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INTERACTION BETWEEN GRAVITATIONAL-CAPILLARY AND MAGNETOHYDRODYNAMIC WAVES

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The uniqueness theorem is proved for the solutions of the hydrodynamic equations for an incompressible strongly conducting ideal liquid. Walen's exact solution for the case in which gravitational and capillary forces act on the liquid is shown to be valid. Relations for the stability conditions, penetration depth etc. are derived for potential and vortex harmonic waves. In the case of potential motion of the liquid the current density in the surface layer is found to be exactly zero.

MAGNETOHYDRODYNAMIC waves are described by the system of equations

$$\rho d\mathbf{v}/dt = -\nabla (p + U) + \mathbf{j}\times\mathbf{B}/c; \quad \operatorname{curl}\mathbf{H} = 4\pi\mathbf{j}/c, \quad \partial\rho/\partial t + \operatorname{div}(\rho\mathbf{v}) = 0; \quad \operatorname{curl}\mathbf{E} = -\partial\mathbf{B}/c\,\partial t; \quad \mathbf{E} + \mathbf{v}\times\mathbf{B}/c = 0, \tag{1}$$

where U is the potential associated with the forces of non-electromagnetic origin, where

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{h}. \tag{2}$$

It will be assumed that the field $\mathbf{H}_0$ is uniform and known and that the liquid is incompressible; these assumptions are justified under the conditions of the present problem. Using the system of equations given in (1) and taking

$$\boldsymbol{\sigma} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}; \quad W = \frac{1}{8\pi}(\varepsilon\mathbf{E}^2 + \mu\mathbf{H}^2), \quad T_{ik} = \frac{1}{4\pi}\{(\varepsilon\mathbf{E}^2 + \mu\mathbf{H}^2)^2\delta_{ik} - \varepsilon E_i E_k - \mu H_i H_k\},$$

it is possible to find the non-relativistic equations for conservation of momentum and energy in a volume τ bounded by a surface S:

$$\frac{d}{dt}\left\{\int_\tau \left(\rho v_i + \frac{\sigma_i}{c^2}\right) d\tau\right\} = -\oint_S (\rho v_i v_k + p\delta_{ik} + T_{ik})\, n_k dS; \tag{3}$$

$$\frac{d}{dt}\left\{\int_\tau \left(\frac{1}{2}\rho v^2 + W\right) d\tau\right\} = -\oint_S \left\{\left(\frac{1}{2}\rho v^2 + p\right) v_i + \sigma_i\right\} n_i dS - \int_\tau \frac{j^2}{\sigma} d\tau. \tag{4}$$

If we are given the quantities:

$$v_n = v_i n_i, \; p, \; E_t \; (\text{or } H_t) \tag{5}$$

on the surface S and the velocity and directions of the fields are given as functions of the coordinates inside the volume τ at time t = 0:

$$\mathbf{v} = \mathbf{v}(x_k), \; \mathbf{E} = \mathbf{E}(x_k), \; \mathbf{H} = \mathbf{H}(x_k), \tag{6}$$

and if the condition of incompressibility is satisfied

$$\operatorname{div}\mathbf{v} = 0 \tag{7}$$

the system of equations in (1) has a unique solution which satisfies the initial conditions and the boundary conditions (5) and (6), provided μ, ϵ, and σ are independent of the coordinates and time. The proof, as in electrodynamics, is based on the fact that the differences of the solutions $\mathbf{v}'$, $\mathbf{E}'$, $\mathbf{H}'$ and $\mathbf{v}''$, $\mathbf{E}''$ and $\mathbf{H}''$, which satisfy Eqs. (5) and (6), are also solutions of the equations in (1), where the continuity equation is taken in the form given in Eq. (7), and the fact that the total energy cannot increase.

To calculate the effects of gravitational and capillary forces, acting on the surface of the liquid S, on the propogation of the magnetohydrodynamic waves, we can use as boundary conditions the equations which describe the discontinuities in the physical quantities at the interface between two media; these relations are derived from Maxwell's equations, the theory of capillary action and the conservation laws

$$[E_t]=0;\ [H_t]=0;\ [\varepsilon E_i n_i]=0;\ [\mu H_i n_i]=0; \tag{8}$$

$$[(\rho v_i v_k + p\delta_{ik} + T_{ik})\, n_k] = -\pi;\quad [\{({}^1/{}_2\rho \mathbf{v}^2 + p)\, v_i + \sigma_i\}\, n_i] = 0, \tag{9}$$

where the square brackets denote the differences in the values of the quantities associated with the second medium and first medium (the given liquid) and

$$\pi = \alpha\,(1/R_1 + 1/R_2) \tag{10}$$

is the Laplace pressure.

