STATIONARY STATES OF ELECTRON-POSITRON SYSTEMS AND ANNIHILATION TRANSITIONS

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A scheme is set up which makes it possible to take into account the influence of an external field and the mutual interaction of electrons and positrons in the interaction of the electron-positron field with the radiation field. The theory is nonrelativistic with respect to the particle momenta in cases in which the photon energy is comparable with the rest energy of the electron. In cases in which the photon energy is much smaller than the rest energy of the electron, some relativistic corrections can be included in the theory by going to higher approximations. This scheme is applied to the stationary states of an electron-positron system and to the two-photon annihilation of a positron in a many-electron system. The equations of the Fock self-consistent field are obtained for an electron-positron system, and an expression is found for the probability of annihilation of a positron in a many-electron system. The special case of annihilation in helium is examined.

THE various processes of interaction of the electron-positron field with the radiation field are ordinarily treated in the approximation of the single-electron problem. A scheme of calculation in the configuration representation, designed for the treatment of many-body problems, has been set up by Shirokov.[1] This scheme was applied by Tumanov and Shirokov to derive the equation of positronium[2] and by Tumanov to the study of the two-photon annihilation of positronium.[3] The only case treated in these papers, however, is that of no external field.

In order to set up a scheme permitting us to include the effect of the external field and of the mutual interaction of the electrons and positrons, we proceed, as in Ref. 1, according to Fock's method.[4]

In the space of second quantization the complete Hamiltonian of a system of electrons subject to an external field and interacting with the radiation field consists of three parts: the Hamiltonian of the particles in the external field, described by vector and scalar potentials $\mathbf{A}_0$ and φ_0, which is of the form (in a system of units in which $\hbar = c = 1$)

$$H_0 = \int \psi^+(x)\,[(\alpha\,(\mathbf{p} - e\mathbf{A}_0)) + \beta m + e\varphi_0]\,\psi(x)\,dx, \tag{1}$$

the Hamiltonian of the Coulomb interaction of the particles, which is of the form

$$H_1 = \frac{e^2}{2}\iint \psi^+(x)\,\psi(x)\frac{1}{|\mathbf{r}-\mathbf{r}'|}\,\psi^+(x')\,\psi(x')\,dx\,dx', \tag{2}$$

and the Hamiltonian of the interaction of the particles with the radiation field described by the vector potential $\mathbf{A}$, which is of the form

$$H' = -e\int \psi^+(x)\,(\alpha\mathbf{A})\,\psi(x)\,dx. \tag{3}$$

Here x denotes a whole set of space coordinates $\mathbf{r}$ and spinor indices. Integration with respect to x means integration with respect to $\mathbf{r}$ and summation over the spinor indices. We have chosen for the radiation field a gauge such that the scalar potential is zero, so that the Lorentz condition reduces to

$$\operatorname{div}\mathbf{A} = 0. \tag{4}$$

Later on we shall expand $\mathbf{A}$ in terms of plane waves, and the condition (4) assures the transversality of the photons. The contribution of the longitudinal and scalar photons has been explicitly separated in the form of the Coulomb interaction between the particles. The radiation field is subjected to quantization, and the external field remains unquantized.

The quantized operator Ψ is an operator for absorption of an electron or emission of a positron, and its Hermitian adjoint Ψ^+ is an operator for absorption of a positron or emission of an electron. Ψ and Ψ^+ satisfy the commutation relations

$$\{\psi_\alpha^+(\mathbf{r}), \psi_\beta(\mathbf{r}')\} = \delta_{\alpha\beta}\,\delta(\mathbf{r}-\mathbf{r}'), \quad \{\psi_\alpha^+(\mathbf{r}), \psi_\beta^+(\mathbf{r}')\} = \{\psi_\alpha(\mathbf{r}), \psi_\beta(\mathbf{r}')\} = 0, \tag{5}$$

where the spinor indices α and β take the values 1, 2, 3, 4.

