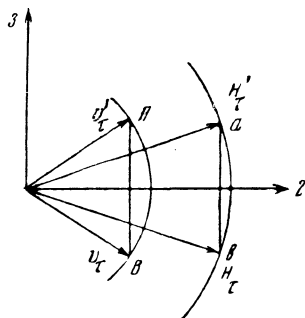


It follows from (47) and (54) that in tangential discontinuities of the type considered the absolute values of the velocity and magnitude of the magnetic field do not change on the discontinuity. All that occurs on the discontinuity is the rotation of those vectors, without a change in their length. We choose axis 2 in such a manner that the component  $H_2$  is not changed by rotation of the vector  $\mathbf{H}_\tau$ . Then  $\{H_2\} = 0$  and  $\{v_2\} = 0$ ,  $\{v_3\} = \frac{v_1}{H_1} \{H_3\}$ . It can be seen



from the figure that thereby the components  $v_3$  and  $H_3$  on the discontinuity change their sign. From the expression for  $E_1 = -v_2 H_3 + v_3 H_2$  it follows that as a result  $E_1$  also changes its sign on the discontinuity.

$$v_3 \rightarrow -v_3, \quad H_3 \rightarrow -H_3, \quad E_1 \rightarrow -E_1.$$

<sup>1</sup>F. Hoffman and E. Teller, Phys. Rev. **80**, 692 (1950).

<sup>2</sup>I. M. Khalatnikov, J. Exptl. Theoret. Phys. (U.S.S.R.) **27**, 529 (1954).

Translated by Mary L. Mahler  
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### Theory of Diffusion and Thermal Conductivity for Dilute Solutions of $\text{He}^3$ in Helium II

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The phenomena of diffusion and thermal conductivity are investigated for dilute solutions of  $\text{He}^3$  in helium II, on the basis of the theory of superfluidity proposed by Landau for helium II. A solution is found for the system of kinetic equations for the elementary excitations in the case of non-zero temperature and concentration gradients within the solution. The temperature dependence of the effective thermal conductivity for the solution is determined. A comparison with experiment is made.

#### 1. INTRODUCTION

THE PROBLEM of the kinetic coefficients for solutions of foreign particles in helium II has been investigated by one of the authors<sup>1</sup>. From phenomenological considerations it was demonstrated that in addition to the single coefficient of first viscosity  $\eta$ , the three coefficients of second viscosity,  $\zeta_1$ ,  $\zeta_2$ ,  $\zeta_3$ , and the coefficient of thermal conductivity  $\kappa$  existing in pure helium II<sup>2</sup> two further kinetic coefficients appear in the case of solutions: the diffusion coefficient  $D$  and the thermal diffusion coefficient  $Dk_T$ , where  $k_T$  is the thermal diffusion ratio. The diffusion of an admixture of

the isotope  $\text{He}^3$  has been investigated experimentally by Beenakker, *et al*<sup>3</sup>, who determined the temperature dependence of the diffusion coefficient in the temperature range from 1.2° K to the  $\lambda$ -point for a concentration  $c \sim 10^{-4}$ . In the present paper we consider the phenomena of diffusion, thermal diffusion, and thermal conductivity for dilute solutions of  $\text{He}^3$  in helium II.

According to Landau's theory<sup>4</sup>, liquid helium in the temperature region below the  $\lambda$ -point (helium II) is to be regarded as a weakly-excited quantum system. This implies that thermal energy of helium II may be represented as a gas of elementary excita-

tions—phonons and rotons<sup>5</sup>. The phonon energy  $\varepsilon_p$  is a linear function of the momentum

$$\varepsilon_p = sp, \quad (1.1)$$

where  $s$  is the velocity of sound in helium II.

The roton energy  $\varepsilon_r$  depends upon the momentum  $p$  in the following way

$$\varepsilon_r = \Delta_r + (p - p_0)^2/2\mu_r, \quad (1.2)$$

where  $\Delta_r$ ,  $p_0$ , and  $\mu_r$  are parameters of the theory;  $\mu_r$  represents the effective mass of the roton. From the most recent data<sup>6</sup>

$$\begin{aligned} \Delta_r/k &= 8,9^\circ \text{K}, \quad p_0/\hbar = 1,95 \cdot 10^8 \text{ cm}^{-1}, \\ \mu_r &= 1,7 \cdot 10^{-24} \text{ g}. \end{aligned} \quad (1.3)$$

As has been shown by Landau and Pomeranchuk<sup>7</sup>, all foreign particles (including He<sup>3</sup> atoms) dissolved in helium II combine with the normal component of the helium II, and do not participate in the superfluid motion.

From the experiments of Lynton and Fairbank<sup>8</sup>, who determined the velocity of second sound in mixtures of He<sup>3</sup> in helium II, it may be concluded<sup>9</sup> that the excitations associated with the He<sup>3</sup> atoms in helium II have the following spectrum:

$$\varepsilon_i = (p^2/2\mu) + \Delta, \quad (1.4)$$

where  $\mu = 8.5 m_1$  ( $m_1$  is the proton mass).

In order to find the dependence of the kinetic coefficients for a solution upon temperature and concentration it is essential to determine the distribution function describing the behavior of gases of elementary excitations when non-zero gradients of temperature  $T$ , concentration  $c$ , and velocity  $\mathbf{v}_n$  are present within the system. The distribution functions are determined by solution of a kinetic equation which we shall now derive.

## 2. THE KINETIC EQUATION

Non-zero temperature and concentration gradients within the solution give rise to motion of the normal and superfluid components of the helium II, leading to the appearance of additional terms on the left-hand side of the kinetic equation. We shall derive these additional terms by means of the method employed in Ref. 2.

The kinetic equation determining the distribution function  $n$  for the elementary excitations in a solution of He<sup>3</sup> in helium II has the form

$$\frac{\partial n}{\partial t} + \frac{\partial H}{\partial \mathbf{p}} \frac{\partial n}{\partial \mathbf{r}} - \frac{\partial H}{\partial \mathbf{r}} \frac{\partial n}{\partial \mathbf{p}} = I(n), \quad (2.1)$$

where  $n = n(\mathbf{r}, \mathbf{p}, t)$  is the distribution function,  $\mathbf{r}$  and  $\mathbf{p}$  are the radius and momentum for the excitation,  $H$  is the Hamiltonian for the excitation, and  $I(n)$  is the collision integral. When superfluid motion of velocity  $\mathbf{v}_s$  takes place in the solution, the Hamiltonian has the form

$$H = \varepsilon(\rho, \mathbf{p}) + \mathbf{p}\mathbf{v}_s, \quad (2.2)$$

where  $\varepsilon(\rho, \mathbf{p})$  is the energy of an elementary excitation in the coordinate system for which  $\mathbf{v}_s = 0$ .

The equilibrium distribution functions for the excitations in a solution in which there is uniform normal fluid motion with velocity  $\mathbf{v}_n$  and superfluid motion with velocity  $\mathbf{v}_s$  have the form<sup>2</sup>:

$$n = \left( \exp \left\{ \frac{\varepsilon_p - \mathbf{p}(\mathbf{v}_s - \mathbf{v}_n)}{kT} \right\} - 1 \right)^{-1} \text{ (phonons);} \quad (2.3)$$

$$N = \exp \left\{ - \frac{\varepsilon_r - \mathbf{p}(\mathbf{v}_n - \mathbf{v}_s)}{kT} \right\} \text{ (rotons);} \quad (2.4)$$

$$N_i = A(c, T) \exp \left\{ - \frac{\varepsilon_i - \mathbf{p}(\mathbf{v}_n - \mathbf{v}_s)}{kT} \right\} \text{ (impurities);}$$

$$A(c, T) = N_3 (2\pi\mu kT)^{-3/2} = (c\rho/m_3) (2\pi\mu kT)^{-3/2}, \quad (2.5)$$

$\rho$  is the solution density,  $c = N_3 m_3 / (N_3 m_3 + N_4 m_4)$  is the concentration,  $N_3$  and  $N_4$  are the numbers of atoms of He<sup>3</sup> and He<sup>4</sup> per unit volume, and  $m_3$  and  $m_4$  are the atomic masses of He<sup>3</sup> and He<sup>4</sup>. For dilute solutions

$$c = N_3 m_3 / \rho = N_3 m_3 / N_4 m_4 = \varepsilon m_3 / m_4, \quad (2.6)$$

where  $\varepsilon$  is the molar concentration.