The surface which divides the two media will be assumed to be approximately the same as the surface

$$S_0(x_i) = 0, \tag{11}$$

which corresponds to the equilibrium configuration of the liquid. We assume that the quantities which appear in the boundary conditions (8) and (9) are given at this surface; because of Eq. (11) these are equivalent to the condition given in (5). Applying the initial conditions in the region which is filled by the liquid and bounded by the surface given by (11), we have a complete formulation of the problem, which has a unique solution.

As is well known, Walen[2] has given an exact solution for Eq. (1) in the case of an incompressible liquid:

$$\mathbf{v} = \pm\mathbf{h}\sqrt{\mu/4\pi\rho};\quad \mathbf{E} = \pm\,\mathbf{h}\times\mathbf{H}_0\sqrt{\mu/4\pi\rho}; \tag{12}$$

$$p + \rho u + \frac{\mu}{8\pi}(\mathbf{H}_0 + \mathbf{h})^2 + \psi(t) = 0, \tag{13}$$

where the vector $\mathbf{h}$ satisfies the equation

$$(\mathbf{H}_0\nabla)\,\mathbf{h} = (4\pi\rho/\mu)^{1/2}\,\partial\mathbf{h}/\partial t, \tag{14}$$

and $\psi(t)$ is an arbitrary function of time.

We now show that there are exact Walen solutions which satisfy the boundary conditions in (8) and (9) thus constituting a solution of the problem stated above. The only quantity assumed given is the pressure p'' in medium 2, which is assumed to be a rarefied gas or even a vacuum. In this case it is sufficient to use only one of the relations in (9); this relation assumes the form:

$$\left[p + \rho\mathbf{v}^2 + \frac{1}{8\pi}(\varepsilon\mathbf{E}^2 + \mu H^2)\right] = -\pi.$$

To make the problem somewhat more concrete we also assume that both media are in a fixed gravitational field

$$U = \rho g z, \tag{15}$$

where the medium fills the half space along the negative z-axis. Then, in view of Eq. (12), assuming that

$$\varepsilon'' = \mu'' = 1,\quad \varepsilon' = \varepsilon,\quad \mu' = \mu,\quad \rho'' v''^2 = 0,$$

the preceding relation can be rewritten in the form

$$p' - p'' + \frac{\mu}{4\pi}(\mathbf{H}' - \mathbf{H}_0)^2 + \frac{1}{8\pi}\{(\varepsilon - 1)\,[E_x'^2 + E_y'^2 - \varepsilon E_z'^2] + (\mu - 1)\,[H_x'^2 + H_y'^2 - \mu H_z'^2]\} = -\alpha\,(\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2). \tag{16}$$

At z = 0 this equation must be compatible with Eq. (1). Hence the boundary condition for the solution in the form given in (12)–(14) can be obtained by differentiating (16) and (13) with respect to time and eliminating dp'/dt (dp''/dt is taken as zero). We find

$$\frac{\partial}{\partial t}\left\{\frac{\mu}{4\pi}(\mathbf{H}' - \mathbf{H}_0)^2 + \frac{1}{8\pi}\left\{(\varepsilon - 1)\,[\mathbf{E}'^2 - (\varepsilon + 1)\,E_z'^2] + (\mu - 1)\,[\mathbf{H}'^2 - (\mu + 1)\,H_z'^2] - \frac{\mu}{8\pi}\mathbf{H}'^2\right\}\right\} \mp g\sqrt{\frac{\mu\rho}{4\pi}}\,h_z'$$

$$\pm\alpha\sqrt{\frac{\mu}{4\pi\rho}}\left(\frac{\partial^2 h_z'}{\partial x^2} + \frac{\partial^2 h_z'}{\partial y^2}\right) + \varphi(t) = 0, \tag{17}$$

where, as before, $\varphi(t)$ is an arbitrary function of t. The Walen solution for gravitational-capillary and magnetohydrodynamic waves which is of interest to us can be obtained in the following manner. We have the solution of Eq. (7) in the form:

$$\mathbf{v} = \mathrm{curl}\, \mathbf{a}, \tag{18}$$

or

$$\mathbf{v} = \mathrm{grad}\, \varphi. \tag{19}$$

Then $\mathbf{h}$, being determined by the first of the equations in (12), automatically satisfies (14). Furthermore, $\mathbf{h}$, and thus $\mathbf{v}$, must be chosen to satisfy the boundary condition in (17) in the plane $z = 0$. The solution in (18) and the $\mathbf{h}$ which corresponds to it satisfy (7) and (14) with the choice of any three functions of the coordinates and time a_i; for the condition in (17), however, we require any two functions of the three a_i.