The setting up of the scheme in the usual representation of the Dirac theory encounters difficulties. Since the positive-frequency and negative-frequency parts of solutions of the Dirac equation, taken separately, do not form complete systems of functions, the operators of absorption and emission of an electron, and also the operators of absorption and emission of a positron, do not satisfy commutation relations of the usual type (5). Moreover, we have a system in which there occur an infinite number of electrons of negative energy. These difficulties prevent the direct application of the Fock method. To avoid these difficulties, we proceed as is done in Ref. 1, i.e., we carry out the canonical transformations considered by Foldy and Wouthuysen.[5] After this we make the transition from the space of second quantization to configuration space. The purpose of the canonical transformations is to represent the wave functions corresponding to states with positive and negative total energy in the form of spinors in which only the upper elements or only the lower elements, respectively, are non-vanishing. Thus after the separation of the positive-frequency and negative-frequency functions the operators of absorption and emission of an electron have only the upper elements different from zero, and the absorption and emission operators of a positron only the lower elements. Then we can write the transformed quantized operators Ψ' and Ψ'^+ in the form

$$\psi' = \begin{pmatrix} \varphi \\ \chi^+ \end{pmatrix}, \quad \psi'^+ = (\varphi^+ \chi), \tag{6}$$

where the two-component functions φ, φ^+, χ, and χ^+ are respectively absorption and emission operators of electron and positron. Since canonical transformations do not change the commutation relations (5), substituting the expressions (6) into Eq. (5) we get the commutation relations for the operators φ and χ in the form:

$$\{\varphi_\alpha^+(\mathbf{r}), \varphi_\beta(\mathbf{r}')\} = \{\chi_\alpha^+(\mathbf{r}), \chi_\beta(\mathbf{r}')\} = \delta_{\alpha\beta}\,\delta(\mathbf{r}-\mathbf{r}'). \tag{7}$$

All other anticommutators of the operators φ, φ^+, χ, and χ^+ are equal to zero.

In the case of presence of a field we carry out a series of canonical transformations

$$\psi'(x) = \exp\{i(S_1+S_2+\ldots)\}\,\psi(x), \qquad \psi'^+(x) = \psi^+(x)\exp\{-i(S_1+S_2+\ldots)\}, \tag{8}$$

each of which leads to the separation of the wave functions corresponding to states with positive and negative energies to a definite order in $1/m$ ($1/m = 3.862 \times 10^{-11}\ \mathrm{cm}^{-1}$). After the performance of a certain number, say n, of these transformations, and with the neglect of terms in the transformed Hamiltonian density of orders higher than $n-1$, the eigenfunctions of this approximate Hamiltonian density of particles in the external field, corresponding to states with positive or with negative energy, will have only upper elements or only lower elements, respectively. S_n is given by the formula

$$S_n = -\left(\frac{i\beta}{2m}\right) \times \begin{pmatrix}\text{lowest-order odd operator in the} \\ \text{transformed Hamiltonian density}\end{pmatrix}^* \tag{9}$$

As the result of the three successive transformations

$$S_1 = -\frac{i}{2m}\beta(\alpha(\mathbf{p}-e\mathbf{A}_0)), \; S_2 = -\frac{i}{2m}\beta\left(-\frac{ie}{2m}\beta(\alpha\,\mathrm{grad}\,\varphi_0)\right), \; S_3 = -\frac{i}{2m}\beta(\mathbf{p}-e\mathbf{A}_0)^2(\alpha(\mathbf{p}-e\mathbf{A}_0)) \tag{10}$$

we get, expressing Ψ and Ψ^+ in terms of φ, φ^+, χ, and χ^+ and changing the order of χ and χ^+ to eliminate

*In Ref. 5 an odd operator is understood to mean an operator, for example α, which, acting on a four-component function, mixes the upper and lower elements. The problem of separating the positive-frequency and negative-frequency functions reduces mathematically to the problem of transforming the Hamiltonian into a form free from odd operators.

