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On Magnetohydrodynamic Waves and Magnetic Tangential Discontinuities in Relativistic Hydrodynamics

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The problem of magnetohydrodynamic waves in relativistic hydrodynamics is discussed. Equations are derived for the velocity of these waves in the presence of a magnetic field making an arbitrary angle with the direction of propagation of the waves in a medium with an arbitrary equation of state. The properties of purely magnetic tangential discontinuities in relativistic hydrodynamics are also discussed.

AN INVESTIGATION by Hoffman and Teller[1] was devoted to problems of relativistic magnetohydrodynamics. In the present note we consider in more detail the problem of magnetohydrodynamic waves in relativistic hydrodynamics. In contrast to Ref. 1 where the absence of a magnetic field along the direction of wave propagation was supposed, we shall assume the presence of a magnetic field whose direction makes an arbitrary angle with the direction of wave propagation. Furthermore, we shall not assume, as was done in Ref. 1, that the ultrarelativistic equation of state $\varepsilon = 3p$ applies. We shall conduct the entire investigation for an arbitrary equation of state. Also, we shall consider the question of purely magnetic tangential discontinuities in relativistic hydrodynamics where the thermodynamic quantities remain continuous.

MAGNETOHYDRODYNAMIC WAVES IN RELATIVISTIC HYDRODYNAMICS

The energy momentum tensor in relativistic hydrodynamics has the following form

$$T^{\mathrm{h}}_{ik} = wnu_iu_k + p\delta_{ik}. \quad (1)$$

Here w is the heat function referred to one particle, n the density of the number of particles, p the pressure, u_i the 4-velocity component ($u_i^2 = -1$). The speed of light is $c = 1$.

We next denote the energy momentum tensor of the electromagnetic field by

$$T^{\mathrm{em}}_{\alpha\beta} = \frac{1}{4\pi}\left\{-H_\alpha H_\beta - E_\alpha E_\beta + \frac{1}{2}\delta_{\alpha\beta}(\mathbf{H}^2+\mathbf{E}^2)\right\},$$
$$T^{\mathrm{em}}_{\alpha 4} = \frac{i}{4\pi}[\mathbf{EH}]_\alpha; \quad T^{\mathrm{em}}_{44} = -\frac{1}{8\pi}(\mathbf{E}^2+\mathbf{H}^2). \quad (2)$$

We consider a medium with an infinite conductivity σ. For such a medium there follows from Ohm's law

$$\mathbf{j} = \sigma(\mathbf{E} + [\mathbf{vH}]) \quad (3)$$

a relationship between the electric and magnetic fields

$$\mathbf{E} = -[\mathbf{vH}]. \quad (4)$$

For one-dimensional motion, all quantities are functions of one spatial coordinate (x_1) and of the time ($x_4 = it$).

The conditions at the discontinuity for such a motion may be written in the form of continuity of the corresponding components of the total energy-momentum tensor, which constitutes the sum of the tensors T^{h}_{ik} and T^{em}_{ik} [Eqs. (1) and (2)]:

$$T_{ik} = T^{h}_{ik} + T^{em}_{ik}. \tag{5}$$

All components of the total tensor having the subscript 1 must remain continuous. This follows directly from the condition

$$\partial T_{ik} / \partial x_k = 0. \tag{6}$$

We thus have the following conditions at the discontinuity (we shall designate the difference between the values of a given quantity on the discontinuity by braces)

$$\left\{ wnu_1^2 + p + \frac{1}{8\pi}(H_\tau^2 - H_1^2 + E_\tau^2 - E_1^2) \right\} = 0, \tag{7}$$

$$\left\{ wnu_1\mathbf{u}_\tau - \frac{1}{4\pi}(H_1\mathbf{H}_\tau + E_1\mathbf{E}_\tau) \right\} = 0, \tag{8}$$

$$\left\{ wnu_1u_4 + \frac{i}{4\pi}(H_3E_2 - H_2E_3) \right\} = 0. \tag{9}$$

Here we denote a vector in the $2-3$ plane by the subscript τ ($\mathbf{u}_\tau$ has the components u_2 and u_3, etc.)

