$$S_e = -\chi_e \rho_0 d\varepsilon/dx.$$

Let us compare this with the flux of radiant energy, Eq. (3), using Eq. (4) to obtain the gradient of the internal energy. We obtain

$$S_e / S_{\text{rad}} = (l_e / l_{\text{rad}})\, v_e / D,$$

where v_e is the thermal velocity of the electrons.

Calculation shows that the fluxes are comparable only at very high temperatures $\sim 300{,}000°$. At lower temperatures: $S_e \ll S_{\text{rad}}$.

The difference between the electronic and the ionic temperature has an essential effect only on the structure of the temperature peak behind the discontinuity, but this in no way influences the behavior of the effective temperature of the wave.

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Translated by D. Lieberman
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SOVIET PHYSICS JETP VOLUME 6, NUMBER 1 JANUARY, 1958

DISPERSION OF SOUND IN A FERMI LIQUID

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Submitted to JETP editor December 11, 1956

J. Exptl. Theoret. Phys. (U.S.S.R.) 33, 110-115 (July, 1957).

On the basis of the theory proposed by Landau for a Fermi liquid the attenuation and dispersion of sound oscillations in such a liquid are investigated. Specific calculations are performed for the case of liquid He^3.

THE characteristics of sound in a Fermi liquid are determined by the kinetic equation for the excitations, which, according to Landau,[1] has the form

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial \mathbf{r}}\frac{\partial \varepsilon}{\partial \mathbf{p}} - \frac{\partial n}{\partial \mathbf{p}}\frac{\partial \varepsilon}{\partial \mathbf{r}} = I(n). \tag{1}$$

Here ϵ is the energy of a single excitation, a function of the excitation density n. For small deviations of the distribution function n from the equilibrium Fermi distribution at $T = 0\,(n_0)$, the function ϵ can be represented in the form

$$\varepsilon = \varepsilon_0 + \int f(\mathbf{p},\mathbf{p}')\, n_1(\mathbf{p}')\, d\tau_{p'} \quad (n_1 = n - n_0). \tag{2}$$

Let oscillations in the excitation density n take place within the liquid with small amplitude ν_1 and frequency ω. n_1 can now be written in the form

$$n_1 = \nu_1 e^{i(\mathbf{kr}-\omega t)}. \tag{3}$$

We substitute (3) in Eq. (1), confining ourselves to terms linear in n_1

$$-i\omega n_1 + i n_1 \mathbf{k}\frac{\partial\varepsilon}{\partial\mathbf{p}} - i\mathbf{k}\frac{\partial n_0}{\partial\mathbf{p}}\int f(\mathbf{p},\mathbf{p}')\, n_1(\mathbf{p}')\, d\tau_{p'} = I(n_1). \tag{4}$$

We now recall that the derivative $\partial n_0/\partial\epsilon$ has the form of a δ-function, and introduce the notation

$$\nu_1 = \nu\,\partial n_0/\partial\varepsilon, \quad f\,(\partial\tau/\partial\varepsilon)_{\varepsilon=\varepsilon_0} = F. \tag{5}$$

We now rewrite Eq. (4) in the form

$$\left\{(\mathbf{kv}-\omega)\,\nu + \mathbf{kv}\int F\nu'\frac{do'}{4\pi}\right\}\frac{\partial n_0}{\partial\varepsilon} = I \quad \left(\mathbf{v} = \frac{\partial\varepsilon}{\partial\mathbf{p}}\right). \tag{6}$$

The function F evidently depends only upon the angle ϑ between the momenta p and p', since in the approximation under consideration these vectors are equal in absolute value to p_0. In the general case F is a complicated function of the angle ϑ. Therefore the function ν is likewise a complicated function of the angle between the vectors k and p.

Landau has shown[2] that at absolute zero, where the collision integral I is equal to zero, Eq. (4) leads to the occurrence of undamped oscillations (zero sound). The velocity of propagation of these oscillations is determined by a transcendental equation (cf. Appendix) involving in a fundamental way the function $F(\vartheta)$. We do not know the form of this function; knowing the effective mass of the excitation and its compressibility, however, we can determine $\overline{F}$ and $\overline{F\cos\vartheta}$ from the formulas (cf. Ref. 1)

$$1/m = (1/m^*)\,(1 + \overline{F\cos\vartheta}), \tag{7}$$

$$c^2 = (p_0^2/3m^2)\,(1+\overline{F})/(1+\overline{F\cos\vartheta}) \tag{8}$$

Specific calculations for He^3 can therefore be carried out, taking the function in the form of a binomial