an infinite background energy (using $\chi\chi^+ \to -\chi^+\chi$ and $\int \chi \mathbf{p} \chi^+ dx \to \int \chi^+ \mathbf{p} \chi dx$)

$$H_0 = \int \varphi^+ \Big[m + e\varphi_0 + \frac{1}{2m}(\mathbf{p} - e\mathbf{A}_0)^2 - \frac{e}{2m}(\boldsymbol{\sigma}\,\mathrm{curl}\,\mathbf{A}_0) + \frac{e}{8m^2}\Delta\varphi_0 + \frac{e}{4m^2}(\boldsymbol{\sigma}[\mathrm{grad}\,\varphi_0(\mathbf{p} - e\mathbf{A}_0)]) + \ldots\Big]\varphi\, dx$$
$$+ \int \chi^+ \Big[m - e\varphi_0 + \frac{1}{2m}(\mathbf{p} + e\mathbf{A}_0)^2 - \frac{e}{2m}(\boldsymbol{\sigma}\,\mathrm{curl}\,\mathbf{A}_0) - \frac{e}{8m^2}\Delta\varphi_0 + \frac{e}{4m^2}(\tilde{\boldsymbol{\sigma}}[\mathrm{grad}\,\varphi_0(\mathbf{p}_0 + e\mathbf{A}_0)]) + \ldots\Big]\chi\, dx \quad (11)$$

(here $\tilde{\boldsymbol{\sigma}}$ is the transposed matrix $\boldsymbol{\sigma}$)

$$H_1 = \frac{e^2}{2}\iint \varphi^+(x)\varphi(x)\frac{1}{|\mathbf{r} - \mathbf{r}'|}\varphi^+(x')\varphi(x')\,dx\,dx' + \frac{e^2}{2}\iint \chi^+(x)\chi(x)\frac{1}{|\mathbf{r} - \mathbf{r}'|}\chi^+(x')\chi(x')\,dx\,dx'$$
$$- e^2 \iint \varphi^+(x)\varphi(x)\frac{1}{|\mathbf{r} - \mathbf{r}'|}\chi^+(x')\chi(x')\,dx dx', \quad (12)$$

$$H' = H'^{\mathrm{even}} + H'^{\mathrm{odd}}, \quad (13)$$

where

$$H'^{\mathrm{even}} = \int \varphi^+ \Big[-\frac{e}{m}(\mathbf{A}(\mathbf{p} - e\mathbf{A}_0)) - \frac{e}{2m}(\boldsymbol{\sigma}\,\mathrm{curl}\,\mathbf{A}) + \frac{ie^2}{2m^2}(\boldsymbol{\sigma}[\mathrm{grad}\,\varphi_0\mathbf{A}]) + \ldots\Big]\varphi dx + \int \chi^+\Big[-\frac{e}{m}(\mathbf{A}(\mathbf{p} - e\mathbf{A}_0)) - \frac{e}{2m}(\tilde{\boldsymbol{\sigma}}\,\mathrm{curl}\,\mathbf{A}) - \frac{ie^2}{2m^2}(\tilde{\boldsymbol{\sigma}}[\mathrm{grad}\,\varphi_0\mathbf{A}]) + \ldots\Big]\chi dx$$

and

$$H'^{\mathrm{odd}} = \int \varphi^+ \mathscr{H}'^{\mathrm{odd}}\chi^+ dx + \int \chi \mathscr{H}'^{\mathrm{odd}}\varphi\, dx. \quad (14)$$

In Eq. (14) $\mathscr{H}'^{\mathrm{odd}}$ has the form

$$\mathscr{H}'^{\mathrm{odd}} = -e(\boldsymbol{\sigma}\mathbf{A}) + \frac{e}{2m^2}(\mathbf{A}(\mathbf{p} - e\mathbf{A}_0))(\boldsymbol{\sigma}(\mathbf{p} - e\mathbf{A}_0)) - \frac{ie}{4m}(\boldsymbol{\sigma}\nabla^*)(\mathbf{A}(\mathbf{p} - e\mathbf{A}_0))$$
$$- \frac{ie^2}{4m}(\boldsymbol{\sigma}[\mathrm{curl}\,\mathbf{A}_0\mathbf{A}]) + \frac{e}{8m^2}(\boldsymbol{\sigma}\,\mathrm{curl}\,\mathrm{curl}\,\mathbf{A}) + \frac{e}{4m^2}(\mathrm{curl}\,\mathbf{A}(\mathbf{p} - e\mathbf{A}_0)) + \ldots, \quad (15)$$

where ∇^* means the operator ∇ acting on $\mathbf{A}$ only.