If  $\nabla c$  and  $\nabla T$  are small,  $\mathbf{v}_n$  and  $\mathbf{v}_s$  are also small, and are proportional to a linear combination of  $\nabla c$  and  $\nabla T$ . Taking this into consideration, it is possible to linearize the hydrodynamic equations for the solution<sup>10, 11</sup> which take the form

$$\begin{aligned} \dot{\rho} &= -\text{div } \mathbf{j}; \quad \dot{\sigma} = -\text{div } \sigma \mathbf{v}_n; \quad \frac{\partial \mathbf{j}}{\partial t} + \nabla p = 0, \\ -\frac{\partial}{\partial t} (\rho c) &= -\text{div } (\rho c \mathbf{v}_n); \\ \dot{\mathbf{v}}_n - \dot{\mathbf{v}}_s &= - \left( 1 + \frac{\rho_s}{\rho_n} \right) \left( \dot{\sigma}_0 + \frac{k c}{m_3} \right) \\ \nabla T - \left( 1 + \frac{\rho_s}{\rho_n} \right) \frac{kT}{m_3} \nabla c, \end{aligned} \quad (2.7)$$

where  $\rho_n$  is the density of the normal component of the solution,  $\rho_n = \rho_{n0} + \rho_{ni}$ ;  $\rho_s$  is the density of the superfluid component;  $\sigma$  is the entropy per unit mass of the solution;  $\sigma_0$  is the entropy per unit mass for pure helium II;  $\rho_{n0}$  is the fraction of the normal density which is associated with the phonons and rotons; and  $\rho_{ni} = (\rho c/m_3)\mu$  is the fraction of the normal density associated with the impurities.

We can then obtain the form of the additional terms in the kinetic equation (2.1) by substituting into the left-hand side of Eq. (2.1) the distribution functions (2.3)–(2.5) and determining  $\dot{v}_n$  and  $\dot{v}_s$  from (2.7). We obtain, as a result, the kinetic equation for the impurity excitations in a dilute solution of He<sup>3</sup> in helium II:

$$\begin{aligned}
 D(N_i) &= N_i + \frac{\partial N_i}{\partial \mathbf{r}} \frac{\partial H}{\partial \mathbf{p}} - \frac{\partial N_i}{\partial \mathbf{r}} \frac{\partial H}{\partial \mathbf{r}} \\
 &= -\frac{N_{i0}}{kT} \left\{ \text{div} (\mathbf{j} - \rho \mathbf{v}_n) \left[ \frac{3}{2} k \left( \frac{c}{\rho} \frac{\partial T}{\partial c} - \frac{\partial T}{\partial \varphi} \right) + \frac{\varepsilon_i}{T} \left( \frac{\partial T}{\partial \varphi} - \frac{c}{\rho} \frac{\partial T}{\partial \varphi} \right) - \frac{\partial \varepsilon_i}{\partial \varphi} \right] \right. \\
 &\quad + \text{div} \mathbf{v}_n \left[ \left( \frac{\varepsilon_i}{T} - \frac{3}{2} k \right) \left( \frac{\partial T}{\partial s} s + \frac{\partial T}{\partial \rho} \rho \right) + kT - \frac{\partial \varepsilon_i}{\partial \varphi} \rho - \frac{1}{3} \frac{\partial \varepsilon_i}{\partial \mathbf{p}} \mathbf{p} \right] \\
 &\quad + \left( \frac{\partial \varepsilon_i}{\partial \rho_l} \rho_l - \frac{1}{3} \delta_{lk} \frac{\partial \varepsilon_i}{\partial \mathbf{p}} \mathbf{p} \right) \left( \frac{\partial v_{nk}}{\partial x_l} + \frac{\partial v_{nl}}{\partial x_k} - \frac{2}{3} \delta_{lk} \frac{\partial v_{ni}}{\partial x_l} \right) \\
 &\quad + \left[ \frac{\rho}{\rho_n} \frac{kT}{m_3} \mathbf{p} + \frac{kT}{c} \frac{\partial \varepsilon_i}{\partial \mathbf{p}} \right] \nabla c \\
 &\quad \left. + \left[ \frac{\rho}{\rho_n} \left( \sigma_0 + \frac{kc}{m_3} \right) \mathbf{p} + \frac{3}{2} k \frac{\partial \varepsilon_i}{\partial \mathbf{p}} - \frac{\varepsilon_i}{T} \frac{\partial \varepsilon_i}{\partial \mathbf{p}} \right] \nabla T - \frac{kT}{\rho} \frac{\partial \varepsilon_i}{\partial \mathbf{p}} \nabla \rho \right\}. \\
 D(N_i) &= I(N_i)
 \end{aligned} \tag{2.8}$$

Neglecting the terms in (2.8) which are associated with the first and second viscosity of the solution, we obtain the kinetic equation in which we are interested:

$$-\frac{N_{0i}}{kT} \left\{ \left( \frac{\rho}{\rho_n} \frac{kT}{m_3} \mathbf{p} - \frac{kT}{c} \frac{\partial \varepsilon_i}{\partial \mathbf{p}_i} \right) \nabla c + \left[ \frac{\rho}{\rho_n} \left( \sigma_0 + \frac{kc}{m_3} \right) \mathbf{p}_i + \frac{3}{2} k \frac{\partial \varepsilon_i}{\partial \mathbf{p}_i} - \frac{\varepsilon_i}{T} \frac{\partial \varepsilon_i}{\partial \mathbf{p}_i} \right] \nabla T \right\} = I(N_i). \tag{2.9}$$

In a similar manner we can derive the equations for the rotons and phonons; these equations have the form

$$\frac{n'}{kT} \left\{ \frac{\rho}{\rho_n} \frac{kT}{m_3} \mathbf{p} \nabla c + \left[ \frac{\rho}{\rho_n} \left( \sigma_0 + \frac{kc}{m_3} \right) \mathbf{p} - \frac{\varepsilon_i}{T} \frac{\partial \varepsilon_i}{\partial \mathbf{p}} \right] \nabla T \right\} = I(n). \tag{2.10}$$

Here,  $n'$  is the distribution function for the rotons or phonons, differentiated with respect to the argument  $(\varepsilon - \mathbf{p} \mathbf{v}_n + \mathbf{p} \mathbf{v}_s)/kT$ . In (2.10), as in (2.9), the terms associated with the viscosity of the helium have been omitted. In order to solve this system of kinetic equations it is necessary to know how the elementary excitations interact with one another. The scattering of phonons by phonons and rotons and of rotons by rotons has already been computed by Landau and Khalatnikov<sup>5</sup> in treating the viscosity of helium II. We shall consider below, therefore, the scattering of phonons by impurities and the scattering of impurities by impurities and rotons.

### 3. COLLISIONS BETWEEN ELEMENTARY EXCITATIONS

#### *Impurity-Roton Scattering*

The impurity-roton interaction law is not known. In selecting the impurity-roton interaction energy, therefore, we shall proceed from the same considerations as those used by Landau and Khalatnikov<sup>5</sup> in treating roton-roton scattering. In order to ascertain the temperature dependence of the kinetic coefficients for the solution it will suffice, in accordance with Ref. 5, to determine the impurity-roton scattering probability as a function of temperature accurately to within some constant multiplier.

This probability is insensitive to the choice of distribution function for the interaction of the impurity-rotor system.

Taking the impurity-rotor interaction energy to be a  $\delta$ -function of the separation, we shall treat it as a perturbation.

$$v = v_{01} \delta(\mathbf{r} - \mathbf{r}_1) \quad (3.1)$$

where  $\mathbf{r}$  and  $\mathbf{r}_1$  are the radius vectors for the impurity and the rotor, respectively, and  $v_{01}$  is a constant whose value may be determined from experiments on the diffusion of  $\text{He}^3$  in helium II containing the  $\text{He}^3$  as an impurity. We shall designate the energy and momentum of the impurity and the rotor, by  $E$ ,  $\mathbf{p}$  and  $E_1$ ,  $\mathbf{p}_1$ , respectively; for the state prior to the collision we shall use unprimed quantities, and for that following the collision, primed. The probability of a transition from the state  $A(\mathbf{p}, \mathbf{p}_1)$  to the state  $F(\mathbf{p}', \mathbf{p}'_1)$  is determined from the perturbation theory formula

$$dw = (2\pi/\hbar) |v_{AF}|^2 \delta(E + E_1 - E' - E'_1) \times (2\pi\hbar)^{-6} d\mathbf{p}' d\mathbf{p}'_1 \Omega^2, \quad (3.2)$$

where  $\Omega$  is the normalization volume.