Let us also take account of the first pair of Maxwell's equations [the second pair is contained in Eq. (6)].

$$\operatorname{curl} \mathbf{E} = -\partial \mathbf{H}/\partial t, \quad \operatorname{div} \mathbf{H} = 0. \tag{10}$$

These equations lead to the two conditions

$$\{\mathbf{E}_\tau\} = 0, \tag{11}$$

$$\{H_1\} = 0. \tag{12}$$

Finally, we also must write the condition for the particle density

$$\{nu_1\} = 0, \tag{13}$$

resulting from the equation of continuity

$$\partial nu_1 / \partial x_1 = 0. \tag{14}$$

The conditions (7) – (9) can be somewhat simplified, using (11), (12), and (4), to read

$$\left\{ wnu_1^2 + p + \frac{1}{8\pi}(\mathbf{H}_\tau^2 - E_1^2) \right\} = 0, \tag{15}$$

$$\{wnu_1\mathbf{u}_\tau\} = \frac{1}{4\pi}H_1\{\mathbf{H}_\tau\} + \frac{1}{4\pi}\mathbf{E}_\tau\{E_1\}, \tag{16}$$

$$\left\{ wnu_1u_4 + \frac{i}{4\pi}v_1\mathbf{H}_\tau^2 \right\} = \frac{iH_1}{4\pi}\{\mathbf{v}_\tau\mathbf{H}_\tau\}. \tag{17}$$

Taking (4) into account, we may write the condition (11) in the form

$$\{\mathbf{v}_\tau\}H_1 = \{v_1\mathbf{H}_\tau\}. \tag{18}$$

The relationships (12), (13), and (15) – (18) constitute the complete system of conditions on the discontinuity.

Making use of these conditions, we shall consider the problem of magnetohydrodynamic waves in relativistic hydrodynamics. Magnetohydrodynamic waves can be regarded as the limiting case of discontinuities of very small intensity. The components v_2 and v_3 in the hydrodynamic wave may be regarded as small. As to v_1, however, that quantity cannot be regarded as small since under relativistic conditions the wave can move along the axis l at a speed approaching the speed of light.

In a magnetohydrodynamic wave all quantities are functions of one parameter. Moreover, in view of the infinitesimal smallness of the jumps, the ratio of the jumps may be replaced by derivatives. We choose the velocity v_1 as the independent variable. Derivatives with respect to this variable will be denoted by primes. Thus we have, for instance

$$p' = \{p\}/\{v_1\} = W'\partial p/\partial W. \tag{19}$$

Below, we shall denote the derivative $\partial p/\partial W$ by a^2. With the aid of the known relationship between the thermodynamic quantities

$$W = \varepsilon + p; \quad W = wn \tag{20}$$

we find the connection between the quantity a^2 and the speed of sound

$$c^2 = \partial p / \partial \varepsilon. \tag{21}$$

By simple differentiation we obtain

$$c^2 = a^2/(1 - a^2). \tag{22}$$

We introduce also the symbol

$$\overline{W} = W/(1 - v_1^2). \tag{23}$$

Considering all this, we can, after eliminating v_τ', reduce, without difficulty, the system of conditions (15) – (18) to the following form

$$\overline{W}(1-a^2)\,2v_1+[v_1^2+a^2(1-v_1^2)]\,\overline{W}'+\mathbf{H}_\tau\mathbf{H}_\tau'/4\pi=0, \tag{24}$$

$$\overline{W}+v_1\overline{W}'+v_1\mathbf{H}_\tau\mathbf{H}_\tau'/4\pi=0, \tag{25}$$

$$\overline{W}H_2v_1=(H_1^2/4\pi-v_1^2H_3^2/4\pi-\overline{W}v_1^2)\,H_2'+v_1^2H_2H_3H_3'/4\pi, \tag{26}$$

$$\overline{W}H_3v_1=(H_1^2/4\pi-v_1^2H_2^2/4\pi-\overline{W}v_1^2)\,H_3'+v_1^2H_2H_3H_2'/4\pi. \tag{27}$$