Let us carry out the further canonical transformation, considered in Ref. 1, to get rid of the transposed matrix $\tilde{\boldsymbol{\sigma}}$:

$$\chi' = \varepsilon\chi, \quad \chi'^+ = \chi^+\varepsilon^+, \quad (16)$$

where

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon^+ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \varepsilon\varepsilon^+ = 1.$$

Using the fact that $\epsilon\tilde{\boldsymbol{\sigma}}\epsilon^+ = -\boldsymbol{\sigma}$, and dropping the primes on the transformed χ' and χ'^+, we get

$$H_0 = \int \varphi^+ \mathscr{H}_e \varphi\, dx + \int \chi^+ \mathscr{H}_p \chi\, dx,$$

where

$$\mathscr{H}_e = m + e\varphi_0 + \frac{1}{2m}(\mathbf{p} - e\mathbf{A}_0)^2 - \frac{e}{2m}(\boldsymbol{\sigma}\,\mathrm{curl}\,\mathbf{A}_0) + \frac{e}{8m^2}\Delta\varphi_0 + \frac{e}{4m^2}(\boldsymbol{\sigma}[\mathrm{grad}\,\varphi_0(\mathbf{p} - e\mathbf{A}_0)]) + \ldots, \quad (17)$$

$$\mathscr{H}_p = m - e\varphi_0 + \frac{1}{2m}(\mathbf{p} + e\mathbf{A}_0)^2 + \frac{e}{2m}(\boldsymbol{\sigma}\,\mathrm{curl}\,\mathbf{A}_0) - \frac{e}{8m^2}\Delta\varphi_0 - \frac{e}{4m^2}(\boldsymbol{\sigma}[\mathrm{grad}\,\varphi_0(\mathbf{p} + e\mathbf{A}_0)]) + \ldots, \quad (18)$$

$$H'^{\mathrm{even}} = \int \varphi^+\Big[-\frac{e}{m}(\mathbf{A}(\mathbf{p} - e\mathbf{A}_0)) - \frac{e}{2m}(\boldsymbol{\sigma}\,\mathrm{curl}\,\mathbf{A}) + \frac{ie^2}{2m^2}(\boldsymbol{\sigma}[\mathrm{grad}\,\varphi_0\mathbf{A}]) + \ldots\Big]\varphi\, dx + \int \chi^+\Big[-\frac{e}{m}(\mathbf{A}(\mathbf{p} + e\mathbf{A}_0)) + \frac{e}{2m}(\boldsymbol{\sigma}\,\mathrm{curl}\,\mathbf{A})$$
$$+ \frac{ie^2}{2m^2}(\boldsymbol{\sigma}[\mathrm{grad}\,\varphi_0\mathbf{A}]) + \ldots\Big]\chi\, dx; \quad H'^{\mathrm{odd}} = \int \varphi^+ H'^{\mathrm{odd}}\varepsilon^+\chi^+ dx + \int \chi\varepsilon H'^{\mathrm{odd}}\varphi dx. \quad (19)$$

The Coulomb interaction Hamiltonian H_1 given by Eq. (12) does not change its form under the transformation (16); the commutation relations (7) also remain unchanged.

Thus we have gone over to a new representation in which positrons are regarded as particles with positive rest mass and positive charge and the operators φ, φ^+, χ, and χ^+ satisfy the commutation relations (7). This provides the necessary conditions for the direct application of the Fock method. The interaction Hamiltonian of the particles and the radiation field falls into two parts. The even part leads to transitions without change of the number of electrons and positrons, and the odd part leads to transitions with change of the number of electrons and positrons, i.e., to the annihilation and creation of pairs.

The transformed Hamiltonian of the system can be used for finding stationary states and for the calculation of probabilities of radiative transitions in cases in which the energy of the photon taking part in the process is much less than the rest energy of the electron. In such cases the theory can include the effects of certain relativistic corrections, by consideration of the higher approximations for H_0 and H'^{even} in powers of $1/m$.

