Excitation of Rotational States in α-Decay of Even-Even Nuclei

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The relative probability of excitation of rotational states in the α-decay of deformed even-even nuclei is computed. The deformation parameter of the nucleus is determined by comparison with the probability for transition to the 2 + state. The values of the deformation parameter and relative probability of transition to a 4 + rotational state obtained in this way are compared with the experimental data.

THE EXCITATION of the rotational states of a nucleus in α-decay can be produced either by the interaction of the α-particle with the non-spherical field of nuclear forces, or by the interaction of the α-particle with the Coulomb field of the non-spherical daughter nucleus. The rotation of the nucleus plays an essential role.[1] The decay of even-even nuclei turns out to be especially simple when the quantity of motion of the whole system is equal to zero. In this case the Schrödinger equation for the system: α-particle + deformed daughter nucleus is greatly simplified – in the Hamiltonian of the system one can exclude the rotational coordinates of the core. For even-even nuclei, the Schrödinger equation in spherical coordinates has the form[1]

$$\frac{\hbar^2}{2m}\frac{\partial^2 u}{\partial r^2} - \hbar^2\left(\frac{1}{2mr^2}+\frac{1}{J}\right)\hat{l}^2 u = (V-E)\, u, \quad (1)$$

where $\psi(\mathbf{r}) = u(\mathbf{r})/r$ is the wave function of the system and depends only on the coordinates $\mathbf{r}$ of the α-particle relative to a coordinate system fixed in the nucleus; m is the mass of the α-particle, J the nuclear moment of inertia, E the energy of the system (equal to the energy of the α-particle at infinity, at the transition to the ground state of the daughter nucleus), $\hat{l}^2$ is the operator of the square of the angular momentum:

$$\hat{l}^2 = -\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta},$$

where ϑ is the angle with respect to the nuclear symmetry axis. In the expression of $\hat{l}^2$ the term $\hat{l}_z^2$ – which contains differenciation with respect to the azimuthal angle φ, has been omitted because the wave function does not depend on φ. If the wave function did contain states with non-zero projections of the angular momentum on the symmetry axis, the nucleus would rotate close to the symmetry axis. This is impossible because the moment of inertia with respect to the symmetry axis is zero ($J_z = 0$, $J_x = J_y = J$).

As it will be shown below, the angular dependence of the wave function is determined, for large distances from the nucleus, principally by the anisotropy of the Coulomb barrier and by the effects of nuclear rotation and centrifugal forces. The effect of nuclear interaction comes as an unknown boundary condition on the nuclear surface and has a relatively small influence on the intensity distribution in the α-spectrum.

In the region outside the nucleus, let us expand the wave function into the quasi-classical series:

$$u = \exp\{\hbar^{-1}[S_0(\mathbf{r}) + \hbar S_1(\mathbf{r}) + \ldots]\}. \quad (2)$$

The method of successive approximations gives for S_1 and S_2 the equations:

$$(\partial y/\partial x)^2 + 4\,(1/x^2+\beta)\,\eta\,(1-\eta)\,(\partial y/\partial\eta)^2 = v - 1, \quad (3)$$

$$\frac{\partial y}{\partial x}\frac{\partial S_0}{\partial x} + 4\left(\frac{1}{x^2}+\beta\right)\eta\,(1-\eta)\frac{\partial S_1}{\partial\eta}\frac{\partial y}{\partial\eta} = -\frac{1}{2}\frac{\partial^2 y}{\partial x^2} + \frac{1}{2}\left(\frac{1}{x^2}+\beta\right)\hat{l}_\eta^2 y, \quad (4)$$

where the following notation has been introduced

$$v(\mathbf{x}) = V(\mathbf{r})/E;\ \mathbf{x} = \mathbf{r}/R_0;\ \beta = mR_0^2/J;$$

$$S_0 = -\hbar k_0 R_0 y(\mathbf{x});\ k_0 = \sqrt{2mE}/\hbar,$$

$\eta = \sin^2\vartheta$ for $\alpha > 0$ (prolate nucleus) and $\eta = \cos^2\vartheta$ for $\alpha < 0$ (oblate nucleus). It is assumed that the nuclear deformation is small and that, therefore, the nuclear surface can be described by the following equation which is exact up to the terms of the order α^2