Taking the wave functions for the system in the form of plane waves normalized over the volume, the matrix element for the transition  $v_{AF}$  can be readily computed and integrated over the impurity momentum. We obtain as a result

$$d\sigma_{ip} = (2\pi\mu/\hbar p) |v_{01}|^2 \delta(E + E_1 - E' - E'_1) \times (2\pi\hbar)^{-3} d\mathbf{p}'_1. \quad (3.3)$$

assuming the rotor to be at rest prior to the collision and taking the total momentum  $\mathbf{Q} = \mathbf{p} + \mathbf{p}_1 = \mathbf{p}' + \mathbf{p}'_1$  along the polar axis, we integrate (3.3) over  $\mathbf{p}'_1$ :

$$I = \int \delta(E + E_1 - E' - E'_1) d\mathbf{p}'_1 \rho_1'^2 \sin \vartheta d\vartheta d\varphi \\ = 4\pi\mu R \int_{(A-B)/L}^{(A+B)/L} \frac{2L^2 y^2 - L^2}{B V y^2 - 1} dy, \quad (3.4)$$

where  $\vartheta$  is the angle made by  $\mathbf{p}'_1$  with the total momentum  $\mathbf{Q}$ ;

$$M = \frac{\mu\mu_r}{\mu + \mu_r}, \quad A = \frac{p_0}{1 + \mu_r/\mu}, \quad B = \frac{Q\mu_r/\mu}{1 + \mu_r/\mu};$$

$$L^2 = \frac{p_0^2 + \mu_r Q^2/\mu - 2\mu_r E}{1 + \mu_r/\mu}, \quad y = (A + B \cos \vartheta)/L.$$

The symbol  $R$  in front of the integral sign indicates that the integration is carried out only over the region in which the integral assumes a real value. It can easily be shown, with the aid of the conservation laws, that  $I$  in (3.4) has a real value only for integration over the region  $1 \leq y \leq (A + B)/L$ . Carrying out the computation and expanding  $B$  and  $L$  in terms of  $p/p_0$  we obtain

$$\sigma_{ir}(\cos \vartheta) \\ = \frac{1}{\pi} \frac{|v_{01}|^2}{\hbar^4} \frac{\mu\mu_r}{\mu + \mu_r} \mu \left(\frac{\mu_r}{\mu}\right)^{-1/2} \left(1 + \frac{\mu_r}{\mu} - \cos^2 \vartheta\right)^{1/2}. \quad (3.5)$$

Averaging over the directions of the incident momentum for the impurity we find

$$\sigma_{ir} = (|v_{01}|^2 \mu/2\pi\hbar^4) \mu\mu_r/(\mu + \mu_r). \quad (3.6)$$

An absolute value for  $\sigma_{ir}$  may be obtained from experimental data<sup>3</sup> on the diffusion of  $\text{He}^3$  in helium II. It turns out to be

$$\sigma_{ir} = 2.1 \cdot 10^{-14} \text{ cm}^2. \quad (3.7)$$

The impurity-rotor interaction constant may be determined from (3.6) and (3.7)

$$|v_{01}| \approx 8 \cdot 10^{-38} \text{ erg-cm}^3.$$

#### Impurity-Impurity Scattering

The interaction law for the impurity particles is not known. Following, therefore, the considerations cited above in the treatment of impurity-rotor collisions, we select an interaction energy in the form of a  $\delta$ -function

$$v = v_{02} \delta(\mathbf{r}_1 - \mathbf{r}_2), \quad (3.8)$$

where  $\mathbf{r}_1$  and  $\mathbf{r}_2$  are the radius vectors of the impurities, and  $v_{02}$  is some constant whose value can be determined experimentally.

It can easily be seen that the computation of the total effective cross-section  $\sigma_{ii}$  for impurity-impurity scattering is completely analogous to the compu-

tation for the scattering of slow neutrons by nuclei<sup>12</sup>, with the single difference that since the impurity particles are identical it is accordingly necessary to symmetrize the wave functions. As a result, we obtain for  $\sigma_{ii}$

$$\sigma_{ii} = |v_{02}|^2 \mu^2 / 4\pi \hbar^4. \quad (3.9)$$

To obtain an approximate value we take

$$|v_{02}|^2 \sim 10^{-76} \text{ erg-cm}^3,$$

whence

$$\sigma_{ii} = 1.4 \cdot 10^{-15} \alpha \text{ cm}^2, \quad (3.10)$$

where the constant  $\alpha$  is included to take account of the fact that  $|v_{02}|$  is not actually known.

In collisions of impurities with impurities and with rotons there can also take place, in addition to scattering, an emission of phonons due to deceleration. Calculation shows, however, that the probability for such processes is small, and hence they need not be considered in the phenomena which we shall treat below.

#### Phonon-Impurity Scattering

We shall treat the impurity as a particle in a phonon field. From this standpoint the internal structure of the impurity is of no significance, since the phonon wavelength is much greater than the de Broglie wavelength for the impurity. Thus it may be shown, as in Ref. 2, that the total perturbation energy for the case under consideration is

$$v = -1/2 (\mathbf{p}\mathbf{v} + \mathbf{v}\mathbf{p}) + \frac{\partial \Delta}{\partial \rho} \rho' + 1/2 \frac{\partial^2 \Delta}{\partial \rho^2} \rho'^2 + \frac{p^2}{2\mu} (a\rho' + 1/2 b\rho'^2), \quad (3.11)$$

where  $\Delta$  is the zero-point energy of the impurity excitation,  $\rho'$  the deviation of the density of the solution from its equilibrium value due to the presence of a phonon,  $\mathbf{v}$  the macroscopic velocity of the medium associated with the presence of a phonon, and  $\mathbf{p} = -i\hbar \nabla$  the momentum operator for the impurity.

Calculations show that the last term in (3.11) is much smaller than the first. The terms involving derivatives of  $\Delta$  with respect to  $\rho$ , however, cannot be evaluated, since the functional form of  $\Delta = \Delta(\rho)$  is not known. We write the excitation energy in the final form

$$v = -1/2 (\mathbf{p}\mathbf{v} + \mathbf{v}\mathbf{p}) + \frac{\partial \Delta}{\partial \rho} \rho' + 1/2 \frac{\partial \Delta}{\partial \rho^2} \rho'^2. \quad (3.12)$$

Further, on carrying out computations of the type performed in Ref. 5, we obtain for the differential effective phonon-impurity cross-section  $\sigma_{pi}$  the expression

$$d\sigma_{pi} = (Pp^2/4\pi\hbar\mu s)^2 \{ \mathbf{m}(\mathbf{n} + \mathbf{n}')(\mathbf{nn}') + \frac{P}{\mu s} [(\mathbf{mn})(\mathbf{mn}')(\mathbf{nn}') + \mathbf{m}(\mathbf{n} + \mathbf{n}')(\mathbf{mn})] + A + B \}^2 dO', \quad (3.13)$$

in which  $P$  is the impurity momentum,  $p$  the phonon momentum,  $s$  the velocity of sound,  $\mathbf{m} = \mathbf{P}/P$ ,  $\mathbf{n} = \mathbf{p}/p$ ,  $\mathbf{n}' = \mathbf{p}'/p'$  is the direction of the phonon momentum following the collision, and

$$A = \frac{p^2}{P_s} \left( \frac{\partial \Delta}{\partial \rho} \right)^2 \frac{1}{\mu s^2} (\mathbf{nn}') + \frac{p^2}{P_s} \left( \frac{\partial \Delta}{\partial \rho} \right) \frac{1}{\rho^2} \left[ (1 - \mathbf{nn}') \frac{s^2}{\rho} - \frac{\partial s^2}{\partial \rho} \right], \quad B = \frac{p^2}{P_s} \frac{\partial^2 \Delta}{\partial \rho^2}. \quad (3.14)$$

Averaging (3.13) over the angles of the incident and scattered impurity particles, we obtain

$$d\sigma_{pi}(P, p, \psi) = \left( \frac{Pp^2}{4\pi\hbar^2\mu s} \right)^2 \left\{ \frac{2}{3} (1 + \cos \psi) \cos^2 \psi + \frac{P}{3\mu s} (A + B) (1 + 2 \cos \psi) + (A + B)^2 \right\} dO', \quad (3.15)$$

where  $\cos \psi = (\mathbf{n}'\mathbf{n})$ . Integrating (3.15) over all scattering angles we find for the total effective cross-section for scattering of a phonon of momentum  $p$  by an impurity particle

$$\sigma_{pi} = \frac{(Pp/\hbar^2\mu s)^2 \bar{\delta}'}{4\pi}. \quad (3.16)$$

$$\bar{\delta}' = \frac{2}{9} + \frac{1}{3} (P/\mu s) \bar{A} (1 + 2 \cos \psi) + \frac{2}{9} (P/\mu s) B + \bar{A}^2 + 2\bar{A}B + B^2,$$

the bar indicating an average over the angles. Substituting into (3.16) the numerical values of all of the known parameters, we obtain

$$\sigma_{pi} = 4.7 \cdot 10^{-20} \delta' (yT) (xT)^4. \quad (3.17)$$

Here  $x$  is the phonon momentum in units of  $kT/s$ , and  $y$  is the energy of the impurity in units of  $3kT/2$ , so that

$$p = xkT/s \text{ and } p^2/2\mu = 3ykT/2.$$

#### 4. TEMPERATURE DEPENDENCE OF THE DIFFUSION COEFFICIENT

We shall treat first the diffusion of the impurity. The kinetic equations for the impurities and the thermal excitations have been derived above (2.9), (2.10); when a non-zero concentration gradient is present in the system they have the form

$$\frac{n_{0i}}{kT} \frac{\rho}{\rho_n} \frac{kT}{m_3} \frac{\rho_{n0}}{\rho_{ni}} \mathbf{p}_i \nabla c = I_{ie} + I_{ii}; \quad (4.1)$$

$$\frac{n'}{m_3} \frac{\rho}{\rho_n} \mathbf{p} \nabla c = I_{ee} + I_{ei}. \quad (4.2)$$

The collision integrals in (4.1) and (4.2) refer to the following processes:  $I_{ie}$ —scattering of impurities by excitations;  $I_{ii}$ —scattering of impurities by impurities;  $I_{ei}$ —scattering of excitations by impurities;  $I_{ee}$ —scattering of excitations by excitations.