In a relativistic process, in particular in annihilation, the energy of the photon is comparable with the rest energy of the electron: $k \sim m$. Since we shall expand $\mathbf{A}$ in plane waves $e^{i(\mathbf{kr})}$, each differential operator acting on $\mathbf{A}$ gives a factor k. Consequently, terms of different orders in $1/m$ can turn out to be of the same order in p/m. For example, in H'^{odd}, Eq. (15), the term $(e/8m^2)(\boldsymbol{\sigma}\text{ curl curl }\mathbf{A})$ is much smaller than the term $-e(\boldsymbol{\sigma}\mathbf{A})$ when $k \ll m$, but they are comparable when $k \sim m$. From this it follows that a theory restricted to an approximation of finite order in $1/m$ is applicable only to nonrelativistic processes, for which the condition $k \ll m$ holds (for example to the emission of photons produced in transitions of atomic electrons).

In order to extend the theory of relativistic processes, it is necessary to: (1) collect the terms belonging to each particular order in p/m after each canonical transformation, (2) find expressions for all the canonical transformations giving the contributions of a given order in $1/m$, and (3) find the expression for the interaction Hamiltonian H' resulting from such transformations. We confine ourselves to the accuracy of the zeroth order in p/m. The result of the transformation given by S_1 is to transform the original interaction Hamiltonian density $\mathcal{H}'^{(0)} = -e\boldsymbol{\alpha}\mathbf{A}$ as follows

$$\mathcal{H}^{(1)} = e^{iS_1}\mathcal{H}'^{(0)}e^{-iS_1} = -e\Big[(\alpha\mathbf{A}) + \frac{1}{2m}\beta\{(\alpha(\mathbf{p}-e\mathbf{A}_0)), (\alpha\mathbf{A})\} - \frac{1}{8m^2}\{(\alpha(\mathbf{p}-e\mathbf{A}_0)), \{(\alpha(\mathbf{p}-e\mathbf{A}_0)), (\alpha\mathbf{A})\}\} - \ldots\Big]. \quad (20)$$

The value of the coefficient of the n-fold anticommutator is given by the expression $\beta^n/n!(2m)^n$. We note the following property of the anticommutator of any operator containing A with the operator $(1/m)(\alpha(\mathbf{p} - e\mathbf{A}_0))$: this anticommutator gives terms that preserve the same order of smallness in p/m as that of the original operator, when in rearranging the operators one lets $\mathbf{p}$ act on $\mathbf{A}$. The other terms are of order of smallness one higher. This enables us to separate out the terms of a prescribed order, since the coefficients of the anticommutators are known.

By means of the formulas

$$\{(\alpha(\mathbf{p}-e\mathbf{A}_0)), (\alpha\mathbf{A})\} = (\sigma\text{curl}\mathbf{A}), \quad \{(\alpha(\mathbf{p}-e\mathbf{A}_0)), (\sigma\text{curl}\mathbf{A})\} = (\alpha\,\text{curl}\,\text{curl}\,\mathbf{A}) \quad (21)$$

valid in zeroth order in p/m, we can get from Eq. (20) the transformed interaction Hamiltonian density:

$$\mathcal{H}'^{(1)} = \mathcal{H}'^{(1)\,\text{odd}} + \mathcal{H}'^{(1)\,\text{even}},$$

where

$$\mathcal{H}'^{(1)\,\text{odd}} = -e\Big((\alpha\mathbf{A}) - \frac{1}{8m^2}(\alpha\text{ curl curl }\mathbf{A}) + \frac{1}{384\,m^4}(\alpha\,\text{curl}^{(4)}\mathbf{A}) \cdot \ldots + (-1)^n\frac{1}{(2n)!\,(2m)^{2n}}(\alpha\text{curl}^{(2n)}\mathbf{A}) + \ldots\Big), \quad (22)$$

$$\mathcal{H}'^{(1)\text{even}} = -\frac{e}{2m}\beta\Big(\sigma, \Big(\text{curl}\mathbf{A} - \frac{1}{24\,m^2}\text{curl}^{(3)}\mathbf{A} + \ldots + (-1)^n\frac{1}{(2n+1)!\,(2m)^{2n+1}}\,\text{curl}^{(2n+1)}\mathbf{A} + \ldots\Big)\Big), \quad (23)$$