Let us multiply Eq. (26) by H_2 and Eq. (27) by H_3 and add the products. We obtain, as a result, the quantity $\mathbf{H}_\tau\mathbf{H}_\tau'$ entering into Eqs. (24) and (25).

$$\mathbf{H}_\tau\mathbf{H}_\tau'=\overline{W}v_1\mathbf{H}_\tau^2/(H_1^2-4\pi\overline{W}v_1^2). \tag{28}$$

We next substitute the expression obtained into (24) and (25) and eliminate the derivative $\overline{W}'$ from these equations. In this way we obtain

$$\left[v_1^2-\frac{a^2}{1-a^2}\right]+\frac{\mathbf{H}_\tau^2v_1^2}{H_1^2-4\pi\overline{W}v_1^2}(1-v_1^2)=0, \tag{29}$$

which determines the velocity of the magnetohydrodynamic waves.

Taking (22) and (23) into account, we rewrite (29) as

$$(v_1^2-c^2)\left(\frac{v_1^2}{1-v_1^2}-\frac{H_1^2}{4\pi W}\right)=(1-v_1^2)\frac{\mathbf{H}_\tau^2v_1^2}{4\pi W}. \tag{30}$$

Here W is the heat function per unit volume in a reference system in which the given volume element is at rest, and $\mathbf{H}_1$, $\mathbf{H}_\tau$ are components of a field in a reference system that moves together with the wave front. The index zero denotes the field components in the reference system that is at rest together with the fluid. We then have

$$H_1=H_{10};\quad \mathbf{H}_\tau=\mathbf{H}_{\tau_0}/\sqrt{1-v_1^2}. \tag{31}$$

Having expressed in (30) the field $\mathbf{H}$ by $\mathbf{H}_0$, we obtain

$$(v_1^2-c^2)\left(\frac{v_1^2}{1-v_1^2}-\frac{H_{10}^2}{4\pi W}\right)=v_1^2\mathbf{H}_{\tau_0}^2/4\pi W. \tag{32}$$

Hoffman and Teller[1] considered the special case where an ultrarelativistic equation of state applies and $H_{10}=0$. In this case

$$\varepsilon=3p,\quad c^2=\partial p/\partial\varepsilon={}^1/_3,\quad W=4\varepsilon/3, \tag{33}$$

and Eq. (32) leads to the result of Hoffman and Teller

$$(v_1^2-{}^1/_3)/(1-v_1^2)=3\mathbf{H}_{\tau_0}^2/16\pi\varepsilon. \tag{34}$$

In the general case, however, the velocity of propagation v_1 of the front of a magnetohydrodynamic wave is determined by the biquadratic equation (32), the roots of which are equal to

$$v_1^2=\frac{c^2(1+r_{10}^2)+r_0^2\pm\sqrt{[c^2(1+r_{10}^2)-r_0^2]^2+4c^2r_{\tau_0}^2}}{2(1+r_0^2)}, \tag{35}$$

$$r_0^2=H_0^2/4\pi W,\quad r_{10}^2=H_{10}^2/4\pi W,\quad r_{\tau_0}^2=H_{\tau_0}^2/4\pi W.$$

In the general case, two waves can thus propagate at different speeds; the sum of their velocities is equal to

$$(v_1^{(1)})^2+(v_1^{(2)})^2=[c^2(1+r_{10}^2)+r_0^2]/(1+r_0^2). \tag{36}$$

In the case where the external field is longitudinal, $\mathbf{H}_\tau=0$, and Eq. (32) breaks up into two. The first root $v_1^2=c^2$ corresponds to hydrodynamic waves not connected with a magnetic field, *i.e.*, to ordinary sound. The second root

$$v_1^2=\frac{H_{10}^2}{4\pi W}\Big/\left(1+\frac{H_{10}^2}{4\pi W}\right) \tag{37}$$

determines the velocity of the magnetohydrodynamic wave.