$$R(\vartheta) = R_0 (1 - {}^1/_5 \alpha^2 + \alpha P_2 (\cos\vartheta))$$
$$\text{or} \quad x(\eta) = x_0 - \varepsilon\eta, \tag{5}$$

where $\varepsilon = {}^2/_3 |\alpha|$, α being the nuclear deformation parameter, while $x_0 = 1 + |\alpha|\gamma - \alpha^2/5$ is the major semi-axis of the nucleus. The parameter γ is equal to 1 for a prolate nucleus, and to ½ for an oblate nucleus. R_0 is the radius of s sphere of volume equal to the nuclear volume: $R_0 = r_0 A^{1/3}$.

Let us expand the functions y, S_1 and v in a power series of η:

$$y = \sum_n y_n \eta^n; \; S_1 = \sum_n z_n (x) \eta^n; \; v = \sum_n v_n (x) \eta^n. \tag{6}$$

From (3) and (4), we obtain the following equations for y_n and z_n:

$$y'_m + 2m (2 - \delta_{1m}) (x^{-2} + \beta) y_1 y_m / y'_0 = g_m (x), \tag{7}$$
$$z'_m + 4m (x^{-2} + \beta) y_1 z_m / y'_0 = h_m (x), \tag{8}$$

where δ_{ab} is the Kronecker symbol. The functions g_m and h_m have the following values for $m = 0$, 1, and 2:

$$g_0 = \sqrt{v_0 - 1}/y'_0, \; h_0 = [-y''_0 - 4 (x^{-2} + \beta) y_1]/2y'_0,$$
$$g_1 = v_1/2y'_0, \; h_1 = [-y'_1 z'_0 - {}^1/_2 y''_1$$
$$+ 8 (x^{-2} + \beta) (y_2 + {}^3/_8 y_1)]/y'_0, \tag{7'}$$
$$g_2 = [-(y'_1)^2 + 4y_1^2 (\beta + x^{-2})]/2y'_0.$$

Here

$$v_0 = b (x^{-1} + 0.4\gamma\varepsilon x^{-3});$$
$$v_1 = -0.6\, \varepsilon b x^{-3}; \; b = 2Ze^2/R_0 E;$$

all the v_m with $m > 1$ are set equal to zero because, in what follows, we will take into account only the quadrupole potential of the nucleus.

The functions g_m always contain $y_{m'}$ with $m' < m$, so that Eq. (7) can be solved consecutively. The same holds for Eq. (8) because the right hand side on the mth equation (8) contains only those $z_{m'}$ for which $m' < m$. The solution of Eq. (7) for $m > 1$, as well as the solution of any of the equations (8), can be written in the form

$$\begin{matrix} y_m \\ z_m \end{matrix} = \exp(-4mF(x)) \left\{ \begin{matrix} y_m (x_0) \\ z_m (x_0) \end{matrix} + \int_{x_0}^{x} \begin{matrix} q_m (t) \\ h_m (t) \end{matrix} \exp(4mF(t))\, dt \right\}, \quad F(x) = \int_{x_0}^{x} \left(\frac{1}{x^2} + \beta \right) \frac{y_1^2}{y'_0} dx. \tag{9}$$

For $m = 1$, Eq. (7) is the Ricatti equation

$$y'_1 + 2 (x^{-2} + \beta) y_1^2 / y'_0 = v_1/2y'_0, \tag{10}$$

and its solution can be found in a general form.