where $\text{curl}^{(n)}\mathbf{A}$ means the result of letting the operator curl act n times on the vector $\mathbf{A}$. By means of the Lorentz condition for the radiation field, div $\mathbf{A} = 0$, we get

$$\text{curl}^{(2)}\mathbf{A} = -\Delta\mathbf{A} = k^2\mathbf{A}, \quad (24)$$

and in general $\text{curl}^{(2n)}\mathbf{A} = k^{2n}\mathbf{A}$, since $\mathbf{A}$ is written in the form of a plane wave. Using Eq. (24), we get a simplified form for $\mathcal{H}'^{(1)}$ in the zeroth approximation in p/m.

$$\mathcal{H}'^{(1)} = -e\Big(c_1^{(1)}(\alpha\mathbf{A}) + c_2^{(1)}\frac{1}{2m}\beta(\sigma\text{curl}\mathbf{A})\Big), \quad (25)$$

where

$$c_1^{(1)} = 1 - \frac{k^2}{8m^2} + \ldots + (-1)^n\frac{k^{2n}}{(2n)!\,(2m)^{2n}} + \ldots = \cos\frac{k}{2m}, \; c_2^{(1)} = 1 - \frac{k^2}{24m^2} + \ldots + (-1)^n\frac{k^{2n}}{(2n+1)!\,(2m)^{2n}} + \ldots = \frac{2m}{k}\sin\frac{k}{2m}.$$

Let us see which of the canonical transformations give a contribution in zeroth order. Obviously these will be only those that contain terms with the same number of operators p as the degree of m in the denominator. By calculation one can verify that these are the transformations with the operators S_{2n+1}, $n = 0, \ldots, \infty$. In fact, on one hand these transformations make the coefficients of even powers of the operator $\boldsymbol{\alpha}(\mathbf{p}-e\mathbf{A}_0)$ in $\mathcal{H}_0$ equal to the coefficients of the expansion of the radical $\{m^2+(\mathbf{p}-e\mathbf{A}_0)^2\}^{1/2}$ in powers of $1/m$, and on the other hand they make the coefficients of the odd powers of the operator $\boldsymbol{\alpha}(\mathbf{p}-e\mathbf{A}_0)$ equal to zero. The general expression for the part of S_{2n+1} that gives the zeroth order contribution is given by the formula

$$S'_{2n+1} = -\frac{i}{2m}\beta\cdot\left((-1)^n\frac{1}{(2n+1)m^{2n}}p^{2n}(\alpha\mathbf{p})\right). \tag{26}$$

We note that Eq. (26) can be found in another way. The transformation

$$e^{iS} = [(1+i\varkappa)/(1-i\varkappa)]^{1/4}, \quad \varkappa = -\frac{i}{m}\beta(\alpha(\mathbf{p}-e\mathbf{A}_0)) \tag{27}$$

found by Fock enables us to separate the solutions of the Dirac equation with positive and negative energies:

$$e^{iS}[(\alpha(\mathbf{p}-e\mathbf{A}_0))+\beta m]e^{-iS} = \beta\sqrt{m^2+(\mathbf{p}-e\mathbf{A}_0)^2}.$$

Consequently S at once contains the principal terms of the canonical transformations, those that give the zeroth order contributions. Regarding κ as a small quantity, one can verify that

$$e^{iS} = \exp\left\{\frac{i\varkappa}{2}+\frac{(i\varkappa)^3}{6}+\ldots+\frac{(i\varkappa)^{2n+1}}{2(2n+1)}+\ldots+\right\}.$$

Consequently, we can write

$$e^{iS} = e^{iS_1+iS_3+\ldots+iS_{2n+1}+\ldots},$$

where

$$S_{2n+1} = -\frac{i}{2m}\beta(-1)^n\frac{1}{(2n+1)m^{2n}}(\alpha(\mathbf{p}-e\mathbf{A}_0))^{2n+1}. \tag{28}$$

From Eq. (28) it follows that the part giving the zeroth order contribution is

$$S'_{2n+1} = -\frac{i}{2m}\beta(-1)^n\frac{1}{(2n+1)m^{2n}}p^{2n}(\alpha\mathbf{p}), \tag{29}$$

which agrees exactly with Eq. (26).

