[1] V. L. Bonch-Bruevich, Fizich. sb. L'vovskovo un-ta (Physics Reports of the Lvov Univ.), 1 (6), 1955.

[2] V. L. Bonch-Bruevich. Report to the 8th All-Union Congress on the Physics of Semiconductors, 1955, Izv. Akad. Nauk SSSR, ser. fiz., 21, 87 (1957).

[3] D. N. Zubarev, J. Exptl. Theoret. Phys. (U.S.S.R.) 25, 548 (1953).

[4] E. M. Conwell, V. F. Weisskopf, Phys. Rev., 77, 388 (1950).

[5] P. P. Debye, E. M. Conwell, Phys. Rev., 93, 693 (1954).

[6] L. E. Gurevich, *Fundamentals of Physical Kinetics*, Gostekhizdat, 1940.

[7] F. Seitz, Modern Theory of Solids.

[8] Cavid Erginsoy, Phys. Rev., 79, 1013 (1950).

[9] G. L. Pearson, J. Bardeen, Phys. Rev., 75, 865 (1949).

Translated by M. Hamermesh
227

SOVIET PHYSICS JETP VOLUME 5, NUMBER 5 DECEMBER, 1957

On the Dependence of the Motion of Bodies in a Gravitational Field on Their Mass

I. G. FIKHTENGOL'TS

Leningrad Institute of Precision Mechanics and Optics

(Submitted to JETP editor March 12, 1956)

J. Exptl. Theoret. Phys. (U.S.S.R.) 32, 1098-1101 (May, 1957)

The Lagrangian function for the motion of a body of small mass, in the fixed field of n other bodies of finite mass, is derived to the second approximation of gravitational theory. This function is compared with the Lagrangian function for the motion of a body of finite mass in the gravitational field of all $n + 1$ bodies. It is found that in the approximation under consideration, the Lagrangian function for the motion of a finite mass depends in an essential way on the value of that mass.

IN ACCORDANCE WITH the principle of the geodesic line, the motion of a body in a fixed gravitational field* is determined by the requirement that

$$\delta \int \mathscr{L}\, dt = 0, \qquad (1)$$

where

$$\mathscr{L} = mc^2\left(1 - \frac{1}{c}\sqrt{g_{00} + 2g_{0i}\dot{x}_i + g_{ik}\dot{x}_i\dot{x}_k}\right) \qquad (2)$$

is the Lagrangian function of the mechanical problem. Here m is the mass of the body under consideration, $x_i(t)$ the Cartesian coordinates of the center of mass of m at the instant t and g_{00}, g_{0i}, g_{ik} the components of the fundamental tensor as determined by the Einstein equations of gravitation; the Latin indices i, k take the values 1, 2, 3 and summation over a repeated index is understood. A superior dot indicates a time derivative.

In order to find an approximate expression for the Lagrangian function $\mathfrak{L}$ for the motion of a small mass in the fixed field of n other finite (not small) masses, we shall use an approximate solution of the Einstein equations of gravitation, obtained by Fock[1].

We consider spherically symmetric, nonrotating bodies, whose linear dimensions are much smaller than the distances between them; and we retain only quantities of order v^2/c^2, where v^2 is the square of the velocity of translational motion of one of the bodies. We then obtain the following expressions for the components of the fundamental tensor:

$$g_{00} = c^2 - 2U + (2U^2 - 2S^*)/c^2, \quad g_{0i} = 4U_i/c^2,$$
$$g_{ik} = -(1 + 2U/c^2)\delta_{ik}. \qquad (3)$$

* The field of a system of bodies is regarded as fixed, with respect to a given body, if the motion of each of the bodies of the system that produces the field is supposed independent of the motion of the given body.