For an arbitrary direction of the field $\mathbf{H}$ there follows from Eqs. (19) and (27) that the existence of special purely magnetic waves is possible. The velocities of these waves are found in the following manner. We multiply Eq. (26) by $\mathbf{H}_3$, Eq. (27) by $\mathbf{H}_2$

and subtract the results; in this way we obtain

$$(H_1^2 - v_1^2 H_\tau^2 - 4\pi \overline{W} v_1^2)(H_2' H_3 - H_3' H_2) = 0. \quad (38)$$

Hence follows an equation for the wave velocity v_1:

$$H_1^2 - v_1^2 H_\tau^2 - 4\pi \overline{W} v_1^2 = 0. \quad (39)$$

Solving this equation with respect to v_1^2 and using (25) and (31), we find

$$v_1^2 = H_{10}^2 / (H_0^2 + 4\pi W). \quad (40)$$

So far we used the equation of continuity for the density of the particles. In the ultrarelativistic case, at high temperatures, processes of particle formation are possible. In this case the law of particle conservation no longer holds. However, in all cases the entropy is continuous simply as the result of Eq. (6). As has been shown in our report on relativistic hydrodynamics[2], results for the ultra-relativistic case are obtained from the above equations by substituting the temperature T for the heat function W.

ON TANGENTIAL DISCONTINUITIES

The properties of purely magnetic tangential discontinuities may be easily obtained from the conditions (13) and (15) – (18). We shall now consider such tangential discontinuities in which the thermodynamic quantities are continuous. We then have from (13)

$$\{u_1\} = \{v_1 / \sqrt{1 - v^2}\} = 0, \quad (41)$$

and since $\{v_1\} = 0$,

$$\{v_\tau^2\} = 0. \quad (42)$$

Next, we introduce the symbol

$$nu_1 = j, \quad (43)$$

take $\{v_\tau\}$ from (18), and substitute it into the condition (16); thus we obtain

$$4\pi j^2 (w / n)\{\mathbf{H}_\tau\} = H_1^2 \{\mathbf{H}_\tau\} + H_1 \mathbf{E}_\tau \{E_1\}. \quad (44)$$

We express the jump $\{E_1\} = \{-v_2 H_3 + v_3 H_2\}$, with the aid of (18), in terms of the jump $\{H_\tau\}$:

$$\{E_1\} = -\mathbf{E}_\tau \{\mathbf{H}_\tau\} / H_1. \quad (45)$$

Substituting this expression into (44), we obtain an equation

$$4\pi j^2 (w / n) \{\mathbf{H}_\tau\} = H_1^2 \{\mathbf{H}_\tau\} - \mathbf{E}_\tau (\mathbf{E}_\tau \{\mathbf{H}_\tau\}). \quad (46)$$

which is homogeneous with respect to $\{\mathbf{H}_\tau\}$. The condition of compatability of the two components of Eq. (46) leads to the following equation for the velocity j of the tangential discontinuities under consideration:

$$\left\{4\pi j^2 \frac{w}{n} - (H_1^2 - E_2^2)\right\}\left\{4\pi j^2 \frac{w}{n} - (H_1^2 - E_3^2)\right\} = E_2^2 E_3^2. \quad (47)$$

This equation has two solutions

$$j^2 = \frac{n}{4\pi w} \begin{cases} H_1^2 \\ H_1^2 - E_\tau^2 \end{cases}. \quad (48)$$

The quantity j^2 is connected with the velocity of the discontinuity v_p^2 by the obvious relationship

$$j^2 / n^2 = v_p^2 / (1 - v_p^2). \quad (49)$$

Let us dwell briefly on the relationships between the jumps of the fields in the tangential discontinuities. According to (15), we have

$$\{H_\tau^2\} = \{E_1^2\} \quad (50)$$

It may easily be shown that each of these jumps is equal to zero.