$$F = F_0 + F_1\cos\vartheta. \tag{9}$$

At temperatures different from zero ordinary sound will be propagated in the Fermi liquid. For low frequencies ω the attenuation of the sound is determined by the usual expression[3]

$$\gamma = \frac{\omega^2}{2\rho c^3}\left\{\left(\frac{4}{3}\eta + \zeta\right) + \frac{\varkappa}{c_p}\left(\frac{c_p}{c_v} - 1\right)\right\}; \tag{10}$$

here γ is the sound attenuation coefficient, η is the coefficient of first viscosity, ζ is the coefficient of second viscosity, κ is the thermal conductivity coefficient, and c is the specific heat. In the case of the ideal gas model, for which the Fermi boundary surface has the form of a sphere, the second viscosity coefficient is very small.[4] The first viscosity coefficient varies with the temperature as $1/T^2$. The ratio κ/c_p, equal in order of magnitude to η, also varies as $1/T^2$ ($\kappa \sim 1/T$, $c_p \sim T$). The term involving κ/c_p, however, is multiplied by the factor $(c_p - c_v)/c_v$. This ratio, as can readily be seen from the well-known relation

$$\frac{c_p}{c_v} - 1 = -\frac{T}{c_v}\left(\frac{\partial p}{\partial T}\right)_v^2 \Big/ \left(\frac{\partial p}{\partial V}\right)_T,$$

varies with the temperature as T^2 ($\partial p/\partial T \sim \partial S/\partial V \sim T$). At low temperatures this factor is extremely small. Therefore at low temperatures the attenuation of sound is determined entirely by the viscosity.

Eq. (10) is valid for the case in which the frequency of the sound is low and the inequality $\omega\tau \ll 1$ is fulfilled; τ is the time between collisions of the excitations. The time τ is inversely proportional to the square of the temperature.[4,5] The above inequality will be violated both at sufficiently low temperatures and at sufficiently high frequencies. If the frequency of the sound is sufficiently high that the reverse inequality $\omega\tau >> 1$ applies, this signifies that collisions between the excitations play no part, and that the collision integral may be neglected. In this case, however, the oscillations investigated by Landau[2] (zero sound) can be propagated in the Fermi liquid. It is of interest to follow the process by which the transition from ordinary sound to zero sound takes place as the frequency ω is increased. For this it is necessary to solve the kinetic equation (4). We shall not solve this difficult problem for the general case, particularly in view of the fact that the interaction law for the excitations is unknown.

It would be possible to simplify the problem by introducing a certain effective time τ and substituting for the integral I in (4) the expression n_1/τ. With such a substitution, however, the conservation laws for number of excitations, momentum, and energy will not come out of the kinetic equation, making the transformation to hydrodynamics impossible. Since the thermal conductivity and the second viscosity are negligibly small in the present case, the terms in the collision integral which involve the zero and first spherical harmonics are absent. We shall therefore replace the collision integral by the following expression:*

$$I(n) \to -\tau^{-1}(n_1 - \overline{n_1} - 3\overline{n_1 \cos\vartheta}\cos\vartheta) \tag{11}$$

It can readily be seen that when integrated over $d\tau_p$ this expression reduces to zero. It also reduces to zero when multiplied by $p\cos\theta$ and integrated over $d\tau_p$.† Thus the conservation equations for the number of particles and for the momentum are automatically fulfilled. Thus, from (4) and (11), the final kinetic equation has the following form

$$(kv\cos\vartheta - \omega)\nu + kv\cos\vartheta\int F\nu'\frac{do'}{4\pi} = -\frac{1}{i\tau}(\nu - \overline{\nu} - 3\overline{\nu\cos\vartheta}\cos\vartheta). \tag{12}$$

Bearing in mind the applicability of these results to He^3 we write the function F in the binomial form of Eq. (9), in order not to complicate the problem.