Here

$$U=\sum_a \frac{\gamma m_a}{|\mathbf{r}-\mathbf{a}|}\left\{1+\frac{1}{2c^2}\left(\mathbf{v}_a^2-U^{(a)}(\mathbf{a})\right)\right\}+\frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\sum_a \gamma m_a|\mathbf{r}-\mathbf{a}|,$$

$$U^{(a)}(\mathbf{r})=\sum_{b}{}' \frac{\gamma m_b}{|\mathbf{r}-\mathbf{b}|},\quad (b\neq a)\qquad U_i=\sum_a \frac{\gamma m_a \dot{a}_i}{|\mathbf{r}-\mathbf{a}|},$$

$$S^*=\sum_a \frac{\gamma m_a \mathbf{v}_a^2}{|\mathbf{r}-\mathbf{a}|}-\frac{1}{2}\sum_{\substack{a,b\\(a\neq b)}}\frac{\gamma^2 m_a m_b}{|\mathbf{r}-\mathbf{a}|\cdot|\mathbf{a}-\mathbf{b}|}. \qquad (4)$$

In these formulas, the radius vectors of the centers of mass of m, m_a, m_b, ... at the instant t are denoted respectively by $\mathbf{r}(x_1, x_2, x_3)$, $\mathbf{a}(a_1, a_2, a_3)$, $\mathbf{b}(b_1, b_2, b_3)$, ... ; $\mathbf{v}_a^2$ is the square of the velocity of mass m_a; a prime on a summation is a reminder that the summation index takes all values except one from 1 to n.

Upon substituting the expressions (3) for the components of the fundamental tensor in the Lagrangian function $\mathfrak{L}$, we find that, in our approximation

$$\mathcal{L}=\frac{m\dot{\mathbf{r}}^2}{2}+\frac{m\dot{\mathbf{r}}^4}{8c^2}+mU-\frac{mU^2}{2c^2}+\frac{mS^*}{c^2}-\frac{4mU_i\dot{x}_i}{c^2}+\frac{3mU\dot{\mathbf{r}}^2}{2c^2},$$

or by use of (4),

$$\mathcal{L}=\frac{m\dot{\mathbf{r}}^2}{2}+\frac{m\dot{\mathbf{r}}^4}{8c^2}+m\sum_a \frac{\gamma m_a}{|\mathbf{r}-\mathbf{a}|}\left\{1-\frac{1}{c^2}(\dot{\mathbf{r}}\cdot\dot{\mathbf{a}})+\frac{3}{2c^2}(\dot{\mathbf{r}}-\dot{\mathbf{a}})^2-\frac{1}{2c^2}\sum_b \frac{\gamma m_b}{|\mathbf{r}-\mathbf{b}|}-\frac{1}{c^2}\sum_b{}' \frac{\gamma m_b}{|\mathbf{a}-\mathbf{b}|}\right\}+\frac{m}{2c^2}\frac{\partial^2}{\partial t^2}\sum_a \gamma m_a|\mathbf{r}-\mathbf{a}|. \qquad (5)$$

The dependence of the Lagrangian function (5) on the accelerations of the finite masses m_a is not an essential feature of the problem; it can easily be removed, thanks to the fact that the variables a_i (and likewise $\dot{a}_i$ and $\ddot{a}_i$) are independent of the variables x_i.

Remembering that

$$\frac{\partial}{\partial t}\sum_a \gamma m_a|\mathbf{r}-\mathbf{a}|=-\sum_a \frac{\gamma m_a}{|\mathbf{r}-\mathbf{a}|}(\dot{\mathbf{a}}\cdot(\mathbf{r}-\mathbf{a})),$$

we get

$$\frac{\partial^2}{\partial t^2}\sum_a \gamma m_a|\mathbf{r}-\mathbf{a}|=\frac{d}{dt}\frac{\partial}{\partial t}\sum_a \gamma m_a|\mathbf{r}-\mathbf{a}|+\sum_a \frac{\gamma m_a}{|\mathbf{r}-\mathbf{a}|}(\dot{\mathbf{r}}\cdot\dot{\mathbf{a}})-\sum_a \frac{\gamma m_a}{|\mathbf{r}-\mathbf{a}|}(\dot{\mathbf{r}}\cdot(\mathbf{r}-\mathbf{a}))(\dot{\mathbf{a}}\cdot(\mathbf{r}-\mathbf{a})).$$

