On the Rarita–Schwinger Method in the Theory of Particles of Half-Integral Spin

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The possibility of generalizing the Rarita–Schwinger method of setting up the Lagrangian for particles with large spins ($s > 3/2$) is examined. It is shown that the result obtained by Moldauer and Case is valid only for particles of spin $3/2$.

IN A WORK ON THE THEORY of particles with high half-integral spin, Rarita and Schwinger[1] have shown in spinor-tensor form a method for constructing the Lagrangian without additional quantities, and have given the Lagrangian for spin-$3/2$ particles in explicit form.

In a recently published work, Moldauer and Case[2] give a Lagrangian which, according to these authors, gives the correct equations for particles with arbitrary half-integral spin $s = n + 1/2$, where $n = 1, 2, 3, \ldots$. On the basis of equations which follow from the variation of their Lagrangian including interactions with the electromagnetic field, conclusions are reached with regard to the magnetic moment of particles with spin $s = n + 1/2$ and the quadrupole moment of particles with spin $3/2$ and $5/2$.

We shall show in this note that the Lagrangian density given by Eq. (1.7) of Moldauer and Case,[2] as well as their field equations (1.9) and subsidiary conditions (1.10) which include interactions with the electromagnetic field are correct only for particles with spin $3/2$ and may not be extended to particles with higher spins. Therefore their results for magnetic and quadrupole moments can be considered valid only for particles with spin* $3/2$.

The Lagrangian function as given by Moldauer and Case is

$$\begin{aligned} L = \Psi^{+}_{\varkappa\nu_2\ldots\nu_n}\Big[&(\gamma_\mu\partial_\mu + m)\,\delta_{\varkappa\lambda} + A\,(\gamma_\varkappa\partial_\lambda + \gamma_\lambda\partial_\varkappa) \\ &+ \left(\frac{3}{2}A^2 + A + \frac{1}{2}\right)\gamma_\varkappa\gamma_\rho\partial_\rho\gamma_\lambda \\ &- (3A^2 + 3A + 1)\,m\gamma_\varkappa\gamma_\lambda\Big]\Psi_{\lambda\nu_2\ldots\nu_n}, \end{aligned} \tag{1}$$

where

$$\partial_\mu = \partial/\partial x_\mu\ (\mu = 1, 2, 3, 4);\quad x_1 = x,$$

$$x_2 = y,\quad x_3 = z,\ x_4 = it;\quad \hbar = c = 1;$$

$\Psi_{\nu_1,\nu_2\ldots\nu_n}$ is an nth rank spin tensor symmetric with respect to all its indices ($\nu_i = 1, 2, 3, 4$);

$$\Psi^{+}_{\nu_1\nu_2\ldots,\,\nu_n} = (-1)^r\Psi^{*}_{\nu_1\nu_2\ldots\,\nu_n}\gamma_4,$$

where r is the number of times the index 4 occurs in $\nu_1, \nu_2, \ldots, \nu_n$; A is a real parameter which may take on arbitrary values other than $-1/2$.

For free particles, L as given by Eq. (1) should lead to the Dirac equation

$$(\gamma_\mu\partial_\mu + m)\,\Psi_{\nu_1\nu_2\ldots\,\nu_n} = 0 \tag{2}$$

with the subsidiary conditions

$$\gamma_\lambda\Psi_{\lambda\nu_2\ldots\,\nu_n} = 0,\quad \partial_\lambda\Psi_{\lambda\nu_2\ldots\,\nu_n} = 0. \tag{3}$$

We have previously[3] used such a Lagrangian to describe particles with spin $3/2$. In the same article we have shown the relation between the equations obtained by variation of the Lagrangian of Eq. (1) with $n = 1$, and the other known forms of writing the equations for spin-$3/2$ particles, namely the Fierz-Pauli-Gupta equations,[4,5] the Petras equations,[6] and the Gel'fand-Iaglom equations;[7] in addition, we discuss the question of allowable linear transformations of the spin-$3/2$ equations. It is found that the arbitrariness in the coefficient A is related to the arbitrariness in normalizing the metric in the spin-$1/2$ subspace. The transformation

$$\Psi_\nu = \Psi'_\nu + k\gamma_\nu\gamma_\lambda\Psi'_\lambda, \tag{4}$$

where

$$k = (A' - A)/2\,(2A + 1)\qquad (k \neq -1/4), \tag{5}$$

*At the same time, of course, part of the Moldauer and Case[2] article which deals with constructing the equations for the independent components of free-particle wave functions with spin $n + 1/2$ in Hamiltonian form is undoubtedly correct, since it is based on the free-particle equations established in the work of Rarita and Schwinger.[1]

leads to the transition from one value of A to another* A'. In the special case $A = -\frac{1}{3}$ the Lagrangian density of Eq. (1) is that given by Rarita and Schwinger.[1]

In order to derive the equations for particles with spin greater than $\frac{3}{2}$, when $n \geqslant 2$, it is important to note that the symmetry of $\Psi_{\nu_1\nu_2\ldots\nu_n}$ with respect to all its tensor indices leads to the fact that the variations $\delta\Psi^+_{\nu_1\nu_2\ldots\nu_n}$ are not independent.