The solution of the kinetic equations for the general case involves extremely tedious computations. For this reason we will consider four limiting cases; the results for the intermediate regions may then be obtained by interpolation. Elementary calculations show that for concentrations of impurity excitations  $c < 10^{-6}$  deviations from the equilibrium values of the roton and impurity distribution functions are determined by the scattering of rotons and impurities by one another. Roton—phonon and impurity—phonon collisions need, therefore, not be considered for  $c < 10^{-6}$  in connection with the establishment of equilibrium in the impurity and roton gases.

On the other hand, the momentum transfer is determined at high temperatures, as can be seen from what follows, by the scattering of impurities by rotons, and at low temperatures by the scattering of phonons by impurities. In view of these circumstances we shall consider first the high-temperature region, in which the diffusion of the impurities may be regarded as taking place in a pure roton gas. We shall then consider the low-temperature region, in which the diffusion of the impurities may be regarded as taking place in a pure phonon gas. The expression for the diffusion coefficients in the gen-

eral case is obtained by joining the solutions obtained at high temperatures with the solutions for the case of low temperatures. The temperature which in a given instance divides the high- and low-temperature regions depends upon the concentration of the impurity excitations in the solution and will be determined below.

It follows from the symmetry of the problem that deviations of the distribution functions from their equilibrium values may be sought in the form

$$\delta n_i = a_i(\varepsilon) (\mathbf{p}_i \nabla c) n_i, \quad (4.3)$$

$$\delta n = a_r(\varepsilon) (\mathbf{p} \nabla c) n (n + 1). \quad (4.4)$$

It is essential here that the function  $a$  be angle-independent.

#### The High-Temperature Region

Under these conditions the fundamental role in the thermodynamics as well as in kinetic phenomena is played by the rotons. Detailed analysis shows that the phonons play no part in transport processes at temperatures above  $0.6^\circ \text{K}$ . In this temperature region we shall consider two limiting cases.

1) The relative number of impurity particles is much smaller than the number of excitations. The probability for collisions between impurity particles is small and such collisions may be ignored. The concentration region for which these conditions hold is determined from the condition  $t_{ir} \ll t_{ii}$ ; *i.e.*, the time characterizing impurity—roton collisions is much smaller than that characterizing collisions between impurities. In this case the kinetic equation for the impurities is highly simplified.

The deviations of the roton distribution function, as a simple calculation shows, are much smaller than the deviations of the impurity functions. The roton gas may thus be taken to be in equilibrium. Impurity particles colliding with a roton are scattered elastically. From these considerations we obtain

$$\begin{aligned} n_i \frac{\rho}{\rho_n m_3} \frac{\rho_{n0}}{\rho_{ni}} (\mathbf{p} \nabla c) &= I_{ir} \\ &= - \int a (\mathbf{p} \nabla c) \sigma_{ir}^* v_i n_r d\tau_r n_i. \end{aligned} \quad (4.5)$$

Here  $\sigma_{ir}^*$  is the transport cross-section for scattering of an impurity by a roton.

$$\sigma_{ir}^* = 1/2 \int \sigma_{ir} (1 - \cos \psi) d \cos \psi. \quad (4.6)$$

Further, from (3.5),  $\sigma_{i\mathbf{r}}^*$  is independent of the energy of the colliding particles. From (4.5), therefore, it follows that

$$a = - (\rho/\rho_n m_3) (\rho_{n0}/\rho_{ni}) / N_r \sigma_{i\mathbf{r}}^* v_i. \quad (4.7)$$

We now compute the impurity current

$$\mathbf{g} = \int a (\mathbf{p}_i \nabla c) \mathbf{p}_i d\tau_i = - \frac{\rho \rho_{n0}}{3 \rho_n \rho_{ni} m_3 N_r} \int \frac{p_i^2 n_i d\tau_i}{\sigma_{i\mathbf{r}}^* v_i} \nabla c.$$

Equating the current  $\mathbf{g}$  to the quantity  $-\rho D \nabla c$  we obtain an expression for the diffusion coefficient

$$D = \frac{\rho_{n0}}{\rho_n} \frac{kT}{N_p m_3} \overline{(1/\sigma_{i\mathbf{r}}^* v_i)}, \quad (4.8)$$

where

$$\overline{(1/\sigma_{i\mathbf{r}}^* v_i)} = \int \frac{1}{\sigma_{i\mathbf{r}}^* v_i} p_i^2 n_i d\tau_i / \int p_i^2 n_i d\tau_i. \quad (4.9)$$

Under the present conditions  $\rho_{ni} \ll \rho_{n0}$ , and the factor  $\rho_{n0}/\rho_n$  in (4.8) is equal to unity. Thus the temperature variation of the diffusion coefficient is largely determined by the temperature dependence of the number of rotons  $N_r$ , which falls exponentially with temperature. Under these conditions the diffusion coefficient rises with falling temperature as  $e^{\Delta T}$ .

2) The number of impurities exceeds the number of thermal excitations (rotons). More precisely, the time characterizing collisions between impurities is much smaller than that for impurity-roton collisions. In this case the distribution function for the impurities may be taken to have its equilibrium form. Since the differences between the roton energy  $\Delta$  and between the momentum and  $p_0$  are slight, the factor  $a$  in Eq. (4.4) can be regarded as energy-independent. For the impurity kinetic equation in this limiting case we write

$$n_i \frac{\rho_{n0} \rho}{\rho_{ni} \rho_n m_3} (\mathbf{p} \nabla c) = I_{i\mathbf{r}} + I_{ii}. \quad (4.10)$$

We now multiply both sides of this equation by the momentum  $\mathbf{p}$  and integrate over the phase volume for the impurity  $d\tau_i$ . Since for impurity-impurity collisions the total impurity momentum is unchanged, the integral

$$\int p_i I_{ii} d\tau_i$$

is equal to zero. Performing the indicated integration we obtain

$$\rho_{n0} \rho / \rho_{ni} \rho_n m_3 = \overline{a_r \sigma_{i\mathbf{r}}^* v_i} N_r. \quad (4.11)$$

Here

$$\overline{\sigma_{i\mathbf{r}}^* v_i} = \int \sigma_{i\mathbf{r}}^* v_i p_i^2 n_i d\tau_i / \int p_i^2 n_i d\tau_i. \quad (4.12)$$

Solving this relation for  $a_r$ , we now calculate the roton current

$$\mathbf{g}_r = \int a_r (\mathbf{p}_r \nabla c) n_r \mathbf{p}_r d\tau = \frac{\rho_0^2 \rho_{n0} \rho}{3 \rho_{ni} \rho_n m_3 \overline{\sigma_{i\mathbf{r}}^* v_i}} \nabla c. \quad (4.13)$$

On the other hand, the total momentum of the liquid does not change when diffusion is present, and the sum of the diffusion currents of the rotons and impurities is equal to zero

$$\mathbf{g}_r + \mathbf{g}_i = 0. \quad (4.14)$$

Consequently

$$\mathbf{g}_r = \rho D \nabla c. \quad (4.15)$$

Comparing (4.15) and (4.13), we obtain for  $D$  the expression

$$D = \frac{\rho_{n0} \rho_0^2}{3 \rho_{ni} \rho_n m_3} \left/ \overline{\sigma_{i\mathbf{r}}^* v_i} \right. = \frac{(\rho_{n0})^2 kT}{\rho_{ni} \rho_n m_3 N_r} \left/ \overline{\sigma_{i\mathbf{r}}^* v_i} \right. \quad (4.16)$$

In this limiting case ( $\rho_n = \rho_{ni}$ ) the temperature variation of the diffusion coefficient is determined by the variation of the normal density, which decreases exponentially in this temperature region.

According to (3.5)  $\sigma_{i\mathbf{r}}^*$  is independent of the energy of the impurity. Performing the elementary integrations in (4.9) and (4.12) we obtain

$$\frac{1}{\overline{\sigma_{i\mathbf{r}}^* v_i}} = \frac{1}{\sigma_{i\mathbf{r}}^*} \cdot \frac{1}{v_i} = \frac{1}{\sigma_{i\mathbf{r}}^*} \frac{32}{9\pi}; \quad \overline{v_i} = \frac{8}{3} \left( \frac{2kT}{\pi\mu} \right)^{1/2}. \quad (4.17)$$