We introduce the notation

$$\overline{\nu} = \nu_0, \quad 3\overline{\nu\cos\vartheta} = \nu_1, \quad \sigma = i\tau kv, \quad \xi = (i\omega\tau - 1)/i\tau kv, \tag{13}$$

following which we readily obtain, from (12)

$$(\cos\vartheta - \xi)\nu + \cos\vartheta\left(F_0\nu_0 + \frac{1}{3}F_1\nu_1\cos\vartheta\right) = \frac{1}{\sigma}(\nu_0 + \nu_1\cos\vartheta). \tag{14}$$

We next solve this equation for ν and compute $\overline{\nu} = \nu_0$ and $\overline{\nu\cos\vartheta} = \nu_1/3$. We thus obtain for the two quantities ν_0 and ν_1 the two equations

$$\nu_0 = F_0\nu_0 w + \frac{1}{3}wF_1\nu_1 - \frac{1}{\sigma}\frac{w+1}{\xi}\nu_0 - \frac{\nu_1}{\sigma}w, \tag{15}$$

$$\frac{1}{3}\nu_1 = F_0\nu_0\xi w - \frac{1}{3}\left(\frac{1}{3} - \xi^2 w\right)F_1\nu_1 - \frac{w}{\sigma}\nu_0 - \frac{\xi w}{\sigma}\nu_1, \tag{16}$$

where

$$w = \frac{1}{2}\ln\frac{\xi+1}{\xi-1} - 1.$$

Solving Eqs. (15) and (16) simultaneously we obtain an equation which determines the complex sound velocity

$$\left(1 + \frac{1}{\xi\sigma}\right)\left(1 + \frac{F_1}{3}\right) - w\left\{\left(1 + \frac{F_1}{3}\right)\left(F_0 - \frac{1}{\xi\sigma}\right) + \xi^2\left(F_1 - \frac{3}{\xi\sigma}\right)\left(1 + \frac{1}{\xi\sigma}\right)\right\} = 0. \tag{17}$$

This is the desired equation expressing the dependence of the velocity of sound upon the frequency, or, in other words, describing the dispersion of sound in a Fermi liquid. Let us first consider the two limiting cases.

a) Low frequencies: $\omega\tau << 1$. In consequence, $\sigma \to 0$, $\xi\sigma \to -1$, $\xi \to \infty$. Expansion of the function w in powers of $1/\xi$ yields

*Here and below we indicate by a superscribed line a quantity averaged over angle.

†In this integration only the range of values for the momentum near p_0 are of importance, since in accordance with (5) the function n_1 includes a δ-function at $p = p_0$.

$$w = 1/3\xi^2 + 1/5\xi^4$$

and, after some simplification, Eq. (17) takes the form

$$\left(1 + \frac{1}{\xi\sigma}\right)^2 = \frac{1}{3\xi^2}\left(1 + \frac{F_1}{3}\right)\left(F_0 - \frac{1}{\xi\sigma}\right) + \frac{1}{5\xi^2}\left(1 + \frac{1}{\xi\sigma}\right)\left(F_1 - \frac{3}{\xi\sigma}\right). \tag{18}$$

From the relations (13) we have

$$\left(\frac{1 + \xi\sigma}{\sigma}\right)^2 = \left(\frac{\omega}{kv}\right)^2.$$

From (18) and (13) we find, to the first order in $i\omega\tau$

$$\left(\frac{\omega}{kv}\right)^2 = \frac{1}{3}(1 + F_0)\left(1 + \frac{F_1}{3}\right) - \frac{4}{15} i\omega\tau\left(1 + \frac{F_1}{3}\right). \tag{19}$$

The first term corresponds to the velocity of ordinary sound in a Fermi liquid. The attenuation of sound in the region $\omega\tau \ll 1$ is obtained in an elementary fashion from (19), as the imaginary component of the wave vector:

$$\gamma = \operatorname{Im} k = (2\omega^2\tau v^2 / 15c^3)(1 + F_1/3). \tag{20}$$

Comparing expressions (20) and (10) we set up the relation

$$\eta = {}^1/_5\, \rho\tau v^2 (1 + {}^1/_3 F_1), \tag{20'}$$

which eneables us to find the time τ from experimental values of the viscosity. At the present time data is lacking on the viscosity of He^3 at temperatures below the degeneracy temperature. There are available only preliminary data (by K. Zinov'eva) in the temperature region near the degeneracy temperature. From these data we obtain the very rough value

$$\tau \approx 0.4 \cdot 10^{-12}\, T^{-2} \text{ sec.}$$

With this we obtain for the attenuation coefficient

$$\gamma \sim 1 \cdot 10^{-18} (\omega/T)^2 \text{ cm}^{-1}.$$

b) We now consider the second limiting case of high frequencies and low temperatures: $\omega\tau \gg 1$. In this case

$$\sigma \to \infty, \quad \xi\sigma \to \infty, \quad \xi = s + i\xi', \quad |\xi'| \ll s.$$

Eq. (17) assumes the form

$$(1 + F_1/3) + w(s)\{(1 + F_1/3)F_0 + s^2 F_1\} = 0. \tag{21}$$

This equation agrees completely with the equation determing the velocity of zero sound [cf. Appendix, Eq. (A8)] $u_0 = sv$.