The solution for $\mathbf{v}$ and $\mathbf{h}$ in the form (19) reduces to the search for $\mathbf{h}$ (or $\mathbf{v}$) from the Laplace equations

$$\triangle \chi = 0, \tag{20}$$

which satisfy the boundary condition in (17) when the following substitution is made:

$$\mathbf{h} = \mathrm{grad}\, \chi. \tag{21}$$

In all cases — for vortex and potential gravitational-capillary and magnetohydrodynamic waves — the velocity in the direction of the fixed external magnetic field $\mathbf{H}_0$ has the usual value

$$u = \pm \mathbf{H}_0 \sqrt{\mu / 4\pi\rho}. \tag{22}$$

It should be noted, however, that in potential waves, as defined by (19) and (21), up to and including the dividing surface between the two media we have

$$\mathbf{j} = 0, \tag{23}$$

this result contradicts the conclusions reached in Ref. 3, in which the boundary conditions $\mathbf{j} \neq 0$ and even $\mathbf{j} = \infty$ have been assumed without justification.

We now consider some simple cases: these correspond to the linear approximation and to fields which are perpendicular and parallel to the surface of the liquid:

$$H_z^0 = H_0 \neq 0; \quad H_x^0 = H_y^0 = 0; \tag{I}$$

$$H_x^0 = H_0 \neq 0; \quad H_y^0 = H_z^0 = 0. \tag{II}$$

Furthermore, for simplicity we assume

$$\varepsilon - 1 = 0, \quad \mu - 1 = 0. \tag{24}$$

First, we consider the potential waves (19) and (21). We have

$$\frac{\partial}{\partial x_i}\left(H_{0k} \frac{\partial}{\partial x_k}\right)\chi = H_{0k} \frac{\partial^2 \chi}{\partial x_i \partial x_k}.$$

On the other hand,

$$\left(H_{0k} \frac{\partial}{\partial x_k}\right) \frac{\partial \chi}{\partial x_i} = H_{0k} \frac{\partial^2 \chi}{\partial x_i \partial x_k}.$$

thus, $(\mathbf{H}_0 \nabla)\mathbf{h} = \nabla(\mathbf{H}_0 \mathbf{h})$. In accordance with these relations and (14), Eq. (13) can be transformed as follows:

$$\frac{\partial \varphi}{\partial t} + gz + \frac{1}{\rho}\left(p + \frac{1}{2}\rho v^2\right) + f(t) = 0, \tag{25}$$

where $f(t)$ is an arbitrary function of the time which may be taken equal to zero. In the linear approximation, using Eqs. (16) and (25), differentiating with respect to time and eliminating $\partial p'/\partial t$, we have in the plane $z = 0$

$$\frac{\partial^2 \varphi'}{\partial t^2} + g \frac{\partial \varphi'}{\partial z} - \frac{\alpha}{\rho} \frac{\partial}{\partial z}\left(\frac{\partial^2 \varphi'}{\partial x^2} + \frac{\partial^2 \varphi'}{\partial y^2}\right) = 0. \tag{26}$$

This boundary condition is in complete agreement with the boundary condition for gravitational-capillary waves.

We consider the solution of (7) in the form

$$\varphi = \varphi_0 e^{i(\omega t - \mathbf{kr})}. \tag{27}$$

Substituting this expression for φ in (7) and (26) we find

$$k_x^2 + k_y^2 + k_z^2 = 0, \tag{28}$$

$$\omega^2 = -ik_z [g + (\alpha/\rho)(k_x^2 + k_y^2)], \tag{29}$$

from which it is apparent that k_z must be an imaginary number, with

$$ik_z = -k = -\sqrt{k_x^2 + k_y^2}, \tag{30}$$

where k is a real positive number and k_x and k_y are also real.

From Eqs. (14) and (29), we have in cases (I) and (II):

$$\omega = -k_z u = -iku, \tag{31.I}$$

$$\omega = -k_z u. \tag{31.II}$$

Thus, in case I a stable wave of the form given in (27) cannot exist. In case II such a wave is possible. It is localized in a layer of thickness $d = 1/k$ and has the velocity of the usual magnetohydrodynamic wave, given by (22), in the direction of the z-axis, which is perpendicular to the surface of the liquid.