Consequently

$$\mathcal{L}=\frac{m\dot{\mathbf{r}}^2}{2}+\frac{m\dot{\mathbf{r}}^4}{8c^2}+m\sum_a \frac{\gamma m_a}{|\mathbf{r}-\mathbf{a}|}\left\{1-\frac{1}{2c^2}(\dot{\mathbf{r}}\cdot\dot{\mathbf{a}})-\frac{1}{2c^2}\frac{(\dot{\mathbf{r}}(\mathbf{r}-\mathbf{a}))(\dot{\mathbf{a}}(\mathbf{r}-\mathbf{a}))}{|\mathbf{r}-\mathbf{a}|^2}+\frac{3}{2c^2}(\dot{\mathbf{r}}-\dot{\mathbf{a}})^2-\frac{1}{2c^2}\sum_b \frac{\gamma m_b}{|\mathbf{r}-\mathbf{b}|}-\frac{1}{c^2}\sum_b{}' \frac{\gamma m_b}{|\mathbf{a}-\mathbf{b}|}\right\}+\frac{d}{dt}f(t, x_1, x_2, x_3); \qquad (6)$$

$$f(t, x_1, x_2, x_3)=-\frac{m}{2c^2}\sum_a \frac{\gamma m_a}{|\mathbf{r}-\mathbf{a}|}(\dot{\mathbf{a}}\cdot(\mathbf{r}-\mathbf{a})).$$

An expression of the form df/dt, where $f(t, x_1, x_2, x_3)$ is an arbitrary differentiable function of its arguments, reduces Lagrange's equations to identities; therefore we may drop the last term in the Lagrangian function (6).

Before writing the final expression for the Lagrangian function of the mechanical problem under consideration, we shall change to a slightly different notation. Denoting the radius vectors of the centers of mass of m and m_k (k = 1, 2, ... n) by $\mathbf{r}(x, y, z)$ and $\mathbf{r}_k(x_k, y_k, z_k)$ respectively, and the Lagrangian function without the term df/dt by L^*, we have

$$L^*=\frac{m\dot{\mathbf{r}}^2}{2}+\frac{m\dot{\mathbf{r}}^4}{8c^2}+m\sum_k \frac{\gamma m_k}{|\mathbf{r}-\mathbf{r}_k|}\left\{1-\frac{1}{2c^2}(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}_k)-\frac{1}{2c^2}\frac{(\dot{\mathbf{r}}\cdot(\mathbf{r}-\mathbf{r}_k))(\dot{\mathbf{r}}_k\cdot(\mathbf{r}-\mathbf{r}_k))}{|\mathbf{r}-\mathbf{r}_k|^2}+\frac{3}{2c^2}(\dot{\mathbf{r}}-\dot{\mathbf{r}}_k)^2-\frac{1}{2c^2}\sum_l \frac{\gamma m_l}{|\mathbf{r}-\mathbf{r}_l|}-\frac{1}{c^2}\sum_l{}' \frac{\gamma m_l}{|\mathbf{r}_k-\mathbf{r}_l|}\right\}. \qquad (7)$$

The equations of motion corresponding to the Lagrangian function $\mathfrak{L}$ (or equivalently L^*) actually do not contain the value of the mass m; they indicate nondependence of the motion of a body (of small mass), in a fixed gravitational field, upon the value of its mass.

In the case of motion of a body of finite mass, it is not permissible to treat the field as fixed with respect to this body. A finite mass, even in the first (Newtonian) approximation, affects the motion of the other masses, changing it; and this change in turn has an influence on the motion of the mass under consideration. However, the dependence of the motion of a body upon its mass, in the first approximation, still does not affect the form of the Lagrangian function.