Let us go on to find the limiting expression for the interaction Hamiltonian H' that results from the infinite succession of canonical transformations (26), in accuracy to the zeroth order. The original expression $\mathcal{H}'^{(1)} = -e(c_1^{(1)}(\boldsymbol{\alpha}\mathbf{A}) + c_2^{(1)}(1/2m)(\boldsymbol{\sigma}\,\text{curl}\,\mathbf{A}))$ is transformed in a "closed" manner, i.e., in all the transformations with the operators S_{2n+1} ($n = 1, \ldots, \infty$) only the coefficients c_1 and c_2 are changed. This can already be seen from Eq. (26), since the operator $p^{2n} = (-1)^n\Delta^n$ leads only to the appearance of a factor k^{2n}, while commutation or anticommutation of the two parts of $\mathcal{H}'^{(1)}$ with $\boldsymbol{\alpha}\mathbf{p}$ leads only to a change of the coefficients $c_1^{(1)}$ and $c_2^{(2)}$, and so on.

Let us set

$$\Pi^{(2n+1)} = (-1)^n\frac{1}{(2n+1)m^{2n}}p^{2n}(\alpha\mathbf{p}), \qquad O^{(2n+1)} = -ec_1^{(2n+1)}(\alpha\mathbf{A}) \quad (\text{odd part of } \mathcal{H}'^{(2n+1)}), \tag{30}$$

$$\mathcal{E}^{(2n+1)} = -ec_2^{(2n+1)}\frac{1}{2m}\beta(\sigma\,\text{curl}\,\mathbf{A}) \quad (\text{even part of } \mathcal{H}'^{(2n+1)}),$$

so that $\mathcal{H}'^{(2n+1)} = O^{(2n+1)} + \mathcal{E}^{(2n+1)}$.

$\mathcal{H}'$ transforms according to the formula

$$\mathcal{H}'^{(2n+1)} = \mathcal{H}'^{(2n+1)} + \frac{1}{2m}\beta[\Pi^{(2n+1)}, \mathcal{E}^{(2n-1)}] - \frac{1}{8m^2}[\Pi^{(2n+1)}, [\Pi^{(2n+1)}, \mathcal{E}^{2n-1)}]] - \ldots$$
$$\ldots + \frac{1}{2m}\beta\{\Pi^{(2n+1)}, O^{(2n-1)}\} - \frac{1}{8m^2}\{\Pi^{(2n+1)}, \{\Pi^{(2n+1)}, O^{(2n-1)}\}\} - \ldots. \tag{31}$$

By means of the following formulas (valid to the zeroth order in p/m):

2m-fold anticommutator of $(\boldsymbol{\alpha}\mathbf{A})$ and $p^{2n}(\boldsymbol{\alpha}\mathbf{p})$

$$k^{4mn+2m}(\alpha\mathbf{A}),$$

(2m + 1)-fold anticommutator of $(\boldsymbol{\alpha}\mathbf{A})$ and $p^{2n}(\boldsymbol{\alpha}\mathbf{p})$

$$k^{2(2m+1)n+2m}(\sigma\,\mathrm{curl}\,\mathbf{A}), \tag{32}$$

2m-fold commutator of $\beta(\boldsymbol{\sigma}\,\mathrm{curl}\,\mathbf{A})$ and $p^{2n}(\boldsymbol{\alpha}\mathbf{p})$

$$k^{4mn+2m}\beta(\sigma\,\mathrm{curl}\,\mathbf{A}),$$

(2m + 1)-fold commutator of $\beta(\boldsymbol{\sigma}\,\mathrm{curl}\,\mathbf{A})$ and $p^{2n}(\boldsymbol{\alpha}\mathbf{p})$

$$-k^{2(2m+1)n+2(m+1)}\beta(\alpha\mathbf{A}),$$

we get

$$\mathcal{H}'^{(2n+1)} = -e\left(c_1^{(2n+1)}(\alpha\mathbf{A}) + c_2^{(2n+1)}\frac{1}{2m}\beta(\sigma\,\mathrm{curl}\,\mathbf{A})\right), \tag{33}$$

where the successive coefficients are connected by the formulas

$$c_1^{(2n+1)} = c_1^{(2n-1)}d_1^{(2n+1)} + c_2^{(2n-1)}d_2^{(2n+1)}, \quad c_2^{(2n+1)} = c_2^{(2n-1)}d_1^{(2n+1)} + c_1^{(2n-1)}d_3^{(2n+1)}. \tag{34}$$