Let us take the boundary condition for the solution of Eq. (3) in the form $S_0(\mathbf{r}) = \text{const} = 0$ on the surface of the nucleus, or $y[x(\eta), \eta] = 0$. Expanding the left hand side of this equality in a power series of the small quantity $\varepsilon\eta$ and equating to zero the coefficients of η^m we obtain the boundary condition for the function y_m at the point x_0:

$$+ \sum_{i_l=1}^{n|2} \sum_{j=n|2+1}^{n} (1 - p_{Ail}) p_{Aj} (e^{w_l | kT} - 1) \Big] . \tag{11}$$

The values of the functions y_m at the point x_0 for $m \leqslant 2$ are:

$$y_0 (x_0) = 0, \; y_1 (x_0) = \varepsilon y'_0 (x_0),$$
$$y_2 (x_0) = \varepsilon y'_1 (x_0) - {}^1/_2 \varepsilon^2 y''_0 (x_0). \tag{11'}$$

The boundary condition chosen for the wave function is, of course, not exact. However, if the true wave function is a smooth function of ϑ on the surface of the nucleus, the possible inaccuracy of the boundary condition on the nuclear surface will lead to an error in $y_m(x_0)$ of the order of $1/k_0 R$ (boundary condition (11) gives for $y_1(x_1)$ a value of the order of $\varepsilon\varkappa/k_0$, where $\varkappa$ is the value of the wave vector of the α-particle in the neighborhood of the nuclear surface). This error is not substantial, because the angular dependence of the wave function in the neighborhood of the sphere of exit from underneath the barrier depends very weakly on the boundary condition on the sphere $x = x_0$. Indeed, the function $F(x)$ of Eq. (9) is positive and monotonically increasing with x in the whole region under the barrier. Therefore, at large distances from the nucleus, the first term in Eq. (9) (which depends on the boundary condition) is strongly attenuated by the exponential factor. This result has a simple physical meaning: close to the nuclear surface, only small angles are important: $\vartheta \lesssim \vartheta^* = (\varepsilon\varkappa R_0)^{-1/2}$. This is true for the functions y_m for $m > 1$. It is easy to convince oneself that, at large distances, y_1 depends also weakly on the boundary condition, but only if $y_1(x_0)$

is not too small. The reason is the following: close to the nuclear surface the wave packet contains α-particle states and, correspondingly, nuclear states with high angular momenta: $l \lesssim 2\,\varepsilon\varkappa R_0$ (see Appendix). During further passage through the barrier, the packet smears out owing to the concurrent action of the nucleus, of the centrifugal forces, and of the quadrupole potential. The first two effects are proportional to the square of the angular momentum, *i.e.*, during passage through the barrier, the smearing of the packet increases as the angular distribution of the α-particles in the neighborhood of the nucleus gets sharper, *i.e.*, as $y_1(x_0)$ increases. As will be shown in the Appendix, it is for the same reason that the anisotropy of the wave function cannot be very large at large distances. In practice, the half width of the wave function cannot be less that $40-50°$.

One cannot set up a correct boundary condition for the functions z_m, because these functions are of the same order of magnitude as the possible inaccuracy of the boundary condition. However, if one takes into account that at large distances the functions z_m depend also weakly on the boundary condition, one can set them equal to zero on the sphere $x = x_0$.

It is easy to see that with such a boundary condition for the functions z_m, and with boundary condition (11) for the function y_m the expansion of the wave function in a power series of $\eta = \sin\vartheta$ or $\cos\vartheta$ is at the same time an expansion in the parameter ε. Indeed, as it can be seen from Eq. (11), the boundary condition on the functions y_m has the property that $y_m(x_0)$ is of the order of ε^m [see also (7′) and (11′)]. The same is true for the right-hand sides of (7). It follows that the solution of these equations themselves will diminish as ε^m. The same holds for the functions z_m with the boundary condition $z_m(x_0) = 0$.* Let us note that the function $F(x)$ in (9) is, for small deformations, proportional to ε, and that all that has been said above about the weak dependence of the angular distribution of α-particles at large distances on the conditions at the nuclear surface is true only for dedormations which are not too small: the function $F(x)$ must be, in the neighborhood of the sphere of exit from underneath the barrier, of the order of unity (actually F turns out to be equal to 0.2–0.4). The relative probability of excitation of a rotational state of the daughter nucleus with spin I is equal to