As regards the attenuation of zero sound, in order to calculate this it is necessary to find the imaginary component of the sound velocity ξ'. From (19) we obtain the equation

$$\xi'\left\{\frac{1}{w(s)}\left(1 + \frac{F_1}{3}\right)\left(\frac{s}{s^2 - 1} - \frac{w(s) + 1}{s}\right) - 2sw(s)F_1\right\} - \frac{1}{\omega\tau}\left\{\left(1 + \frac{F_1}{3}\right)(1 + w(s)) + w(s)s^2(3 - F_1)\right\} = 0. \tag{22}$$

With the aid of Eq. (13) we find the attentuation coefficient

$$\gamma = \operatorname{Im} k = 1/s\tau v - \omega\xi'/s^2 v. \tag{23}$$

If we substitute in this the values of the parameters for He^3 ($s = 1.84$, $v = 1.13 \times 10^4$ cm/sec) we obtain

$$\xi' = 1.5/\omega\tau, \quad \gamma = 2.2 \cdot 10^7 T^2 \text{ cm}^{-1}. \tag{24}$$

The attenuation of zero sound is thus independent of frequency and increases with increasing temperature as $1/\tau$; i.e., proportionately to T^2.

In conclusion, we shall comment upon the attenuation of sound in a Fermi liquid at extremely low temperatures, for which the inequality $\hbar\omega \gg kT$ holds. It is evident that in this region the classical treatment is inapplicable. The attenuation process must here be treated quantum-mechanically.

Detailed calculations performed by Landau[2] yield for the attenuation of sound in this limiting case the following results:

$$\gamma = \gamma_{cl}(1 + (\hbar\omega/2\pi kT)^2), \tag{25}$$

γ_{cl} is determined from Eq. (23). Since τT^2 = const., we have in the limit $\hbar\omega >> kT$

$$\gamma \sim 10^{-15}\omega^2 \text{ cm}^{-1}.$$

In the quantum region the attenuation of zero sound is independent of temperature and is proportional to the square of the frequency.

APPENDIX

The integral equation for the velocity of zero sound derived by Landau[2] has the following form:

$$(v\cos\theta - u)\nu(\theta,\varphi) + v\cos\theta\int F(\vartheta)\nu(\theta',\varphi')\frac{do'}{4\pi} = 0, \tag{A1}$$

$\vartheta = (\theta'\varphi', \theta\varphi)$. In Ref. 2 the solution of this equation is given for the case in which the function $F(\vartheta)$ is independent of the angle ϑ. We shall derive here the solution of Eq. (4) at T = 0 for an arbitrary function $F(\vartheta)$.

In the general case $F(\vartheta)$ can be represented as a sum of spherical harmonics

$$F(\vartheta) = \sum F_n P_n(\vartheta), \tag{A2}$$

Here the F_n are the coefficients in the expansion of $F(\vartheta)$ in Legendre polynomials. We substitute (A2) in (A1), using the addition theorem for Legendre polynomials

$$P_n(\vartheta) = \sum_{m=-n}^{n} P_n^m(\theta)P_n^m(\theta')e^{im[\varphi-\varphi']}(n-|m|)!/(n+|m|)!,$$

where $P_n^m = P_n^{-m}$ are the associated Legendre polynomials. After making the substitutions indicated above we obtain

$$(v\cos\theta - u)\nu + v\cos\theta\sum\frac{(n-|m|)!}{(n+|m|)!}P_n^m(\theta)F_n e^{im\varphi}\int P_n^m(\theta')\nu(\theta',\varphi')e^{-im\varphi'}\frac{do'}{4\pi} = 0. \tag{A3}$$

We introduce the notation

$$F_n\frac{(n-|m|)!}{(n+|m|)!}\int P_n^m(\theta')\nu(\theta',\varphi')e^{-im\varphi'}\frac{do'}{4\pi} = \Phi_{nm} \tag{A4}$$

and solve (A3) for ν ($s = u/v$);

$$\nu = -\frac{\cos\theta}{\cos\theta - s}\sum\Phi_{nm}P_n^m(\theta)e^{im\varphi} \tag{A5}$$

Setting this expression into the relation (A4) and carrying out the integration over φ', we obtain

$$F_n\frac{(n-|m|)!}{(n+|m|)!}\int\sum_k P_n^m(\theta')\frac{\cos\theta'}{\cos\theta' - s}P_k^m(\theta')\frac{do'}{4\pi}\Phi_{km} = \sum_k\Phi_{km}\delta_{kn}. \tag{A6}$$