Thus the quantities  $\overline{(1/\sigma_{i\mathbf{r}}^* v_i)}$  and  $1/\overline{\sigma_{i\mathbf{r}}^* v_i}$  appearing in Eqs. (4.8) and (4.16) differ only by the factor  $32/9\pi$ , which is very near to unity. This makes it

possible to write an interpolation formula for the diffusion coefficient which covers the whole high-temperature region:

$$D = (\rho_{n0}/\rho_n)^2 (kT/m_3) / N_r \sigma_{ir}^* v_i. \quad (4.18)$$

For the case in which  $t_{ir} \ll t_{ii}$ ,  $\rho_{n0} \approx \rho_n \gg \rho_{ni}$  Eq. (4.18) reduces to Eq. (4.8). In the other limiting case  $t_{ir} \gg t_{ii}$ ,  $\rho_{n0} \ll \rho_{ni} \approx \rho_n$  and Eq. (4.18) reduces to Eq. (4.16). Equation (4.18) may also be rewritten in the form

$$D = (\rho_{n0}/\rho_n)^2 (kT/m_3) t_{ir}, \quad (4.19)$$

where the time  $t_{ir}$  characterizes the scattering of impurities by rotons:

$$\frac{1}{t_{ir}} = N_r \sigma_{ir}^* \bar{v}_i = \sigma_{ir}^* N_r \frac{8}{3} \left( \frac{2kT}{\pi\mu} \right)^{1/2}; \quad (4.20)$$

$$N_r = \frac{2p_0^2 (\mu_r kT)^{1/2} e^{-\Delta/kT}}{(2\pi)^{3/2} \hbar^3}.$$

Making use of the experimental values for the diffusion coefficient we can with the aid of (4.19) find  $\sigma_{ir}^*$  and, consequently, the unknown impurity-roton interaction constant in Eq. (3.8). Thus we find

$$1/t_{ir} = 1.0 \cdot 10^{13} T e^{-\Delta/kT}. \quad (4.21)$$

Inserting this value into Eq. (4.19) we obtain an expression for the diffusion coefficient.

$$D = 2.8 \cdot 10^{-6} e^{-\Delta/kT} (\rho_{n0}/\rho_n)^2, \quad (4.22)$$

$$\rho_{n0} = \rho_{nr} = p_0^2 N_r / 3kT,$$

which is applicable over the whole high-temperature region, in which the phonons play no role.

As regards the time  $t_{ii}$ , which characterizes collisions between two impurities, this cannot be obtained from the experimental data. Extremely crude estimates for this time yield

$$\frac{1}{t_{ii}} = \overline{\sigma_{ii} v_i} N_i = \frac{1}{\pi} \frac{|v_{02}|^2 m^2}{\hbar^4} \sqrt{\frac{kT}{\pi\mu}} N_i \quad (4.23)$$

$$= 3 \cdot 10^{11} \cdot T^{1/2} c$$

( $v_{02}$  is taken equal to  $10^{-38}$  erg/cm<sup>3</sup>, approximately the same rate as for the interaction between two rotons). The concentration for which the times  $t_{ir}$  and  $t_{ii}$  become comparable in order of magnitude is

$$c \sim 10^2 T^{1/2} e^{-\Delta/kT}.$$

### The Low-Temperature Region

In this temperature region the part played by the rotons, both in transport processes and in the thermodynamics, is negligible. These phenomena are now governed entirely by the phonon portion of the excitation spectrum. As in the high-temperature region, we shall consider two limiting cases.

1) The case in which there are few phonons as compared with the number of impurities. More precisely, the time  $t_{ii}$  between collisions of two impurities is small as compared with the phonon-impurity collision time  $t_{ip}$ . Evaluation of the times  $t_{ii}$  and  $t_{ip}$  with the aid of (3.9) and (3.18) shows that the inequality  $t_{ii} \ll t_{ip}$  limits the region of applicability to that indicated by the inequality

$$c > 10^{-5} T^{7.5}. \quad (4.24)$$

This condition (4.24) indicates that in solving the kinetic equations for the present case the impurity excitations may be taken to be in equilibrium for concentrations down to  $c > 10^{-6}$ . The phonons, however, are scattered elastically, and their distribution does not change (the phonons form a light gas, the impurities a heavy gas). Calculation shows that, despite the low phonon concentration, phonon-phonon collisions are extremely important in the establishment of equilibrium with regard to energy in the phonon gas. The time required to establish energy equilibrium in a phonon gas has been calculated<sup>5</sup> and is

$$1/t_p = 10^5 T^7 x (x + 6)^3 \quad (4.25)$$

( $x = \epsilon/kT$  is the phonon energy in units of  $kT$ ). The time  $t_{pi}$  characterizing the scattering of a phonon by the impurity gas is

$$\frac{1}{t_{pi}} = \frac{6!sN_i}{8\pi} \left[ \frac{(3\mu kT)^{1/2} (kT/s)^2}{\hbar^2 \rho_s} \right]^2 \delta, \quad (4.26)$$

$$\delta = \frac{8}{45} + L_0 + \frac{L_{-2}}{3\mu kT},$$

$\delta$  is a complicated function of the parameters describing the impurity energy spectrum; in our calculations it will be taken of order unity (*cf.* Sec. 3). In transport processes the important phonons are those having energies on the order of 6 to 7  $kT$ . For such phonons the inequality  $t_p < t_{pi}$  assumes the form



$$T^2/c > 1.$$

This latter inequality shows that at temperatures down to 0.1° K for concentrations  $c < 10^{-2}$  equilibrium with respect to energy will be established rapidly as compared with the scattering of phonons by impurities. As regards the time  $\theta_p$  for establishment of equilibrium with respect to the number of phonons (for a more detailed discussion, cf. Ref. 5), this will in the region of interest be comparable

with the time  $t_{ip}$ . Taking values for  $\theta_p$  from the expression  $\theta_p = 2 \times 10^8 T^9$  we obtain

$$t_{pi}/\theta_p \sim T^4/15c. \quad (4.26')$$

The solution of the kinetic equation for this limiting case follows a procedure completely analogous to that involved in the calculation of the phonon viscosity<sup>5</sup>. Omitting the simple computations, we give directly the expression for  $\delta n_p$

$$\delta n_p = n_p (n_p + 1) \frac{\rho}{\rho_n} \frac{kT}{m_3} \frac{t_{pi}}{s} \frac{31.2 - (\rho s / kT) (3.5 - 3.7 t_{pi} / \theta_p) \rho \nabla c}{(1 + 8 t_{pi} / \theta_p) \rho}. \quad (4.27)$$

The time  $t_{pi}$  for scattering of phonons by the impurity is determined from Eq. (4.26). Further, with the aid of (4.27) we compute the diffusion component of the phonon current

$$\mathbf{g}_p = \int \mathbf{p} \delta n_p d\tau_p$$

and, using the fact that  $\mathbf{g}_i = -\mathbf{g}_p = -\rho D \nabla c$ , we find the diffusion coefficient to be

$$D = 5.1 \frac{\rho_{np} kT}{\rho_n m_3} t_{pi} \left\{ \frac{1 + 0.75 t_{pi} / \theta_p}{1 + 8 t_{pi} / \theta_p} \right\}. \quad (4.28)$$

Inasmuch as the temperature dependence of  $\rho_{np}$  is given by a  $T^4$  law, the product  $\rho_{np} kT t_{pi}$  is, in view of (4.26), temperature-independent. In this region, therefore, the diffusion depends only weakly upon temperature. The entire temperature dependence is embodied in the factor enclosed in the braces.

We shall now treat the final limiting case.

2) The number of impurities is small compared with the number of phonons. Here only collisions between impurities and phonons are significant; collisions of the impurities with one another are not important. Simple analysis shows that in this case the deviations of the phonon distribution functions from their equilibrium values are small and may be neglected in the kinetic equation.

The deviation of the impurity distribution function from its equilibrium value due to the presence of a concentration gradient can be written in the form

$$\delta n_i = a_i(\varepsilon) (\mathbf{p}_i \nabla c) n_i \quad (4.29)$$

The quantity  $a_i$  depends upon the energy of the im-

purity. We can also write down the kinetic equations for the impurities and phonons

$$(n_i \rho / \rho_n m_3) (\mathbf{p}_i \nabla c) = I_{ip}, \quad (4.30)$$

$$(n(n+1) \rho / \rho_n m_3) (\mathbf{p}_p \nabla c) = I_{pi} + I_{pp}. \quad (4.31)$$

We multiply the second of these equations by  $\mathbf{p}_p$  and integrate over the phonon phase volume; taking account of the fact that in collisions between two phonons the total phonon momentum does not change, we obtain

$$-(\rho kT / m_3) \nabla c = \int \mathbf{p}_p I_{pi} d\tau_p. \quad (4.32)$$

In what follows we shall disregard the weak dependence of the scattering cross-section  $\sigma_{pi}$  upon the phonon scattering angle. Further, it follows from the conservation of energy

$$p_i^2 / 2\mu + sp_p = p_i'^2 / 2\mu + sp_p'$$

that the change in the energy of an impurity resulting from a collision will be small compared with the magnitude of the energy  $\varepsilon_i$

$$\Delta \varepsilon = \varepsilon_i' - \varepsilon_i = \mathbf{p}_i (\mathbf{p}_p - \mathbf{p}_p') / \mu + (\mathbf{p}_p - \mathbf{p}_p')^2 / 2\mu.$$

This is connected with the fact that the momentum of a phonon is small compared with that of the impurity. We expand the difference  $a_i' p_i' - a_i p_i$  appearing in the collision integral in a power series to terms of the second order of  $\Delta \varepsilon$ . After a simple integration over angle the integral equation (4.30) assumes the following form:

$$= \left( \frac{5}{3} \frac{\partial a_i}{\partial \varepsilon} + \frac{2}{3} \frac{\partial^2 a_i}{\partial \varepsilon^2} \varepsilon \right) \int d\tau_p n_p (n_p + 1) \sigma_{p_i s_p} \frac{\rho_p^2}{\mu}. \quad (4.33)$$

We thus reduce our problem to one of solving a differential equation of the Fokker-Planck type.