$$\xi_I = \frac{(2I+1)\left(\int\limits_0^\pi \sin\vartheta\, d\vartheta\, \psi(\vartheta)\, P_I(\cos\vartheta)\right)^2}{\int\limits_0^\pi \sin\vartheta\, d\vartheta\, \psi(\vartheta),} \tag{12}$$

where $P_I(\cos\vartheta)$ is the Legendre polynomial and for $\psi(\vartheta)$ one can take the wave function in the neighborhood of the sphere of exit from underneath the barrier. In order to determine $\psi(\vartheta)$ in first approximation with respect to α, one has to solve the Ricatti equation (10). In this approximation we simply have

$$\psi(\vartheta) \sim \exp\left\{-a\binom{\sin^2\vartheta}{\cos^2\vartheta}\right\}, \quad a = k_0R_0y_1(x)^*,$$

and the integrals in (12) can be evaluated in a general manner. For an oblate nucleus we obtain:

$$\xi_2 = \frac{5}{16}\{3e^a\chi^{-1}(\sqrt{a}) - 3a^{-1} - 2\}^2,$$
$$\chi(a) = \int\limits_0^a e^{t^2}\,dt. \tag{13}$$

The solution of the Ricatti equation can be simplified if one takes into account that the contribution of the quadrupole potential is small* with respect to the contribution of the term proportional to y_1^2.* The accuracy of the first approximation is, however, not sufficient, because ξ_I is a sharp function of α and, if the determination of the wave function is not accurate enough, the error in ξ_I can be too large. Because of this, the wave function has been computed taking into account terms of the order α^2 in the quasi-classical part (in S_0) and a term of the order α in the quantum correction.† The wave function has the form

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*The solutions of the mth equations of (7) and (8) and the functions y_m and z_m contain also terms of order higher than ε^m. These terms are substantial at large distances from the nucleus.

*Formally, v_1 of (10) is a small quantity of lower order, but since the quadrupole potential falls off rapidly with distance, its contribution at large distances becomes numerically small relative to the contribution of the term containing y_1^2.

†An actual calculation of terms of higher order (y_3 and z_3) for the case of the decay of U^{234} and $\alpha = 0.15$ has shown that the contribution from these terms is indeed small (the correction to the exponent is $\lesssim 0.1$); practically, one can therefore limit oneself to the terms mentioned.

$$\psi \sim \exp\left\{-a_1 \frac{\sin^2 \vartheta}{\cos^2 \vartheta} - a^2 \frac{\sin^4 \vartheta}{\cos^4 \vartheta}\right\},$$

$$\alpha \gtreqless 0, \quad a_1 = k_0 R y_1 - z_1, \quad a_2 = k_0 R_0 y_2,$$

$(a_2/a_1 \sim \varepsilon)$. For $y_{1,2}$ and z_1, their value at the point $x^* = b - 1$ was used, *i.e.*, somewhat short of the exit sphere. One has to do that because in the immediate neighborhood of the exit sphere the quasi-classical approximation is not valid. But one can show that in the region $x \gtrsim x^*$ the angular dependence of the wave function is practically constant.

The calculations were performed on an M-2 electronic computer. Equations (7) and (8) were solved using variable increments chosen to make the accuracy of the functions at the point x^* of the order of 10^{-3}. The integrals in Eq. (12) were computed for $l = 2$ and 4 using Simpson's rule with a spacing of 1/512. The probability of transition into a state $l \geqslant 6$ was not determined because the probability of transition into these states depends appreciably on the boundary condition as well as on other factors which are difficult to take into account (the octupole potential and higher order potentials, a more accurate characteristic of the nuclear shape, the finite dimensions of the α-particle, *etc.*)

The calculation was performed for all the even-even nuclei mentioned in the reviews, Refs. 1 and 2, for $r_0 = 1.2$, 1.3, and 1.4×10^{-13} cm, for positive and negative values of α from 0.05 to 0.25 in steps of 0.05. In order to illustrate the convergence of the series, the dependence of the functions y_1, y_2 and z_1 on α at the point $x^* = b - 1$ has been plotted on Fig. 1. The series converges faster at smaller distances from the nucleus. The parameter involved in Eq. (7) and (8) (the energy E and the rotational constant $B = \hbar^2/2J$) and the experimental value of ξ_2 and ξ_4 were determined from the compiled experimental data.[2,3] The experimental values of these parameters are tabulated together with the results of the calculations. The references to the experimental work are given in the mentioned reviews and are therefore not shown in the table. The value of ξ_4 given for the case of decay of Ra^{226} is obtained from Ref. 4.

