[5] T. Holstein, Phys. Rev. 88, 1427 (1952).
[6] R. B.Dingle, Physica 19, 348 (1953).
[7] M. Ia. Azbel' and M. I. Kaganov, Dokl. Akad. Nauk SSSR 95, 41 (1954).
[8] M. I. Kaganov and M. Ia. Azbel', Dokl. Akad. Nauk SSSR 102, 49 (1955).
[9] E. Kaner, M. I. Kaganov, J. Exptl. Theoret. Phys. (U.S.S.R.) 31, 459 (1956); Soviet Phys. JETP 4, 393 (1957).
[10] V. Heine, Proc. Phys. Soc. (London) 69 A, 505 (1956).

Translated by L. B. Leder
292

SOVIET PHYSICS JETP VOLUME 5, NUMBER 6 DECEMBER 15, 1957

Bremsstrahlung of Ultra-Relativistic Particles in a Central Field

A. G. SITENKO
Khar'kov State University
(Submitted to JETP editor November 19, 1956)
J. Exptl. Theoret. Phys. (U.S.S.R.) 32, 1505-1508 (June, 1957)

Bremsstrahlung of an ultra-relativistic particle of spin one-half in an arbitrary field with central symmetry is considered. A relation between the bremsstrahlung cross section and elastic scattering cross section is obtained for ultra-relativistic particles.

1. AS IS KNOWN, the bremsstrahlung in the collision of an ultra-relativistic charged particle with a nucleus occurs principally at large distances from the nucleus. The cross section for the process is therefore determined by the assymptotic form of the wave function of the particle in the nuclear field.[1-3] The assymptotic form of the particle wave functions may be found by describing the scattering effect of the nucleus by the scattering matrix. In the ultra-relativistic case, it is possible to establish a general relationship between bremsstrahlung and elastic scattering cross sections. This relationship does not depend on the character of the interaction between the particles and the scattering nuclear field.

Let us first of all consider the elastic scattering of fast particles with spin ½ in a field with central symmetry. The free motion of a spin-½ particle of momentum $\mathbf{p}$ is described by the spinor plane wave

$$\psi_0 = u_p e^{i\mathbf{pr}},$$

where u_p is the unit amplitude of the spinor wave. The scattering of particles in an external central field will be characterized by the scattering matrix S. The wave function describing the stationary states of the particles in the external field will then obviously be determined by the product of the matrix S by ψ_0

$$\psi_{\mathbf{p}}^{(+)} = S\psi_0. \tag{1}$$

At large distances from the center of the field, the wave function (1) will be of the form of a sum of a plane wave and an outgoing spherical wave. To verify this, let us use the Huygens principle as formulated by Akhiezer[4] for spinor waves. This principle establishes a relationship between the value of the wave function at a certain point and the value of the wave function on a closed surface surrounding this point. Let us choose for this surface an infinite plane perpendicular to the momentum of the impinging particle and passing through the center of the external field

$$\begin{aligned}\psi_{\mathbf{p}}^{(+)}(\mathbf{r}) &= \frac{1}{4\pi}\int\left(\gamma\frac{\partial}{\partial\mathbf{r}} - \gamma_4 E - m\right)\gamma\mathbf{n}\,\frac{\exp(ip\,|\,\mathbf{r}-\boldsymbol{\rho}\,|)}{|\,\mathbf{r}-\boldsymbol{\rho}\,|}\,S u_p e^{i\mathbf{p}\boldsymbol{\rho}}\,d\boldsymbol{\rho} \\ &= u_p e^{i\mathbf{pr}} - \frac{1}{4\pi}\int\left(\gamma\frac{\partial}{\partial\mathbf{r}} - \gamma_4 E - m\right)\gamma\mathbf{n}\,\frac{\exp(ip\,|\,\mathbf{r}-\boldsymbol{\rho}\,|)}{|\,\mathbf{r}-\boldsymbol{\rho}\,|}\,\{1-S\}\,u_p e^{i\mathbf{p}\boldsymbol{\rho}}\,d\boldsymbol{\rho}.\end{aligned} \tag{2}$$