The relation (4.32) yields a condition which must be satisfied by the functions  $a_i$ ; this condition has the form

$$-\frac{\rho kT}{m_3} = \iint \sigma_{p_i s} \left( a_i + \frac{2}{3} \frac{\partial a_i}{\partial \varepsilon} \varepsilon \right) n_p d\tau_p \frac{\rho_p^2}{3} n_i d\tau_i. \quad (4.34)$$

The cross-section  $\sigma_{p_i}$  depends linearly upon  $\varepsilon_i$ . The solution of Eq. (4.33) is readily found

$$a_i = A \ln \varepsilon + \left( B - \frac{2}{3} A \right) + C \varepsilon^{-1/2}, \quad (4.35)$$

where

$$A = \frac{\mu \rho}{\rho_{ni} m_3} \left/ \int \left( \frac{\sigma}{\varepsilon} \right) \rho_p^2 n_p (n_p + 1) d\tau_p \right. \\ = \mu \rho / 2 \rho_{ni} m_3 \rho_{np} \sigma_{p_i s}, \quad (4.36)$$

$$\frac{1}{\sigma_{p_i s}} = \frac{s \int \sigma_{p_i} n_p (n_p + 1) \rho_p^2 d\tau_p n_i d\tau_i}{\int \rho_p^2 n_p (n_p + 1) d\tau_p n_i d\tau_i}. \quad (4.37)$$

The characteristic time previously introduced can be expressed in terms of the quantities of (4.37) by means of

$$1/t_{p_i} = 0.47 \overline{\sigma_{p_i s}} N_i, \quad (4.38)$$

$B$  and  $C$  are constants of integration. The constant  $C$  must necessarily be set equal to zero, since the number of phonons moving in any given direction must be finite. The constant  $B$  is determined by the condition (4.32)

$$\overline{\sigma_{p_i s}} \int (A \ln \varepsilon + B) \varepsilon n_i d\tau = -\rho kT / m_3 \rho_{n0}. \quad (4.39)$$

From this we obtain

$$B = -2A - A \overline{\ln \varepsilon}, \\ \overline{\ln \varepsilon} = \int \ln \varepsilon \cdot \varepsilon n_i d\tau_i / \int \varepsilon n_i d\tau_i. \quad (4.40)$$

We now compute the impurity diffusion current

$$\mathbf{g}_i = -\rho D \nabla c \\ = \int A \left( \ln \varepsilon - \overline{\ln \varepsilon} - \frac{8}{3} \right) \rho_i (\rho_i \nabla c) n_i d\tau_i. \quad (4.41)$$

From this, taking (4.40) into account, we obtain

$$D = 8AkT \rho_{ni} / 3\rho = 4\rho_{ni} kT / 3m_3 \rho_{n0} \overline{N_i \sigma_{p_i s}}. \quad (4.42)$$

Thus we have, to within a factor of order unity

$$D = (\rho_{ni} kT / \rho_{n0} m_3) / \overline{N_i \sigma_{p_i s}}. \quad (4.43)$$

The coefficient  $D$  thus obtained increases with falling temperature according to a  $T^{-8}$  law.

An interpolation formula embracing both limiting cases at low temperatures may readily be written. For this we rewrite Eq. (4.28), which holds in the region  $\rho_{np} \ll \rho_n \approx \rho_{ni}$ , in the following form:

$$D = 11 \left( \frac{\rho_{np}}{\rho_{ni}} \right)^2 \frac{kT}{m_3} \frac{1}{\sigma_{p_i s} N_p (4kT / \mu s^2)} \frac{1 + 0.75 t_{p_i} / \theta_p}{1 + 8 t_{p_i} / \theta_p}. \quad (4.43')$$

Here we make use of the relation (4.38) and the following well-known formulas:

$$\rho_{np} = 4E_p / 3s^2 \approx (4kT / s^2) N_p, \quad \rho_{ni} = \mu N_i. \quad (4.44)$$

Equation (4.43), which holds for the region in which  $\rho_n = \rho_p \gg \rho_{ni}$ , may likewise be rewritten in the form

$$D = \frac{kT}{m_3} \left/ \overline{\sigma_{p_i s}} N_p \frac{4kT}{\mu s^2} \right. . \quad (4.45)$$

We note further that in this region the impurity concentrations are extremely small, and, in accordance with (4.26'),

$$t_{p_i} / \theta_p \gg 1.$$

It is now readily seen that Eqs. (4.43) and (4.45) can be combined into a single equation

$$D = \left( \frac{\rho_{np}}{\rho_n} \right)^2 \frac{kT}{m_3} \frac{1}{\sigma_{p_i s} N_p (4kT / \mu s^2)} \frac{11 + 8 t_{p_i} / \theta_p}{1 + 8 t_{p_i} / \theta_p}. \quad (4.46)$$

Comparing Eq. (4.18), which applies in the high-temperature region, with Eq. (4.46), applying in the

low-temperature region, we note that both of these equations may be combined into a single expression

$$D = (\rho_{n0}/\rho_n)^2 (kT/m_3) t_i, \quad (4.47)$$

in which the time  $t_i$  characterizes the collisions of the impurity with the thermal excitations is given by

$$t_i = \left\{ \frac{\sigma_{ir} v_i N_r}{\sigma_{ip} S N_p} + \frac{\sigma_{ip} S N_p}{\mu s^2} \frac{4kT}{11 + 8t_{pi}/\theta_p} \right\}^{-1} \quad (4.48)$$

In the limit of high temperatures, for which the gas of thermal excitations consists principally of rotons,  $\rho_{n0} = \rho_{np}$ , and all of the terms due to phonons may be neglected. We obtain as a result Eq. (4.18). In the low-temperature region, on the other hand, we neglect the rotons, and obtain Eq. (4.46).

Making use now of Eqs. (4.21), (4.26) and (4.38), we obtain an expression for  $t_i$  in a form convenient for numerical calculation:

$$t_i = \left\{ 1 \cdot 10^{13} T e^{-\Delta/kT} + 1.2 \cdot 10^6 T^9 \frac{1 + T^4/15c}{11 + T^4/15c} \right\}^{-1}. \quad (4.49)$$

It should be noted that the second term in (4.49), due to the phonons, becomes important in this equation at temperatures below  $0.5^\circ \text{K}$ . This can easily be understood if we recall that in the thermodynamic treatment the phonon component of the normal density predominates only for  $T < 0.5^\circ \text{K}$ . From the kinetic standpoint the times characterizing scattering of the impurity by rotons  $t_{ir}$  and by the phonon gas  $t_{ip}$  are comparable only for  $T \sim 0.5^\circ \text{K}$ . For temperatures higher than  $0.5^\circ \text{K}$ ,  $t_{ir} \ll t_{ip}$ , and, as a consequence, the diffusion is in this case determined by scattering of impurities by rotons. Thus for  $T > 0.5^\circ \text{K}$  we may limit ourselves to the first term in Eq. (4.49) at all concentrations:

$$1/t_i = 10^{13} T e^{-\Delta/kT}. \quad (4.50)$$

## 5. THERMAL DIFFUSION

In general, when non-zero temperature and concentration gradients exist in a solution (under constant pressure) the impurity current  $\mathbf{g}$  is defined by the expression

$$\mathbf{g} = -\rho D [\nabla c + (k_T/T) \nabla T], \quad (5.1)$$

where  $D$  is the diffusion coefficient as calculated above (4.38), and  $k_T$  is the thermal diffusion ratio. Simple calculation shows that the thermal diffusion contribution to the diffusion current is negligible when the number of impurity excitations is small compared with the number of thermal excitations.