The values of α tabulated in the third column were obtained by comparison with the experimental value of the relative probability of transition into the state with $l = 2$. For each nucleus and each sign of deformation the upper column corresponds to $r_0 = 1.4 \times 10^{-13}$ cm and the lower to $r_0 = 1.2 \times 10^{-13}$ cm. The difference between these values of r_0 is

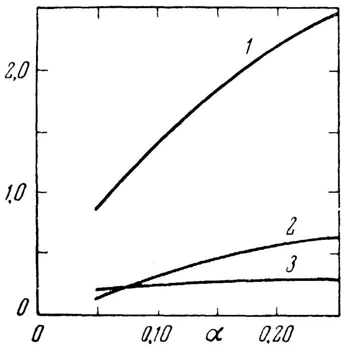

FIG. 1. Plot illustrating the convergence of the series. Decaying nucleus Pu^{236}, $r_0 = 1.4 \times 10^{-13}$ cm. Curve 1 – $k_0 R_0 y_1(x^*)$, 2 – $k_0 R_0 y_2(x^*)$, 3 – $z_1(x^*)$. The relative order of the functions y_1, y_2 and z_1 is α, α^2 and α, respectively.

small and one cannot practically make a choice between them. For $r_0 = 1.3 \times 10^{-13}$ cm the value of α is always in between the values corresponding to $r_0 = 1.4 \times 10^{-13}$ and 1.2×10^{-13} cm. These values of α are not tabulated. For an oblate nucleus with a given deformation, the probability of transition into a 2+ state is always considerably smaller than for a prolate nucleus. For an oblate nucleus with any value of $\alpha \leqslant 0.25$, ξ_2 is, in most of the cases, smaller than its experimental value. On the other hand, the quadrupole moment data indicate a small deformation ($\alpha \lesssim 0.20$); therefore, the data on intensity distribution in the α-spectrum exclude the assumption of oblate nuclei. This is in agreement with the experimental results on $\alpha - \gamma$ angular correlation in the case of α-decay of Am^{241} from which it follows that the nuclei Am^{241} and Np^{237} are prolate.*[5]

The few exceptions are the nuclei for which the probability of transition into a 2+ state is relatively small (isotopes of Rn, Cm, and Cf). The assumption that these nuclei are oblate can be made to agree – generally speaking – with the intensity distribution of the α-spectrum although this yields deformations. The quadrupole moment data on these nuclei would permit one to make a definite conclusion about their shape. The table shows the negative values of α only in those cases for which the assumption on oblate nucleus can be made to agree with the intensity of transition into the 2+ state.

*An adiabatic rotation of the nucleus was assumed in Ref. 5. This assumption does not agree with what actually happens, but one can show that the nuclear rotation does not substantially change the relative phase between the $l = 2$ and $l = 0$ waves, a phase which is important for the determination of the sign of the deformation.