Far from the center of the field ($r \to \infty$), this function has the form

$$\psi_{\mathbf{p}}^{(+)}(\mathbf{r}) \to u_p e^{i\mathbf{pr}} + f(\vartheta_e)\, u_{p_e} e^{ipr}/r,$$
$$f(\vartheta_e) = \frac{ip}{2\pi} u_{p_e}^{*} u_p \int \{1 - e^{2i\eta(\rho)}\} \exp\{-i(\mathbf{p}_e - \mathbf{p})\boldsymbol{\rho}\}\, d\boldsymbol{\rho}, \tag{3}$$

where $\mathbf{p}_e = p\mathbf{r}/r$; the eigenvalues of the scattering operator S are expressed through the phase shifts $\eta(\rho)$. (It is assumed that the phase shift η is identical for $l + ½$ and $l - ½$, $l = \rho p$). The multiplier diverging wave $f(\vartheta_e)$ is the elastic-scattering amplitude.

Averaging the square of the modulus of the elastic-scattering amplitude over the initial polarizations, and summing it over the final polarizations, we obtain the following expression for the elastic scattering cross section of spin-½ particles

$$d\sigma_e = \left(1 - v^2 \sin^2 \frac{\vartheta_e}{2}\right) \left| \int_0^\infty \{1 - e^{2i\eta(\rho)}\} J_0(p \sin \vartheta_e \rho)\, \rho d\rho \right|^2 p^2 do_e. \tag{4}$$

This expression differs from the expression for elastic scattering cross section of particles without spin by an additional factor $[1 - v^2 \sin^2(\vartheta_e/2)]$. (We use a system of units where $c = \hbar = 1$.)

2. Let us now examine the scattering of a spin-½ particle with emission of a γ-quantum. Assuming the interaction between the charge and the electromagnetic field of the γ-quantum to be small we obtain the following expression for the transition matrix element

$$u_{i \to f} = -\frac{ie}{\sqrt{2\omega}} \int \psi_{\mathbf{p}'}^{(-)*} \gamma_4 \hat{e} e^{-i\mathbf{kr}} \psi_{\mathbf{p}}^{(+)}\, d\mathbf{r}, \tag{5}$$

where ω, $\mathbf{k}$ and $\mathbf{e}$ are the frequency, momentum and polarization of the emitted γ-quantum, and $\mathbf{p}'$ is the particle momentum after emission of the γ-quantum. At large distances from the center, the wave function of the particle in the final state $\psi_{\mathbf{p}'}^{(-)}$ has the form of a sum of a plane wave and a converging spherical wave

$$\psi_{\mathbf{p}'}^{(-)}(\mathbf{r}) = u_{p'} e^{i\mathbf{p}'\mathbf{r}} + \frac{1}{4\pi} \int \left(\gamma \frac{\partial}{\partial \mathbf{r}} - \gamma_4 E' - m\right) \gamma \mathbf{n}' \frac{\exp\{-ip' |\mathbf{r} - \boldsymbol{\rho}|\}}{|\mathbf{r} - \boldsymbol{\rho}|} \{1 - S^*\} u_{p'}\, d\boldsymbol{\rho}. \tag{6}$$

In computing the matrix element (5), let us note that, in the ultra-relativistic case ($E, E' \gg m$), the spherical parts of the wave functions $\psi_{\mathbf{p}}^{(+)}$ and $\psi_{\mathbf{p}'}^{(-)}$ practically do not overlap. Noting also that, in the ultra-relativistic case, the main role is played by the small angles between k, p′ and p, we obtain the matrix element in the form

$$u_{i \to f} = -\frac{2\pi ie}{\sqrt{2\omega}} \int_0^\infty \{1 - e^{2i\eta(\rho)}\} J_0(|k\boldsymbol{\vartheta} + p'\boldsymbol{\vartheta}'|\rho)\, \rho d\rho\, \bar{u}_{p'} \left\{ \frac{\hat{e}(i\hat{f}' - m)\hat{n}}{\mathbf{p}^2 - \mathbf{f}'^2} + \frac{\hat{n}'(i\hat{f} - m)\hat{e}}{\mathbf{p}'^2 - \mathbf{f}^2} \right\} u_p, \tag{5'}$$