1) The case for which the number of impurities greatly exceeds the number of rotons. Following a procedure completely analogous to that used in calculating the diffusion coefficient in the second limiting case, we find for the thermal diffusion ratio

$$k_T = c (1 - \rho_{ni} \sigma_0 m_3 / \rho_{n0} k c). \quad (5.2)$$

2) The case of low temperatures, for which the momentum is transported by the phonons and the contribution of the rotons to the diffusion process may be neglected. In this limiting case, the thermal diffusion ratio is

$$k_T = c (1 + \sigma_0 m_3 / k c - \rho_n \sigma_p m_3 / \rho_{np} k c). \quad (5.3)$$

For sufficiently low temperatures ( $T \ll 0.6^\circ \text{K}$ ),  $\rho_{n0} \sim \rho_{np}$  and  $\sigma_0 \sim \sigma_p$ , and the thermal diffusion ratio takes on the simpler form

$$k_{Tp} = c (1 - \rho_{ni} \sigma_p m_3 / \rho_{np} k c) = c (1 - \mu s^2 / kT). \quad (5.3')$$

Comparison of  $k_T$  for the roton region (5.2) with  $k_T$  for the deep phonon region (5.3') shows that Eq. (5.2) can serve as a useful interpolation formula for the thermal diffusion ratio of the solution over the whole range of temperatures and concentrations:

$$k_T = c (1 - \rho_{ni} \sigma_0 m_3 / \rho_{n0} k c). \quad (5.4)$$

## 6. THERMAL CONDUCTIVITY

The mechanism for thermal transport in the solution can be twofold. On one hand, heat can be transferred due to motion of the normal component of the helium. On the other hand, the presence within the solution of a non-zero temperature gradient leads to an additional thermal current which arises, as in ordinary condensed media, from the transfer of heat by diffusion of thermal excitations. This thermal current is characterized by a definite thermal conductivity coefficient. The thermal conductivity coefficient for a solution of  $\text{He}^3$  in helium II is defined in such a way that for zero impurity diffusion current  $\mathbf{g} = 0$  the heat current  $\mathbf{q}$  will be

$$-q = \kappa \nabla T. \quad (6.1)$$

The thermal conductivity coefficient  $\kappa$  for the solution is a sum of roton, phonon and impurity components. Calculations analogous to those carried out for the thermal conductivity of helium II<sup>6</sup> yield the following result:

$$\kappa = \kappa_r + \kappa_p + \kappa_i, \quad (6.2)$$

$$\kappa_r = t_r \frac{\Delta_r^2 N_r}{3T \mu_r} \left[ 1 - \frac{3\sigma_0 \sigma_r T^2 \mu_r \rho^2}{\Delta_r^2 N_r \rho_{n0}} \right]. \quad (6.3)$$

$$\kappa_p = 1.7\theta N_p s^2 k \frac{11 + 8\theta/\theta_p}{1 + 8\theta/\theta_p} \left\{ 1 - \frac{\sigma_0 T \rho}{\rho_{n0} s^2} \right\}, \quad (6.4)$$

$$\kappa_i = \frac{c\rho}{\mu} t_i \frac{kT}{m_3} \frac{5}{2} k, \quad (6.5)$$

where  $s$  is the velocity of sound,  $\sigma_0$  is the entropy per unit mass for pure helium II, and  $\sigma_r$  is the entropy per unit mass of the roton gas. The effective time  $\theta$  is determined by the expression

$$\begin{aligned} 1/\theta &= 1/t_{pr} + 1/t_{pi}, \\ \frac{1}{t_{pr}} &= 0,85 \frac{N_r 6!}{4\pi s} \left[ \frac{\rho_0 (kT/s)^2}{\rho \hbar^2} \right]^2, \\ \frac{1}{t_{pi}} &= \frac{6! N_i}{8\pi s} \left[ \frac{(3\mu kT)^{1/2} (kT/s)^2}{\rho \hbar^2} \right]^2 \delta. \end{aligned} \quad (6.6)$$

In the transition to the pure helium II case  $c \rightarrow 0$ , and  $\kappa_r$  and  $\kappa_p$  go over into the corresponding quantities for helium II as computed in Ref. 6.

It can easily be seen that for concentrations  $c > 10^{-6}$ ,  $\kappa_i \ll \kappa_r + \kappa_p$  over the whole temperature range. Substituting numerical values for the quantities in (6.3)–(6.5) and dropping the negligible second term in  $\kappa_r$  we obtain

$$\begin{aligned} \kappa &= \frac{2.0 \cdot 10^3}{T} \frac{1}{1 + 0.2ce^{8.9/T}} \\ &+ \frac{7.6 \cdot T^{-1/2} e^{8.9/T}}{1 + 8.28 \cdot 10^{-2} T^{1/2} c e^{8.9/T}} \frac{11 + 8\theta/\theta_p}{1 + 8\theta/\theta_p} \left[ 1 - \frac{\sigma_0 T \rho}{\rho_{n0} s^2} \right]. \end{aligned} \quad (6.7)$$

The temperature dependence of  $\kappa$  for various concentrations is illustrated in Fig. 1.

## 7. EFFECTIVE THERMAL CONDUCTIVITY OF SOLUTIONS

Experiment gives a direct determination not of the diffusion coefficient, but of the effective thermal conductivity coefficient

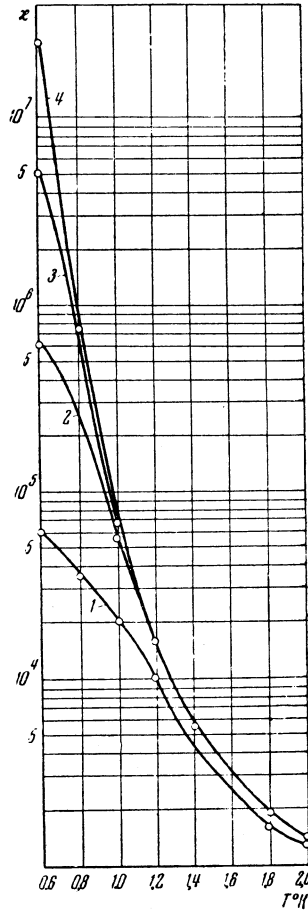


FIG. 1. Dependence of the thermal conductivity coefficient for a solution of He<sup>3</sup> in He<sup>4</sup> on the temperature  $T$  and the concentration  $c$  [Eq. (6.7)]: 1— $c = 10^{-2}$ , 2— $c = 10^{-3}$ , 3— $c = 10^{-4}$ , 4— $c = 10^{-5}$ .

$$\mathbf{Q} = -\kappa_m \nabla T, \quad (7.1)$$

where  $\mathbf{Q}$  is the heat flow through the solution per unit area. The diffusion coefficient  $D$  is directly related to  $\kappa_m$ . We shall derive an expression for  $\kappa_m$  under the restriction that the experiment be performed under stationary conditions. The heat current per unit area  $\mathbf{Q}$  in the solution is<sup>8</sup>:

$$\mathbf{Q} = Zc\mathbf{v}_n + \sigma\rho T\mathbf{v}_n + \mathbf{q}' = -\kappa_m \nabla T, \quad (7.2)$$

where  $\mathbf{v}_n$  is the velocity of the normal component of the solution,  $Z = \rho(\mu_3/m_3 - \mu_4/m_4)$ , and  $\mu_3$  and  $\mu_4$  are the chemical potentials of the pure He<sup>3</sup> and He<sup>4</sup> isotopes. In (7.2) the thermal current  $\mathbf{q}'$  has the form

$$\mathbf{q}' = T^2 \left[ \frac{k_T}{T} \frac{\partial}{\partial c} \frac{Z}{\rho T} - \frac{\partial}{\partial T} \frac{Z}{\rho T} \right] \mathbf{g} - \kappa \nabla T, \quad (7.3)$$

where  $\kappa$  is the thermal conductivity coefficient for the solution and  $g$  is taken from (5.11). Under stationary conditions there follows from the constancy of the potential  $\Phi$  the equation

$$\frac{\partial}{\partial c} \left( \frac{Z}{\rho} \right) \nabla c = c \frac{\partial}{\partial c} \left( \frac{\sigma}{c} \right) \nabla T, \quad (7.4)$$

the equation of motion for the impurities being given by the expression

$$g + \rho c v_n = 0. \quad (7.5)$$

Using the thermodynamic relations which follow from  $d\Phi = (dp/\rho) - \sigma dT + (Z/\rho)dc$  and solving simultaneously Eqs. (7.2)–(7.5), we can determine the effective thermal conductivity coefficient

$$\begin{aligned} \kappa_m = \kappa + \left( \rho DT / \frac{\partial}{\partial c} \frac{Z}{\rho} \right) \left\{ c \frac{\partial}{\partial c} \left( \frac{\sigma}{c} \right) \right. \\ \left. + \frac{k_T}{T} \frac{\partial}{\partial c} \left( \frac{Z}{\rho} \right) \right\}. \end{aligned} \quad (7.6)$$

For a weak solution

$$c \frac{\partial}{\partial c} \left( \frac{\sigma}{c} \right) = -\frac{\sigma_0}{c} - \frac{k}{m_3}, \quad \frac{\partial}{\partial c} \frac{Z}{\rho} = \frac{kT}{m_3 c}.$$