Daughter nucleus	E, Mev; B, kev; ξ_2 (exp.)	α	α_Q	ξ_4 (calc.)	ξ_4 (exp.)
Rn^{220}	5.68 40 0.05	+0.07 +0.08 −0.14 −0.15		10^{-4} $7\cdot10^{-5}$	
Rn^{222}	4.78 31 0.06	+0.08 0.10 −0.15 0.20		$5\cdot10^{-5}$ $3\cdot10^{-5}$	
Ra^{222}	6.34 18.2 0.27	+0.13 0.15		$4\cdot10^{-3}$	$8\cdot10^{-3}$
Ra^{224}	5.42 14 0.40	+0.17 0.20		10^{-2}	$2.8\cdot10^{-3}$
Ra^{226}	4.68 11.2 0.30	+0.15 0.17		$6\cdot10^{-3}$	$2.6\cdot10^{-3}$
Ra^{228}	3.99 10 0.32	+0.17 0.21		$6\cdot10^{-3}$	
Th^{226}	5.89 11.6 0.45	+0.17 0.20		$1.4\cdot10^{-2}$	
Th^{228}	5.32 10 0.47	+0.18 0.21		$1.6\cdot10^{-2}$	$4.4\cdot10^{-3}$
Th^{230}	4.76 8.7 0.31	+0.14 0.15		$6\cdot10^{-3}$	$4\cdot10^{-3}$
Th^{232}	4.50 $\sim$8.3 $\sim$0.37	+0.16 0.18	0.07[6]	10^{-2}	
Th^{234}	4.18 $\sim$8.3 $\sim$0.30	+0.14 0.15		$6\cdot10^{-3}$	
U^{230}	6.19 $\sim$8.3 0.16	+0.08 0.09			
U^{232}	5.75 7.5 0.25	+0.10 0.12	U^{233} 0.17 [7]	$4\cdot10^{-3}$	
U^{234}	5.50 7.8 0.39	+0.14 0.16	U^{235} 0.12 [7]	10^{-2}	$1.2\cdot10^{-3}$
U^{236}	5.16 7.5 0.32	+0.13 0.15		$8\cdot10^{-3}$	$1.2\cdot10^{-3}$

Daughter nucleus	E, Mev; B, kev ξ_2 (exp.)	α	α_Q	ξ_4 (calc.)	ξ_4 (exp.)
U^{238}	4.90 7.3 0.25	+0.12 0.13	0.11 [7] 0,07 [6]	$3\cdot10^{-3}$	
Pu^{238}	6.11 7.3 0 36	+0,12 0.13		10^{-2}	$5\cdot10^{-4}$
Pu^{240}	5.80 7.2 0.33	+0.12 0.13		$1.3\cdot10^{-2}$	
Cm^{242}	6.75 7.0 0.28	+0.10 0.11 —0.20 0,25		$6\cdot10^{-3}$ $1.5\cdot10^{-2}$	$>10^{-3}$
Cm^{246}	6.15 6.2 0.11	+0.07 0.08 —0.09 0.10		$2\cdot10^{-3}$ $1.5\cdot10^{-3}$	
Cm^{248}	6.12 6,2 0,11	+0.07 0.08 —0.12 0.14		10^{-3} 10^{-3}	
Cf^{250}	7.22 7.0 0.18	+0.08 0.08 —0.12 0,14		$4\cdot10^{-3}$ $3\cdot10^{-3}$	$>3\cdot10^{-3}$

The fourth column (α_Q) of the table shows the values of α determined from quadrupole moments obtained from experiments on Coulomb excitation of rotational levels, for $r_0 = 1.4 \times 10^{-13}$ cm. The parameter α_Q is obtained using the formula:[8]

$$Q_0 = {}^6\!/_5\, Z R_0^2\, \alpha_Q .$$

At the present time the quadrupole moment of the following even-even nuclei has been measured: Th^{232} ($Q_0 = 5.6$ barns[6]) and U^{238} ($Q_0 = 6.9$ barns[6] and 9.5 barns[7]). The table shows both values for U^{238}. $Q_0 = 6.9$ barns corresponds to $\alpha = 0.07$. This value differs appreciably from the α determined from α-decay. The discrepancy is also very large in the case of Th^{232}. In the other case, the deformation parameter has been determined from the quadrupole moment of the neighboring odd isotopes.[7] These isotopes are shown in the table. The dependence of the deformation parameter on the quadrupole moment agrees qualitatively with the measured nuclear quadrupole moments. The following columns list the calculated and experimental values of the transition probability into the 4+ state. In all the cases the agreement with the experimental data on decay into a 4+ state is satisfactory. The discrepancy between the experimental and the theoretical values of ξ_4 is within reasonable limits and can be fully accounted for by the inaccuracy of the boundary condition (it is shown in the appendix that the inaccuracy of the boundary condition can change ξ_4 by a factor of $\sim e$). Fig. 2 shows the dependence of ξ_2 on α for the decays of Cm^{242} (experimental value $\xi_2 = 0.35$) and Ra^{222}, for $r_0 = 1.4 \times 10^{-13}$ cm (for the latter nucleus the intensity of transition into a 2+ state has not been determined experimentally). Let us note that the curves showing the dependence of ξ_2 on α turn out to be close to each other for all the isotopes of the same element, but are quite different for different elements.