where $f = p - k$, $f' = p' + k$ and the two-dimentional angular vectors $\boldsymbol{\vartheta}$ and $\boldsymbol{\vartheta}'$ are determined by the relations

$$\mathbf{p}' = (\mathbf{p}'\mathbf{n})\,\mathbf{n} + p'\boldsymbol{\vartheta}', \quad \mathbf{k} = (\mathbf{kn})\,\mathbf{n} + k\boldsymbol{\vartheta}.$$

The differential cross section for the emission of a γ-quantum by the particle is determined from the formula

$$d\sigma_\gamma = (2\pi/v)\, |u_{i \to f}|^2\, \delta(E - E' - \omega)\, d\mathbf{p}'\, d\mathbf{k}/(2\pi)^6, \tag{7}$$

where v is the velocity of the incident particle.

Averaging the cross section over the initial particle polarizations and summing it over the final particle polarizations and over polarizations of the emitted γ-quantum, we obtain

$$d\sigma_\gamma = \frac{e^2p'}{2\omega p}\left|\int_0^\infty \{1 - e^{2i\eta(\rho)}\} J_0(|k\vartheta + p'\vartheta'|\rho)\rho\, d\rho\right|^2 \frac{1}{4}\,\mathrm{Sp}\, F\, \frac{\omega^2 d\omega}{(2\pi)^3}\, do\, do',$$

$$F = \left\{\frac{\hat{n}'(i\hat{f} - m)\gamma_i}{\mathbf{p}'^2 - \mathbf{f}^2} + \frac{\gamma_i(i\hat{f}' - m)\hat{n}}{\mathbf{p}^2 - \mathbf{f}'^2}\right\}(i\hat{p} - m)\left\{\frac{\gamma_i(i\hat{f} - m)\hat{n}'}{\mathbf{p}'^2 - \mathbf{f}^2} + \frac{\hat{n}(i\hat{f}' - m)\gamma_i}{\mathbf{p}^2 - \mathbf{f}'^2}\right\}(i\hat{p}' - m). \tag{8}$$

Calculating the trace the usual way, we finally obtain the following expression for the differential cross section for emission of a γ-quantum by an ultra-relativistic spin-½ particle in an external central field

$$d\sigma_\gamma = \frac{e^2p'}{4\pi^3 p}\left|m\int_0^\infty \{1 - e^{2i\eta(\rho)}\} J_0(m|\mathbf{x} + \mathbf{y}|\rho)\rho\, d\rho\right|^2 \left\{\left(\frac{\mathbf{x}}{1 + x^2} + \frac{\mathbf{y}}{1 + y^2}\right)^2 + \frac{\omega^2}{2pp'}\frac{(\mathbf{x} + \mathbf{y})^2}{(1 + x^2)(1 + y^2)}\right\}\frac{d\omega}{\omega}\, d\mathbf{x}\, d\mathbf{y}, \tag{9}$$

where

$$\mathbf{x} = (p/m)\,\vartheta \text{ and } \mathbf{y} = (p'/m)(\vartheta' - \vartheta).$$

In the ultra-relativistic case ($E \gg m$), the elastic scattering will also principally occur at small angles; the cross section (4) can therefore be presented in the form

$$d\sigma_e = \sigma_e(u)\, d\mathbf{u}, \quad \sigma_e(u) = \left|m\int_0^\infty \{1 - e^{2i\eta(\rho)}\} J_0(mu\rho)\rho\, d\rho\right|^2,$$

where

$$\mathbf{u} = (p/m)\,\vartheta_e. \tag{10}$$

Comparing (9) and (10), one can establish the following general relationship between the elastic scattering cross section σ_e and bremsstrahlung from charged ultra-relativistic spin-½ particles:

$$d\sigma_\gamma = \sigma_e(|\mathbf{x} + \mathbf{y}|)\frac{e^2p'}{4\pi^3 p}\left\{\left(\frac{\mathbf{x}}{1 + x^2} + \frac{\mathbf{y}}{1 + y^2}\right)^2 + \frac{\omega^2}{2pp'}\frac{(\mathbf{x} + \mathbf{y})^2}{(1 + x^2)(1 + y^2)}\right\}\frac{d\omega}{\omega}\, d\mathbf{x}\, d\mathbf{y}. \tag{11}$$