If we also introduce into (7.6) the value of the thermal diffusion ratio from (5.4) we obtain the simpler expression

$$\kappa_m = (\rho D m_3 \sigma_0^2 / kc) (\rho_n / \rho_{n0})^2 + \kappa, \quad (7.7)$$

where  $D$  is the diffusion coefficient determined from Eq. (4.52). Setting Eq. (4.47) in place of  $D$ , we obtain

$$\kappa_m = \rho T \sigma_0^2 t_i / c + \kappa. \quad (7.8)$$

As we have already noted, the time  $t_i$  is practically independent of the concentration of the solution. Thus the first term in Eq. (7.8) depends upon the concentration as  $1/c$ . As regards the second term—the thermal conductivity coefficient for the solution—this, in accordance with (6.7), depends in a rather complicated fashion upon the concentration of the solution. As can be seen from the graph (Fig. 2), the second term in (7.8) is negligible in the high temperature region for sufficiently low concentrations. We then have ( $\rho_n \approx \rho_{n0}$ )

$$\kappa_m = \rho T \sigma_0^2 t_i / c. \quad (7.9)$$

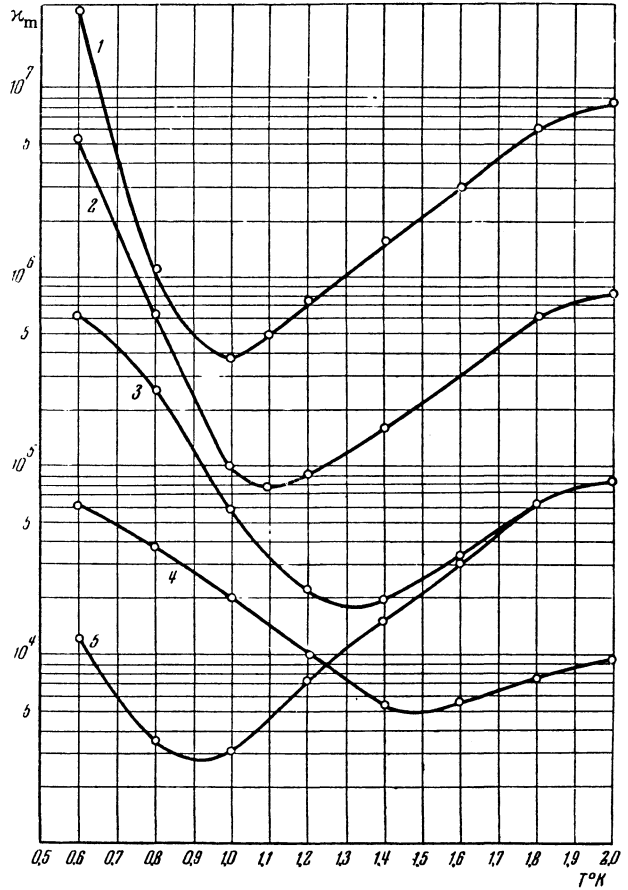


FIG 2. Dependence of the effective thermal conductivity for a solution of  $\text{He}^3$  in  $\text{He}^4$  on the temperature  $T$  and concentration  $c$  [Eq. (7.9)]:  $\kappa_m = K(T)/c + \kappa$ ; 1— $c = 10^{-3}$ , 2— $c = 10^{-4}$ , 3— $c = 10^{-3}$ , 4— $c = 10^{-2}$ , 5— $c = K \times 10^{-3}$ .

The temperature distribution in a weak solution of  $\text{He}^3$  in  $\text{He}^4$  in the presence of a thermal current is determined as follows. We write the expression for the thermal current (taking the one-dimensional case for simplicity)

$$q = -\kappa_m \partial T / \partial x \quad (7.10)$$

and the condition that the potential  $\Phi$  be constant along the solution

$$\Phi_0 - ckT/m_3 = \text{const} \quad (7.11)$$

( $\Phi_0$  is the solvent potential, that for helium II, which depends upon temperature but is independent of the concentration  $c$ ). Further, we express the concentration  $c$  from (7.11) in terms of the temperature and an unknown constant. We substitute this

expression into the equation for  $\kappa_m$  and integrate Eq. (7.10)

$$x = -\frac{1}{q} \int_{T_0}^T \kappa_m dT, \quad (7.12)$$

$T_0$  is the temperature at the cold end of the vessel ( $x = 0$ ). We then find, with the aid of (7.12) and (7.11), the dependence of the concentration  $c$  upon the coordinate  $x$ . This dependence includes an unknown constant, which can be determined from the condition that the impurities must be conserved.

$$c_0 = \frac{1}{d} \int_0^d c dx, \quad (7.13)$$

We shall illustrate this procedure, taking as our example an extremely weak solution, for which Eq. (7.10) is applicable; we write the latter in the form [with  $K(T)$  a function of temperature]

$$\kappa_m = K(T)/c. \quad (7.14)$$

It is essential here to bear in mind that for sufficiently weak solutions it follows from the condition (7.11) ( $kc/m_3 \ll \sigma_0$ ) that:

$$\sigma_T \nabla T = -(kT/m_3) \nabla c. \quad (7.15)$$

Thus

$$|\nabla c/c| \gg |\nabla T/T|,$$

and, consequently, under these conditions the temperature varies only slightly along the helium, while the concentration changes quite appreciably. Integrating (7.15), we obtain

$$c = -(m_3/kT) \sigma_0 (T - T_c), \quad (7.16)$$

where  $T_c$  is a constant of the integration. Equation (7.15) represents an alternative form for the condition (7.11). We now integrate Eq. (7.10), taking (7.14) and (7.16) into account, and obtain

$$\begin{aligned} x &= -\frac{1}{q} \int_{T_0}^T \frac{K(T) dT}{c} = \frac{1}{m_3 q} \int_{T_0}^T \frac{kTK(T) dT}{\sigma_0 (T - T_c)} \\ &= \frac{kTK(T)}{m_3 q \sigma_0} \ln \frac{T - T_c}{T_0 - T_c}. \end{aligned} \quad (7.17)$$

Since the temperature variation is small, the coefficient preceding the logarithm in (7.17) can be computed for the mean temperature of the helium. From (7.17) we obtain the temperature

$$\begin{aligned} T &= T_c + (T_0 - T_c) e^{x/x_0}, \\ 1/x_0 &= kTK(T)/m_3 q \sigma_0. \end{aligned} \quad (7.18)$$

Further, using (7.16) we express the concentration  $c$  as

$$c = -(m_3/kT) \sigma_0 (T_0 - T_c) e^{x/x_0}, \quad (7.19)$$

and, finally, we obtain the constant of integration  $T_c$  from the condition (7.13)

$$-(T_0 - T_c) = c_0 \frac{d}{x_0} \frac{kT}{m_3 \sigma_0} \frac{1}{e^{d/x_0} - 1}. \quad (7.20)$$

Ultimately, we obtain from (7.18) and (7.20) the formula for temperature distribution along the solution:

$$T - T_0 = \frac{c_0 d}{x_0} \frac{kT}{m_3 \sigma_0} \frac{e^{x/x_0} - 1}{e^{d/x_0} - 1}. \quad (7.21)$$

Beenakker *et al.*<sup>3</sup> obtained experimental values of the diffusion coefficient for the impurities in a solution. In analyzing their results, however, the authors used in place of the entropy  $\sigma$  appearing in Eq. (7.21) another quantity  $f$ , which they took from the results of unpublished experiments. We have recalculated the data of these authors, using known data for the value of the entropy. This recalculation changes the value of the diffusion coefficient somewhat. Under the conditions of the experiment— $T \geq 1.20^\circ$ ,  $c \sim 10^{-4}$ —the part played by the phonons was negligible (the quantity  $\kappa$  could also be neglected), and the diffusion of the impurities was determined solely by their interactions with the rotons. The unknown impurity–roton interaction constant was determined from the experimental value of the diffusion coefficient at  $T = 1.5^\circ$  K. The effective time  $t_{i\tau}$  then takes the value given by Eq. (4.20).

It should be emphasized that in experiments conducted under stationary conditions diffusion of the impurities cannot be considered alone. What is actually determined experimentally, in accordance with (7.7), is a certain effective thermal conductivity coefficient for the solution, representing a com-

bination of the diffusion and thermal conductivity coefficients and the thermal diffusion ratio.

The results obtained in the present work are correct for the temperature region  $T \leq 1.6 - 1.8^\circ \text{K}$ , in which the rotons may be regarded as constituting an ideal gas. At low temperatures the applicability of the theory is limited by the mean free paths of the excitations associated with the transport phenomena. As usual, the mean free paths must be much shorter than the characteristic dimensions of the containers. For high impurity concentrations, for which the mean free paths are short down to the lowest temperatures, the theory is applicable down to the temperatures at which the Fermi degeneracy of the impurities becomes significant.

In conclusion, the authors wish to express their deep indebtedness to Academician L. D. Landau for his helpful discussions.

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## Shock Waves of Large Amplitude in Air

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The state of air compressed by strong shock waves is examined by taking dissociation and ionization into account. Approximate expressions are given for the density and temperature in this region. The radiation from the front of the shock wave is considered. With increasing shock-wave amplitude, the observed surface temperature passes through a maximum, owing to the formation of an opaque layer of air, preheated by the radiation, ahead of the front of the wave. A proof is given of the non-existence of a continuous solution and of the unavailability of discontinuities in the velocity, density, and temperature in a strong shock wave with radiative heat exchange. The wave structure is investigated in a strongly-ionized gas with allowances for the slow energy transfer between the ions and electrons.

**P**HENOMENA OCCURRING in strong shock waves are very interesting from many points of view. In practice we encounter shock waves during explosions and during the motion of bodies at supersonic speeds. The principal interest lies in the peculiarities of the compression in the shock wave: the compression occurs rather rapidly, is accom-

panied by a sharp increase in the gas entropy, and is irreversible. Gas compression in a shock wave produces high temperatures, considerably higher than adiabatic compression to the same pressure.

It was already noted by Muraour<sup>1</sup> that the glow observed in an explosion is neither the chemiluminescence reaction of the decomposition of the ex-