The situation changes in an essential way when we go on to a consideration of the motion of finite masses in a higher approximation than the first (the Newtonian). In this case, in the derivation of the equations of motion it is necessary to start directly from the Einstein equations of gravitation. The equations of motion that emerge from them, for a system of finite masses, can also be expressed in Lagrangian form; but the Lagrangian function of this more general problem* has the form[2]

$$L = \sum_i L_{0i} - V, \quad L_{0i} = {}^1/_2\, m_i\dot{\mathbf{r}}_i^2 + m_i\dot{\mathbf{r}}_i^4/8c^2,$$

$$\begin{aligned} V = -\frac{1}{2} \sum_{\substack{i,\,k \\ (i\neq k)}} \frac{\gamma m_i m_k}{|\mathbf{r}_i - \mathbf{r}_k|}\Big\{1 - \frac{1}{2c^2}(\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_k) \\ - \frac{1}{2c^2}\frac{(\dot{\mathbf{r}}_i\cdot(\mathbf{r}_i-\mathbf{r}_k))\,(\dot{\mathbf{r}}_k\cdot(\mathbf{r}_i-\mathbf{r}_k))}{|\mathbf{r}_i-\mathbf{r}_k|^2} \\ + \frac{3}{2c^2}(\dot{\mathbf{r}}_i-\dot{\mathbf{r}}_k)^2 - \frac{1}{c^2}\sum_l{}' \frac{\gamma m_l}{|\mathbf{r}_i-\mathbf{r}_l|}\Big\}. \end{aligned} \tag{8}$$

It follows that when we examine the motion of the ith mass in the gravitational field of all the bodies (in the present case the number of bodies is supposed equal to $n + 1$), the Lagrangian function of the appropriate mechanical problem can be reduced to the following form:

$$\begin{aligned} L_i = \frac{m_i\dot{\mathbf{r}}_i^2}{2} + \frac{m_i\dot{\mathbf{r}}_i^4}{8c^2} \\ + m_i\sum_k{}' \frac{\gamma m_k}{|\mathbf{r}_i-\mathbf{r}_k|}\Big\{1 - \frac{1}{2c^2}(\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_k) \\ - \frac{1}{2c^2}\frac{(\dot{\mathbf{r}}_i\cdot(\mathbf{r}_i-\mathbf{r}_k))\,(\dot{\mathbf{r}}_k\cdot(\mathbf{r}_i-\mathbf{r}_k))}{|\mathbf{r}_i-\mathbf{r}_k|^2} + \frac{3}{2c^2}(\dot{\mathbf{r}}_i-\dot{\mathbf{r}}_k)^2 \\ - \frac{1}{2c^2}\sum_l{}' \frac{\gamma m_l}{|\mathbf{r}_i-\mathbf{r}_l|} \\ - \frac{1}{c^2}\sum_{\substack{l \\ (l\neq k,\ l\neq i)}}{}'' \frac{\gamma m_l}{|\mathbf{r}_k-\mathbf{r}_l|} - \frac{1}{2c^2}\frac{\gamma m_i}{|\mathbf{r}_i-\mathbf{r}_k|}\Big\}. \end{aligned} \tag{9}$$

In order to compare the functions L_i of (9) and L^* of (7), we set

$$L^*|_{m=m_i,\mathbf{r}=\mathbf{r}_i} = L_i^*, \tag{10}$$

then

$$L_i - L_i^* = -\frac{m_i^2}{2c^2}\sum_k{}' \frac{\gamma^2 m_k}{|\mathbf{r}_i-\mathbf{r}_k|^2}. \tag{11}$$

Consequently, the Lagrangian function L_i differs from L_i^* by an expression proportional to the square of the mass m_i (whereas L_i^* is proportional to the first power of m_i). Only when the mass m_i is small does the function L_i in fact agree with L_i^*.

Thus in a treatment of the motion of a body of finite mass, with quantities of order v^2/c^2 taken into account, the dependence of this motion on the value of the mass expresses itself in an essential way, in the actual form of the Lagrangian function.

Note added in proof (April 12, 1957). In a recently published article[3], Shirokov and Brodovskii attempt to prove that "the center of inertia of any body (not necessarily of small mass—I. F.) of the system moves along a geodesic line in the gravitational field of the other bodies," and that the equations of motion derived by the principle of the geodesic line agree with the equations of motion of a system of bodies obtained by Petrova[4] by Fock's method.

From relation (11) of the present article, it follows that both these assertions made are incorrect (for more details on this, *cf.* Ref. 5, p. 34).

---

*We consider only translational motion of the bodies, and we take no account of the dependence of this motion on their shape and on other parameters.