From Eqs. (29), (30) and (31.II) we obtain the equation for the damping factor

$$k_x^2 u^2 = k(g + \alpha k^2/\rho). \tag{32}$$

If $k_x = k$,

$$k = (\rho u^2/2\alpha)(1 \pm \sqrt{1 - 4\alpha g/u^4\rho}). \tag{33}$$

Whence, from Eq. (22) it is also obvious that the stability condition for waves of this type is:

$$H_0^4 \geqslant 64\pi^2 \alpha g \rho. \tag{34}$$

In mercury, for example, the smallest magnetic field for which this type of magnetohydrodynamic potential wave is possible is $H_{0m} = 246.2$ gauss. When $H_0 \gg H_{0m}$ the thickness of the surface layer is given by the formula

$$d = \alpha/\rho u^2. \tag{35}$$

In the long-wave case

$$d = u^2/g. \tag{36}$$

In mercury with $H_0 = 10^4$ gauss, i.e., with a field approximately 100 times larger than the minimum value H_{0m}, we have $d \approx 10^{-2}$ cm from Eq. (35) and $d \approx 10^3$ cm from Eq. (36).

Thus, the strongest effect on the concentration of the potential magnetohydrodynamic waves in the surface layer is that due to the capillary forces. However, when these forces are present the thickness of the surface-wave layer, in general, is smaller than for the skin effect; moreover, the current density is exactly zero for the potential surface waves, in contrast to the skin effect and in contradiction with the statements of Ref. 3. In the case of vortex waves (18), in the linear approximation using (I), (II), and (24) the boundary condition in (17) assumes the form

$$\frac{\partial h_z'}{\partial t} + \frac{g}{u} h_z' - \frac{\alpha}{\rho u}\left(\frac{\partial^2 h_z'}{\partial x^2} + \frac{\partial^2 h_z'}{\partial y^2}\right), \tag{17.I}$$

$$\frac{\partial h_x'}{\partial t} + \frac{g}{u} h_z' - \frac{\alpha}{\rho u}\left(\frac{\partial^2 h_z'}{\partial x^2} + \frac{\partial^2 h_z'}{\partial y^2}\right). \tag{17.II}$$

We consider a solution in the form

$$\mathbf{v} = \pm \frac{1}{\sqrt{4\pi\rho}} \operatorname{curl} \mathbf{a} = \pm \frac{h}{\sqrt{4\pi\rho}}, \quad \mathbf{a} = \mathbf{a}_0 e^{i(\omega t - kr)}, \tag{37}$$

which is known to satisfy Eq. (1) and the condition given in (7). In cases (I) and (II) we have

$$\omega = i\left[\left(\frac{g}{u} + \frac{\alpha}{\rho u}\right)(k_x^2 + k_y^2)\right] = -uk_z, \tag{38.I}$$

$$k_y = 0;\ k_x = ik_z = k \leqslant 0, \qquad \omega = \frac{1}{u}\left(g + \frac{\alpha}{\rho}k^2\right) = -uk_x. \tag{38.II}$$

In case I we do not obtain stable waves of the form given in (37). In case II we obtain a wave which decays in the z-direction and which is characterized by a penetration depth

$$d = -1/k = (\rho u^2/2\alpha)(1 \pm \sqrt{1 - 4\alpha g/\rho u^4}). \tag{39}$$

Thus, the vortex magnetohydrodynamic surface waves have the same propogation and decay relations as the potential waves with the one exception that the current density is not zero; these waves are similar to the skin effect in this respect.

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ON THE DYNAMICS OF A BOUNDED PLASMA IN AN EXTERNAL FIELD

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Some problems relating to the dynamics of a quasi-neutral plasma formation situated in the field of a plane electromagnetic wave are considered. The method of successive approximations is employed. It is shown that within the limits of the assumptions underlying the analysis a plasma bunch tends to spread out.

In connection with the proposal of a radiation method of accelerating particles[1] the question has arisen of the behavior of a quasi-neutral plasma condensation situated in the field of an electromagnetic wave and, in particular, the question of the stability of such a condensation. A more or less rigorous solution of this problem encounters formidable mathematical difficulties. Therefore it is not without interest to consider a simplified problem which would nevertheless allow one to obtain approximate estimates of the magnitude and the nature of the forces acting on the condensation. The present paper is devoted to the examination of one of these simplest cases.

We consider a system of electrons and ions subjected to an incident plane electromagnetic wave with a propagation vector **k** parallel to the z axis. By using the hydrodynamic description of plasma,* which is

* We shall not consider here questions concerning the permissibility of applying the hydrodynamic approximation.