Generally speaking, the nuclear deformation can also be determined from the absolute probability of

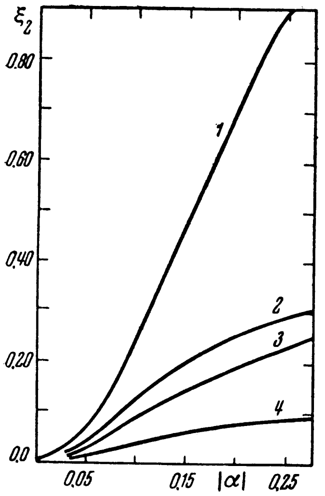

FIG. 2. Relative probability of transition into an $I = 2$ state *vs.* deformation of the daughter nucleus. The following parent nuclei are shown: 1 – Cm^{242}, $\alpha > 0$; 2 – Cm^{242}, $\alpha < 0$; 3 – Ra^{222}, $\alpha > 0$; 4 – Ra^{222}, $\alpha < 0$.

α-decay – the presence of deformation leads to an increase of the Coulomb barrier penetrability. This effect is, however, relatively small as compared to the spherical case (the decay probability increases by a factor of ~ 10). The factor in front of the exponential is determined either with the same accuracy or less accurately; this method does not, therefore, permit the determination of the deformation with any accuracy.

In conclusion the author expresses his gratitude to C. B. Mostinskaia who carried out the calculations with an electronic computer.

APPENDIX

1. DEPENDENCE OF THE WAVE FUNCTION ON THE BOUNDARY CONDITION AT THE NUCLEAR SURFACE

In the neighborhood of the nucleus one can neglect the curvature of the α-particle orbit. On a sphere Σ surrounding the nucleus, of radius $x_0 = 1 + \alpha$ equal to the major semi-axis of the nucleus (we consider prolate nuclei), the wave function can be written in the form

$$\psi|_\Sigma = \chi(\vartheta) \exp\left\{-\int_{x_0-\Delta x}^{x_0} \varkappa\, dx\right\} \simeq \chi(\vartheta) \exp\{-\varkappa R_0 \Delta x(\vartheta)\}, \quad (A.1)$$

where $\chi(\vartheta) \equiv \chi[x(\eta), \eta]$ is the wave function at the nuclear surface and $\Delta x(\vartheta) = \varepsilon \sin^2 \vartheta$. Let us expand the function (A.1) is Legendre polynomials:

$$\psi|_\Sigma = \sum_l a_l P_l(\cos\vartheta),$$
$$a_l \sim (2l+1) \int_0^\pi \psi|_\Sigma P_l(\cos\vartheta) \sin\vartheta\, d\vartheta. \quad (A.2)$$

Only small angles are important in (A.2), because practically $\varepsilon\varkappa R \sim 4-5 \gg 1$. Substituting, in the exponent, ϑ^2 for $\sin^2\vartheta$ and a Bessel function for the Legendre polynomial, we find:

$$a_l \sim (2l+1) \int_0^\infty e^{-c\vartheta^2} J_0\left[\left(l + \frac{1}{2}\right)\vartheta\right] \vartheta\, d\vartheta \sim (2l+1) \exp[-l(l+1)/4c], \quad (A.3)$$

where $c = \varepsilon\varkappa R + \delta$, δ being a correction due to the function $\chi(\vartheta)$. If the function $\chi(\vartheta)$ is not too abrupt, $\delta \lesssim 1$. The quadrupole potential can be neglected in the analysis of the dependence of the intensity of rotational state excitation on the boundary condition. In the region under the barrier one can therefore write the wave function in the form