A similar relation holds for spin-zero particles†

$$d\sigma_\gamma^0 = \sigma_e^0(|\mathbf{x} + \mathbf{y}|)\frac{e^2p'}{4\pi^3 p}\left(\frac{\mathbf{x}}{1 + x^2} + \frac{\mathbf{y}}{1 + y^2}\right)^2\frac{d\omega}{\omega}\, d\mathbf{x}\, d\mathbf{y}. \tag{12}$$

Integrating (11) over the angle x and y, we obtain the spectral distribution of the radiation

$$d\sigma_\gamma(\omega) = \frac{4e^2p'\, d\omega}{\pi p\omega}\int_{q_m}^\infty \sigma_e(2q)\left\{\frac{2q^2 + 1}{q\sqrt{1 + q^2}}\ln(q + \sqrt{1 + q^2}) - 1 + \frac{\omega^2}{pp'}\frac{q}{\sqrt{1 + q^2}}\ln(q + \sqrt{1 + q^2})\right\} q\, dq, \tag{13}$$

where $q = ½\,|\mathbf{x} + \mathbf{y}|$, and the minimum value q_m is determined by the conservation law: $q_m = m\omega/4EE'$.

3. As an illustration, let us apply the obtained formulas to the case of scattering of fast protons by absorbing nuclei.* In this case, the scattering matrix can be written in the form

---

* The relationship between $d\sigma_\gamma$ and the elastic scattering amplitude for spin zero particles was also obtained in Ref. 3.

† It should be noted that, in the case of a proton, a big role should be played by the radiation due to the anomalous magnetic moment; this effect has not been taken into account, however.

$$S=\begin{cases}0, & \rho\leqslant R\\ e^{2i\eta(\rho)}, & \rho>R,\end{cases}$$

where R is the nuclear radius, $\eta(\rho)\simeq n\ln p\rho\,(p\rho\gg 1)$ and $n=Ze^2E/4\pi p$.

The elastic scattering cross section is equal to

$$d\sigma_e=\left(1-v^2\sin^2\frac{\vartheta}{2}\right)\left|\frac{RJ_1(pR\sin\vartheta)}{\sin\vartheta}+\frac{2in\,R}{\sin\vartheta}\int\limits_1^\infty J_1(pR\sin\vartheta\zeta)\,\zeta^{2in}\,d\zeta\right|^2do. \tag{14}$$

In the limiting case $n\ll 1$

$$d\sigma_e=\left(1-v^2\sin^2\frac{\vartheta}{2}\right)\left\{\frac{R^2J_1^2(pR\sin\vartheta)}{\sin^2\vartheta}+\frac{4n^2}{p^2}\frac{J_0^2(pR\sin\vartheta)}{\sin^4\vartheta}\right\}do. \tag{15}$$

The first term in the braces describes the differential scattering of particles by an absolutely black nucleus. The second term describes the scattering of charged particles in the Coulomb field of a nucleus of finite dimensions.