[1] V. A. Fock, J. Exptl. Theoret. Phys. (U.S.S.R.) 9,

375(1939). *Cf.* also: V. A. Fock, *Theory of Space, Time, and Gravitation*, GTTI, 1955.

[2] I. G. Fikhtengol'ts, J. Exptl. Theoret. Phys. (U.S.S.R.) **20**, 233 (1950).

[3] M. F. Shirokov and V. B. Brodovskii, J. Exptl. Theoret. Phys. (U.S.S.R.) **31**, 1027 (1956). Soviet Phys. **4**, 904 (1957).

[4] I. M. Petrova, J. Exptl. Theoret. Phys. (U.S.S.R.) **19**, 989 (1949).

[5] I. G. Fikhtengol'ts, *Dissertation*, Leningrad State Univ., 1952.

Translated by W. F. Brown, Jr.
228

SOVIET PHYSICS JETP VOLUME 5, NUMBER 5 DECEMBER, 1957

On Magnetohydrodynamic Waves and Magnetic Tangential Discontinuities in Relativistic Hydrodynamics

I. M. KHALATNIKOV
Institute for Physical Problems, Academy of Sciences, U.S.S.R.

(Submitted to JETP editor June 8, 1957)
J. Exptl. Theoret. Phys. (U.S.S.R.) **32**, 1102-1107 (May 1957)

The problem of magnetohydrodynamic waves in relativistic hydrodynamics is discussed. Equations are derived for the velocity of these waves in the presence of a magnetic field making an arbitrary angle with the direction of propagation of the waves in a medium with an arbitrary equation of state. The properties of purely magnetic tangential discontinuities in relativistic hydrodynamics are also discussed.

AN INVESTIGATION by Hoffman and Teller[1] was devoted to problems of relativistic magnetohydrodynamics. In the present note we consider in more detail the problem of magnetohydrodynamic waves in relativistic hydrodynamics. In contrast to Ref. 1 where the absence of a magnetic field along the direction of wave propagation was supposed, we shall assume the presence of a magnetic field whose direction makes an arbitrary angle with the direction of wave propagation. Furthermore, we shall not assume, as was done in Ref. 1, that the ultrarelativistic equation of state $\varepsilon = 3p$ applies. We shall conduct the entire investigation for an arbitrary equation of state. Also, we shall consider the question of purely magnetic tangential discontinuities in relativistic hydrodynamics where the thermodynamic quantities remain continuous.

MAGNETOHYDRODYNAMIC WAVES IN RELATIVISTIC HYDRODYNAMICS

The energy momentum tensor in relativistic hydrodynamics has the following form

$$T^{\mathrm{h}}_{ik} = wnu_i u_k + p\delta_{ik}. \tag{1}$$

Here w is the heat function referred to one particle, n the density of the number of particles, p the pressure, u_i the 4-velocity component ($u_i^2 = -1$). The speed of light is $c = 1$.

We next denote the energy momentum tensor of the electromagnetic field by

$$T^{\mathrm{em}}_{\alpha\beta} = \frac{1}{4\pi}\left\{-H_\alpha H_\beta - E_\alpha E_\beta + \frac{1}{2}\delta_{\alpha\beta}(\mathbf{H}^2 + \mathbf{E}^2)\right\},$$
$$T^{\mathrm{em}}_{\alpha 4} = \frac{i}{4\pi}[\mathbf{EH}]_\alpha; \qquad T^{\mathrm{em}}_{44} = -\frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{H}^2). \tag{2}$$

We consider a medium with an infinite conductivity σ. For such a medium there follows from Ohm's law

$$\mathbf{j} = \sigma(\mathbf{E} + [\mathbf{vH}]) \tag{3}$$

a relationship between the electric and magnetic fields

$$\mathbf{E} = -[\mathbf{vH}]. \tag{4}$$

For one-dimensional motion, all quantities are functions of one spatial coordinate (x_1) and of the time ($x_4 = it$).

The conditions at the discontinuity for such a motion may be written in the form of continuity of the corresponding components of the total energy-