$$\psi(x,\vartheta) = \sum_l a_l R_l(x) P_l(\cos\vartheta), \quad (A.4)$$

where R_l are the usual Coulomb radial functions $[R_l(x_0) = 1]$. The relative probability of excitation of a spin I rotational level is equal to

$$\xi_I = a_I^2 \frac{R_I^2(x^*)}{R_0^2(x^*)} \approx (2I+1) \exp\left[-I(I+1)\left(\frac{1}{2c} + d\right)\right] \approx (2I+1) \times \exp\left\{-I(I+1)\left(\frac{1}{2\varepsilon\varkappa R_0} + \frac{\delta}{2(\varepsilon\varkappa R_0)^2} + d\right\}, \quad (A.5)$$

$$d \approx \frac{1}{k_0 R_0} \int_{x_0}^{x^*} \frac{(\beta + x^{-2})}{(v_0 - 1)^{1/2}} dx = \frac{1}{k_0 R_0}\left(\frac{2}{\sqrt{b}} + \frac{\pi}{2}\beta b\right).$$

For most nuclei $\beta \sim 0.1$, $b \sim 5$. Here $d = 1.5/k_0 R_0$. The factor $\delta/2(\varepsilon\varkappa R_0)^2$ which depends on the boundary condition on the nuclear surface is substantial for $l \gtrsim \sqrt{2\varepsilon\varkappa R_0}$. For $\delta \lesssim 1$ and $I = 2$, the correction which is due to the inaccuracy of the boundary condition does not exceed $\sim 20\%$. For $I = 4$, the ξ_I calculated with an approximate boundary condition can be several times larger or smaller than the true value. The probability of transition into higher states depends appreciably on the boundary condition.

2. MINIMUM ANGULAR WIDTH OF THE WAVE FUNCTION

The action of centrifugal forces and the energy loss of the α-particle (nuclear rotation) lead to a finite width of the wave function on the sphere of exit from underneath the barrier, even for a δ-function distribution of α-particles on the sphere Σ. The intensities of the fine-structure lines in the α-spectrum for a deformed nucleus cannot, therefore, exceed the values given by the simple Gamow formula, which does not take into account the nuclear deformation. The minimum width of the wave function can be obtained either by using formula (A.5), putting $c = \infty$, or directly from Eq. (10), by letting $v_1 = 0$ and $y_1(x_0) = \infty$ in the Ricatti equation. We obtain

$$y_1(x^*)\big|_{\max} = \Big[2\int_{x_0}^{x^*} \frac{(\beta + x^{-2})}{(v_0 - 1)^{1/2}} \Big]^{-1} = \frac{1}{2\, dk_0 R_0}. \quad (A.6)$$

The angular half width ϑ^* of the wave function is determined from $\sin\vartheta^* \approx \sqrt{k_0 R_0 y_1(x^*)}$; for the values of d mentioned above and for $k_0R_0 \approx 8$, it is equal to $\sim 40°$. As can be seen from Eq. (10), the quadrupole potential can only decrease the value of $y_1(x^*)$ and leads, therefore, to an additional increase of the angular width of the wave function and to a decrease of the probability of transition into excited rotational states.

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Motion of a Charge Parallel to the Axis of a Cylindrical Channel in a Dieletric

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The field produced by a charge moving parallel to the axis of a cylindrical channel in a dielectric is determined. The field and energy losses of the charge are computed for various assumptions concerning the medium.

THE PASSAGE OF A CHARGE along a channel in a dielectric was first considered by Ginzburg and Frank.[1] These authors calculated the field produced by a point charge moving with uniform motion along the axis of a cylindrical channel of radius a, filled with a dielectric $\varepsilon_1(\omega)$ in a medium of dielectric constant $\varepsilon_2(\omega)$.

Problems connected with the passage of a charge along the axis of a channel in a dielectric have also been treated by Bohr,[2] Schoenberg,[3] Huybrechts,[3,5] and Sitenko[6] (problems of this type have also been considered in Ref. 7).

In problems concerning the generation of electromagnetic radiation, focusing of charged particles in a cylindrical channel, and the theory of Cerenkov counters, it is of interest to consider the case in