In fact, according to (17) and (18)

$$v_1 \{H_\tau^2\} = H_1 (H_\tau v_\tau\}, \quad (51)$$

$$\{v_\tau H_1 - v_1 H_\tau\} = 0. \quad (52)$$

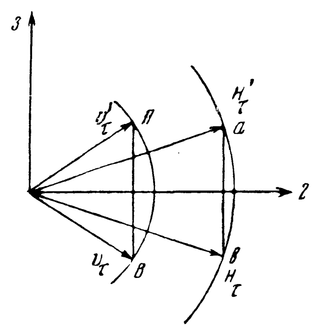
The square of the quantity in braces in (52) is also continuous, as is the quantity itself

$$\{v_\tau^2\} H_1^2 + v_1^2 \{H_\tau^2\} - 2 v_1 H_1 \{v_\tau H_\tau\} = 0. \quad (53)$$

Eliminating the quantity $\{v_\tau H_\tau\}$ from (51) and (53) and taking (42) and (50) into consideration, we obtain

$$\{H_\tau^2\} = \{E_1^2\} = (H_1 / v_1)^2 \{v_\tau^2\} = 0. \quad (54)$$

It follows from (47) and (54) that in tangential discontinuities of the type considered the absolute values of the velocity and magnitude of the magnetic field do not change on the discontinuity. All that occurs on the discontinuity is the rotation of those vectors, without a change in their length. We choose axis *2* in such a manner that the component H_2 is not changed by rotation of the vector $\mathbf{H}_\tau$. Then $\{H_2\} = 0$ and $\{v_2\} = 0$, $\{v_3\} = \frac{v_1}{H_1}\{H_3\}$. It can be seen



from the figure that thereby the components v_3 and H_3 on the discontinuity change their sign. From the expression for $E_1 = -v_2 H_3 + v_3 H_2$ it follows that as a result E_1 also changes its sign on the discontinuity.

$$v_3 \rightarrow -v_3, \quad H_3 \rightarrow -H_3, \quad E_1 \rightarrow -E_1.$$

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Theory of Diffusion and Thermal Conductivity for Dilute Solutions of He^3 in Helium II

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The phenomena of diffusion and thermal conductivity are investigated for dilute solutions of He^3 in helium II, on the basis of the theory of superfluidity proposed by Landau for helium II. A solution is found for the system of kinetic equations for the elementary excitations in the case of non-zero temperature and concentration gradients within the solution. The temperature dependence of the effective thermal conductivity for the solution is determined. A comparison with experiment is made.

1. INTRODUCTION

THE PROBLEM of the kinetic coefficients for solutions of foreign particles in helium II has been investigated by one of the authors[1]. From phenomenological considerations it was demonstrated that in addition to the single coefficient of first viscosity η, the three coefficients of second viscosity, ζ_1, ζ_2, ζ_3, and the coefficient of thermal conductivity $\varkappa$ existing in pure helium II[2] two further kinetic coefficients appear in the case of solutions: the diffusion coefficient D and the thermal diffusion coefficient Dk_T, where k_T is the thermal diffusion ratio. The diffusion of an admixture of the isotope He^3 has been investigated experimentally by Beenakker, *et al*[3], who determined the temperature dependence of the diffusion coefficient in the temperature range from 1.2° K to the λ-point for a concentration $c \sim 10^{-4}$. In the present paper we consider the phenomena of diffusion, thermal diffusion, and thermal conductivity for dilute solutions of He^3 in helium II.

According to Landau's theory[4], liquid helium in the temperature region below the λ-point (helium II) is to be regarded as a weakly-excited quantum system. This implies that thermal energy of helium II may be represented as a gas of elementary excita-