We have thus obtained a system of homogeneous equations determining the quantity Φ_{km}. This system is separated into independent subsystems corresponding to various values of m. It follows from Eq. (A6) that in a Fermi liquid at absolute zero there can be propagated oscillations of various types, distinguished basically by a differing dependence of amplitude upon the angle φ. To the value m = 0 there correspond oscillations for which ν is isotropic in the plane perpendicular to **k**. For $m \neq 0$ the oscillations are polarized in a definite manner in this plane. The number of types of oscillations is determined by the number of possible values of m ($|m| \leq n$). The propagation velocity for the oscillations is determined from the condition that the determinant of the corresponding system be equal to zero:

$$|\delta_{kn} + F_n\Omega_{kn}^m(s)| = 0 \quad (N \geqslant n,\ k \geqslant |m|), \quad \Omega_{kn}^m(s) = \frac{(n-|m|)!}{(n+|m|)!}\int P_k^m(\theta')\frac{\cos\theta'}{\cos\theta' - s}P_n^m(\theta')\frac{do'}{4\pi}. \tag{A7}$$

In view of the fact that $P_n^m = P_n^{-m}$ the coefficients Ω_{kn}^m are independent of the sign of m, so that oscillations differing in the sign of m are propagated with the same velocity.

From (A7) it is evident that the equations for the velocity are transcendental. In the general case they do not always posses real roots. Cases are possible, however, in which there are some real roots. These correspond to certain types of oscillations having identical polarization in the plane perpendicular to **k**.

We shall consider as an example the case for which the function F (ϑ) includes only the zero and first harmonic [Eq. (9)]. Here the coefficients Ω_{kn}^m are

$$\Omega_{00}^0 = \frac{1}{2}\int_{-1}^{1}\frac{x\,dx}{x-s} = 1 - \frac{s}{2}\ln\frac{s+1}{s-1} = -w; \qquad \Omega_{10}^0 = \Omega_{01}^0 = \frac{1}{2}\int_{-1}^{1}\frac{x^2dx}{x-s} = -sw,$$

$$\Omega_{11}^0 = \frac{1}{2}\int_{-1}^{1}\frac{x^3dx}{x-s} = \frac{1}{3} - s^2w, \quad \Omega_{11}^1 = \frac{1}{4}\int_{-1}^{1}\frac{(1-x^2)\,x\,dx}{x-s} = \frac{1}{2}\left[(s^2-1)\,w - \frac{1}{3}\right].$$

For the velocity of propagation of oscillations of the type m = 0 we obtain, after substitution into the determinant (A7), the equation

$$w = \frac{1 + F_1/3}{F_0 + F_0F_1/3 + F_1s^2}. \tag{A8}$$

For the case m = 1 we obtain the equation

$$w = (F_1 - 6)\,/\,3F_1\,(s^2 - 1). \tag{A9}$$

This equation has one real root for $F_1 > 6$.

We shall now conclude with the application of these formulas to the case of liquid He^3. We do not possess any extensive information concerning the function F (ϑ); knowing the effective mass and the compressibility, however, we can determine $\overline{F}$ and $\overline{F\cos\vartheta}$ from Eqs. (7) and (8) of the text. It is therefore reasonable to confine ourselves to the approximation (9) for F. It follows from the temperature dependence of the entropy, in accordance with Ref. 6, that

$$m^* = 1.43m\,(\mathrm{He}^3).$$

Data on the compressibility yield c = 195 m/sec. From this we find $F_0 = 5.2$ and $F_1 = 1.3$. Using these values it follows that to the oscillations of the type m = 0 there corresponds the single velocity (the root of Eq. (A8))

$$s = u/v = 1.84, \quad u = 206 \text{ m/sec.}$$

Oscillations of the type m = 1 are absent (as are all those for m > 1). It is of course possible that this result is due to the crudity of the approximation we have chosen for F (ϑ), but we see no reason to draw such a conclusion.

From Eqs. (A8) and (8) we can obtain for the velocity of sound c the following approximation relation characterizing the width of the region of dispersion:

$$(u - c)/c \approx 2/5\,(F_0 + 1).$$

For the case of He^3 the width of this region is on the order of 10 m/sec for a velocity c ~ 200 m/sec.

In conclusion the authors wish to express their gratitude to L. D. Landau for numerous discussions.

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Translated by S. D. Elliott
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