The cross section for emission of a γ-quantum during scattering of a proton by an absorbing nucleus is, taking into account the Coulomb interaction, equal to

$$d\sigma_\gamma=\frac{e^2p'}{4\pi^3p}\left|\frac{RJ_1(m\,|\mathbf{x}+\mathbf{y}|\,R)}{|\mathbf{x}+\mathbf{y}|}+\frac{2in}{|\mathbf{x}+\mathbf{y}|}\int\limits_R^\infty e^{2i[\eta(\rho)-\eta(R)]}J_1(m|\mathbf{x}+\mathbf{y}|\,\rho)\,d\rho\right|^2$$
$$\times\left\{\left(\frac{\mathbf{x}}{1+x^2}+\frac{\mathbf{y}}{1+y^2}\right)^2+\frac{\omega^2}{2pp'}\frac{(\mathbf{x}+\mathbf{y})^2}{(1+x^2)(1+y^2)}\right\}\frac{d\omega}{\omega}\,d\mathbf{x}\,d\mathbf{y}.$$

Setting $R=0$ in (16) and noting that

$$\int\limits_0^\infty e^{2i[\eta(\rho)-\eta(0)]}J_1(m\,|\mathbf{x}+\mathbf{y}|\,\rho)\,d\rho=(2p)^{2in}/(m\,|\mathbf{x}+\mathbf{y}|)^{2in+1}, \tag{16}$$

we obtain the Bethe-Heitler formula for bremsstrahlung from a proton in the Coulomb field of a nucleus

$$d\sigma_\gamma^{\mathrm{B-H}}=\frac{n^2e^2}{\pi^3}\frac{p'}{p}\frac{1}{m^2\,|\mathbf{x}+\mathbf{y}|^4}\left\{\left(\frac{\mathbf{x}}{1+x^2}+\frac{\mathbf{y}}{1+y^2}\right)^2+\frac{\omega^2}{2pp'}\frac{(\mathbf{x}+\mathbf{y})^2}{(1+x^2)(1+y^2)}\right\}\frac{d\omega}{\omega}\,d\mathbf{x}\,d\mathbf{y}. \tag{17}$$

Setting $n=0$ in (16), we obtain the Akhiezer formula[4] for the diffraction radiation of γ-quanta by protons

$$d\sigma_\gamma^{\mathrm{A}}=\frac{e^2}{4\pi^3}\frac{p'}{p}\frac{R^2J_1^2(m\,|\mathbf{x}+\mathbf{y}|\,R)}{|\mathbf{x}+\mathbf{y}|^2}\left\{\left(\frac{\mathbf{x}}{1+x^2}+\frac{\mathbf{y}}{1+y^2}\right)^2+\frac{\omega^2}{2pp'}\frac{(\mathbf{x}+\mathbf{y})^2}{(1+x^2)(1+y^2)}\right\}\frac{d\omega}{\omega}\,d\mathbf{x}\,d\mathbf{y}. \tag{18}$$

Equation (16) describes bremsstrahlung due to Coulomb interaction as well as to the presence of diffraction. In the limiting case $n\ll 1$, this cross section is equal to

$$d\sigma_\gamma=\frac{e^2p'}{4\pi^3p}\left\{\frac{R^2J_1(m\,|\mathbf{x}+\mathbf{y}|\,R)}{|\mathbf{x}+\mathbf{y}|^2}+\frac{4n^2}{m^2}\frac{J_0^2(m\,|\mathbf{x}+\mathbf{y}|\,R)}{|\mathbf{x}+\mathbf{y}|^4}\right\}$$
$$\times\left\{\left(\frac{\mathbf{x}}{1+x^2}+\frac{\mathbf{y}}{1+y^2}\right)^2+\frac{\omega^2}{2pp'}\frac{(\mathbf{x}+\mathbf{y})^2}{(1+x^2)(1+y^2)}\right\}\frac{d\omega}{\omega}\,d\mathbf{x}\,d\mathbf{y}. \tag{19}$$

I express my deep gratitude to A. I. Akhiezer for discussing this work.

[1] L. D. Landau and I. Ia. Pomeranchuk, J. Exptl. Theoret. Phys. (U.S.S.R.) 24, 505 (1953).

[2] E. L. Feinberg, Uspekhi Fiz. Nauk 58, 193 (1956).

[3] Iu. A. Vdovin. Dokl. Akad. Nauk SSSR 105, 947 (1955).

[4] A. I. Akhiezer, Dokl. Akad. Nauk SSSR 94, 651 (1954).

Translated by E. S. Troubetskoy
293