It is particularly important to account for the symmetry of $\Psi^+_{\varkappa\nu_2\ldots\nu_n}$ with respect to the index $\varkappa$ when performing the variation of L. When the variation of Eq. (1) is correctly performed, it leads to the following field equations for particles with spin $s = n + \frac{1}{2}$:

$$\begin{aligned}(\gamma_\mu\partial_\mu + m)\Psi_{\nu_1\nu_2\ldots\nu_n} &+ \frac{1}{n}A(\gamma_{\nu_1}\partial_\lambda\Psi_{\lambda\nu_2\ldots\nu_n} + \gamma_{\nu_2}\partial_\lambda\Psi_{\nu_1\lambda\nu_3\ldots\nu_n} + \ldots \\ &+ \gamma_{\nu_n}\partial_\lambda\Psi_{\nu_1\nu_2\ldots\nu_{n-1}\lambda} + \gamma_\lambda\partial_{\nu_1}\Psi_{\lambda\nu_2\ldots\nu_n} + \gamma_\lambda\partial_{\nu_2}\Psi_{\nu_1\lambda\nu_3\ldots\nu_n} + \ldots \\ &\ldots + \gamma_\lambda\partial_{\nu_n}\Psi_{\nu_1\nu_2\ldots\nu_{n-1}\lambda}) + \frac{1}{n}B(\gamma_{\nu_1}\gamma_\rho\partial_\rho\gamma_\lambda\Psi_{\lambda\nu_2\ldots\nu_n} + \gamma_{\nu_2}\gamma_\rho\partial_\rho\gamma_\lambda\Psi_{\nu_1\lambda\nu_3\ldots\nu_n} + \ldots \\ &\ldots + \gamma_{\nu_n}\gamma_\rho\partial_\rho\gamma_\lambda\Psi_{\nu_1\nu_2\ldots\nu_{n-1}\lambda}) + \frac{1}{n}Cm(\gamma_{\nu_1}\gamma_\lambda\Psi_{\lambda\nu_2\ldots\nu_n} + \gamma_{\nu_2}\gamma_\lambda\Psi_{\nu_1\lambda\nu_3\ldots\nu_n} + \ldots \\ &\ldots + \gamma_{\nu_n}\gamma_\lambda\Psi_{\nu_1\nu_2\ldots\nu_{n-1}\lambda}) = 0,\end{aligned} \tag{6}$$

where we have written

$$B = {}^3/_2 A^2 + A + {}^1/_2, \quad C = -(3A^2 + 3A + 1). \tag{7}$$

Let us first note that the form of (6) depends strongly on the value of n, so that it is impossible to obtain an equation of a single form for arbitrary n, as was done by Moldauer and Case. Secondly, for $n \geqslant 2$, *i.e.*, for free particles with spin $s \geqslant \frac{5}{2}$, it is not only impossible to obtain the subsidiary conditions (3) with the coefficients given by Eq. (7), but in general, with any values of A, B, and C.

Detailed investigation shows that so many algebraic conditions must be fulfilled by A, B, and C that for values of n higher than 1 the number of equations is greater than the number of unknowns, and the equations become inconsistent. The mistake made by Moldauer and Case[2] is that they did not take account of symmetry in varying L.

Let us consider the question of the applicability of the Rarita–Schwinger method to particles with spin $\frac{5}{2}$. We may attempt to extend L as given by Eq. (1). The most general Lagrangian density with no additional quantities for particles with spin $\frac{5}{2}$, which leads to first-order equations is of the form

$$\begin{aligned}L = \Psi^+_{\varkappa\nu}\{(\gamma_\mu\partial_\mu + m)\delta_{\varkappa\lambda} + 2a_1(\gamma_\varkappa\partial_\lambda + \gamma_\lambda\partial_\varkappa) + 2a_2\gamma_\varkappa\gamma_\rho\partial_\rho\gamma_\lambda + 2a_3 m\gamma_\varkappa\gamma_\lambda\}\Psi_{\lambda\nu} \\ + 2a_4(\Psi^+_{\varkappa\varkappa}\partial_\lambda\gamma_\nu\Psi_{\lambda\nu} + \Psi^+_{\varkappa\nu}\partial_\varkappa\gamma_\nu\Psi_{\lambda\lambda}) + \Psi^+_{\varkappa\varkappa}(a_5\gamma_\rho\partial_\rho + a_6 m)\Psi_{\lambda\lambda}.\end{aligned} \tag{8}$$

When

$$2a_1 = A, \quad 2a_2 = B, \quad 2a_3 = C, \quad a_4 = a_5 = a_6 = 0 \tag{9}$$

the function of Eq. (8) becomes the Lagrangian density of Eq. (1) for $n = 2$.

