum of type (2).

As was shown by us in Ref. 2, for a spectrum of type (3), we must consider not only the low temperature range but also the Boltzmann region. This is quite easy, as in this region, the kinetic coefficients ought to depend on the temperature. Actually, in this case, the situation is the same as for the rotons in He II. In accord with the calculations of Landau and Khalatnikov[5], the roton viscosity does not depend on the temperature. Thus in the Boltzmann region, we must expect η = const. To find the coefficient of thermal conductivity, we must make use of the relation $\varkappa/\eta \sim c/m^*$, where c is the heat capacity for a single particle, and m^* is the effective mass of the excitation. But the effective mass of the roton, as is well known (see Ref. 6) is equal to $p_0^2/3kT$. Thus in this case we must expect $\varkappa \sim T$. In other words, the curve $\varkappa(T)$ will have a minimum.

If we apply limitation (40) to the case under examination, it then appears that the calculations in the Boltzmann region can be valid if the temperature is higher than some limit. An estimate shows that this limit corresponds to several degrees. Hence the theory which pertains to the Boltzmann region can, in the best case, indicate a definite tendency in the temperature dependence of a quantity, but cannot delineate this dependence exactly. The same applies to the calculation of thermodynamic quantities carried out in Ref. 2.

Therefore, on the basis of data on thermodynamic quantities and the coefficients η and $\varkappa$ we cannot select any definite spectra. The fact is that in the case of a spectrum of type (3), the second viscosity ζ ought to have a value comparable with the first. We can convince ourselves of this, just as before, by taking the corresponding interval from Eq. (12).

In zeroth order, the integral does not vanish, in contrast to the case of a spectrum of type (2). Thus, whenever the second viscosity is finally measured in the region of very low temperatures (for example, by sound absorption), we can expect more or less of a verification of the form of the He³ spectrum.

In conclusion, we note that there are lacking at the present time any sort of measurements which could be compared with the theory developed above. Measurements of viscosity recently carried out by Zinov'eva* apply to the temperature region from 1 to 3° K. In this region, the viscosity is slightly temperature dependent and changes approximately from 2.2×10^{-5} to 1.7×10^{-5} poise. If we compare this with Eq. (29), an impression is created that the spectrum of type (2) leads to too small a value of the viscosity, although naturally Zinov'eva's data refer to a very remote temperature region, and the accuracy of the estimate of the coefficient in Eq. (29) is not large. On the other hand, such a slow change in the coefficient of viscosity and its magnitude is not in bad agreement with the results for spectrum (3). However, because of what was pointed out above, this cannot be considered as proof of the correctness of such a spectrum.

In conclusion, the authors express their gratitude to Acad. L. D. Landau for discussion of the results of the research.

[1] L. D. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) **30**, 1058 (1956); Soviet Phys. JETP **3**, 920 (1957).

[2] I. M. Khalatnikov and A. A. Abrikosov, J. Exptl. Theoret. Phys. (U.S.S.R.) **32**, 915 (1957); Soviet Phys. JETP **5**, 745 (1957).

[3] I. Ia. Pomeranchuk, J. Exptl. Theoret. Phys. (U.S.S.R.) **20**, 919 (1950).

[4] G. K. Walters and W. M. Fairbanks, Phys. Rev. **103**, 263 (1956).

[5] L. D. Landau and I. M. Khalatnikov, J. Exptl. Theoret. Phys. (U.S.S.R.) **19**, 637, 709 (1949).

[6] E. M. Lifshitz, Uspekhi Fiz. Nauk **34**, 512 (1948).

* The authors are grateful to K. N. Zinov'eva for a report on the viscosity data prior to publication.