To obtain the subsidiary conditions (3) for free particles it becomes necessary, as a consequence of the field equations which follow from the Lagrangian density of Eq. (8), to operate with $\gamma_\varkappa$ and $\partial_\varkappa$ and sum over $\varkappa$ and then to operate with $\delta_{\nu\varkappa}$, $\partial_\nu\gamma_\varkappa$, and $\partial_\nu\partial_\varkappa$ and sum over $\varkappa$ and ν. We then obtain two sets of equations. The first two operations give two differential equations for $\gamma_\lambda\Psi_{\lambda\nu}$, $\partial_\lambda\Psi_{\lambda\nu}$, $\Psi_{\lambda\lambda}$, $\partial_\rho\partial_\lambda\Psi_{\lambda\rho}$ and $\partial_\rho\gamma_\lambda\Psi_{\lambda\rho}$. The three final operations lead to equations of the form

$$\begin{aligned}A_{33}\Psi_{\lambda\lambda} + A_{35}\partial_\rho\gamma_\lambda\Psi_{\lambda\rho} &= 0, \\ A_{43}\Psi_{\lambda\lambda} + A_{44}\partial_\rho\partial_\lambda\Psi_{\lambda\rho} + A_{45}\partial_\rho\gamma_\lambda\Psi_{\lambda\rho} &= 0, \\ A_{53}\Psi_{\lambda\lambda} + A_{54}\partial_\rho\partial_\lambda\Psi_{\lambda\rho} + A_{55}\partial_\rho\gamma_\lambda\Psi_{\lambda\rho} &= 0,\end{aligned} \tag{10}$$

where the A_{ik} (with $i, k = 1, 2, 3, 4, 5$) are certain expressions containing differential operators and depending on the coefficients $a_1 - a_6$ as parameters.

*This makes the results for the quadrupole moment of spin-$\frac{5}{2}$ particles as obtained by Moldauer and Case seem even stranger, as it depends on the unspecified parameter A, and thus on the choice of the representation.

The condition that under which the system (10) has the only required vanishing solution is that the operator determinant of the set of equations become a nonzero constant. This requirement leads to three algebraic equations for the coefficients $a_1 - a_6$ and to the condition ($m \neq 0$)

$$(1 + 6a_3)(1 + 2a_3 + 4a_6) \neq 0. \tag{11}$$

Inserting

$$\Psi_{\lambda\lambda} = 0, \quad \partial_\rho\gamma_\lambda\Psi_{\lambda\rho} = 0, \quad \partial_\rho\partial_\lambda\Psi_{\lambda\rho} = 0,$$

into the equations obtained with the aid of the first two operations, we obtain

$$\begin{aligned} A_{11}\gamma_\lambda\Psi_{\lambda\nu} + A_{12}\partial_\lambda\Psi_{\lambda\nu} &= 0, \\ A_{21}\gamma_\lambda\Psi_{\lambda\nu} + A_{22}\partial_\lambda\Psi_{\lambda\nu} &= 0. \end{aligned} \tag{12}$$

From the condition that the system (12) has only a vanishing solution, we obtain two other algebraic equations for the coefficients a_1, a_2, and a_3. Solving the five equations obtained, we obtain the coefficients a_2, a_3, a_4, a_5, and a_6 as functions of the arbitrary parameter a_1:

$$\begin{aligned} a_2 &= {}^1/_4\,(5a_1^2 + 2a_1 + 1); \\ a_3 &= -{}^1/_8\,(15a_1^2 + 10a_1 + 3); \\ a_4 &= -{}^1/_8\,(5a_1^2 + 6a_1 + 1); \\ a_5 &= a_6 = {}^1/_{16}\,(15a_1^2 + 10a_1 - 1). \end{aligned} \tag{13}$$

Condition (11), however, is not fulfilled for any value of a_1. The determinant of (10) vanishes identically and therefore there exists no set of values for the coefficients $a_1, \ldots, a_6$ for which both sets (10) and (12) have simultaneously only vanishing solutions, *i.e.*, the subsidiary conditions (3) cannot follow from the L function of Eq. (8).*

We can thus conclude that the Rarita–Schwinger method does not give positive results for spin-$^5/_2$ particles. One may suppose that this is related to the fact that in analogy with the spin-$^3/_2$ case, the Lagrangian function for spin-$^5/_2$ particles should have two arbitrary parameter related to the choice of normalization in the subsidiary subspaces with spin $^1/_2$ and spin $^3/_2$. It was just this number of arbitrary coefficients which appeared in the Lagrangian function for spin-$^5/_2$ particles as constructed previously by the author[8] using a method involving subsidiary quantities of lower rank.

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*If we turn to L of Eq. (1) ($n = 2$), then setting $a_4 = a_5 = a_6 = 0$ in agreement with Eq. (9) we obtain two incompatible equations for $a_1 = A/2$.