Here

$$d_1^{(2n+1)} = \sum_{l=0}^{\infty}(-1)^l\frac{1}{(2l)!\,(2)^{2l}}\left(\frac{1}{2n+1}\right)^{2l}\left(\frac{k}{m}\right)^{4ln+2l} = \cos\left((-1)^n\frac{1}{2(2n+1)}\left(\frac{k}{m}\right)^{2n+1}\right),$$

$$d_2^{(2n+1)} = \sum_{l=0}^{\infty}(-1)^l\frac{1}{(2l+1)!\,(2)^{2l}}\left(\frac{1}{2n+1}\right)^{2l+1}\left(\frac{k}{m}\right)^{2(2l+1)n+2l} = -\frac{k}{2m}\sin\left((-1)^n\frac{1}{2(2n+1)}\left(\frac{k}{m}\right)^{2n+1}\right), \tag{35}$$

$$d_3^{(2n+1)} = -\frac{1}{4}\left(\frac{k}{m}\right)^2 d_2^{(2n+1)} = \frac{2m}{k}\sin\left((-1)^n\frac{1}{2(2n+1)}\left(\frac{k}{m}\right)^{2n+1}\right).$$

From this it follows that

$$c_1^{(2n+1)} = \cos\left(\frac{k}{2m}\sum_{l=0}^{2n}(-1)^l\frac{1}{2l+1}\left(\frac{k}{m}\right)^{2l}\right), \quad c_2^{(2n+1)} = \frac{2m}{k}\sin\left(\frac{k}{2m}\sum_{l=0}^{2n}(-1)^l\frac{1}{2l+1}\left(\frac{k}{m}\right)^{2l}\right). \tag{36}$$

We can find the limiting expressions $c_1^{(\infty)}$ and $c_2^{(\infty)}$, using

$$\sum_{l=0}^{\infty}(-1)^l\frac{1}{2l+1}\left(\frac{k}{m}\right)^{2l+1} = \tan^{-1}\frac{k}{m}\ \text{for}\ k\leqslant m, \quad c_1^{(\infty)} = \cos\left(\frac{1}{2}\tan^{-1}\frac{k}{m}\right), \quad c_2^{(\infty)} = \frac{2m}{k}\sin\left(\frac{1}{2}\tan^{-1}\frac{k}{m}\right). \tag{37}$$

For the case $k = m$

$$c_1^{(\infty)} = \cos\frac{\pi}{8}, \quad c_2^{(\infty)} = 2\sin\frac{\pi}{8}, \quad c_1^{(\infty)}c_2^{(\infty)} = \frac{1}{\sqrt{2}}. \tag{38}$$

The transformation (16) brings the interaction Hamiltonian of the particles and the radiation field into the form

$$H' = H'^{\text{even}} + H'^{\text{odd}},$$

where

$$H'^{\text{even}} = -\frac{c_2^{(\infty)}e}{2m}\left[\int\varphi^+(\sigma\,\mathrm{curl}\mathbf{A})\varphi\,dx + \int\chi^+(-\sigma\,\mathrm{curl}\mathbf{A})\chi\,dx\right], \quad H'^{\text{odd}} = -c_1^{(\infty)}e\left[\int\varphi^+(\sigma\mathbf{A}\varepsilon^+\chi^+)\,dx + \int\chi\varepsilon(\sigma\mathbf{A})\varphi\,dx\right], \tag{39}$$

expressed to zeroth order in p/m. Consequently we can apply to some relativistic processes of quantum electrodynamics $(k \sim m)$ the theory with the interaction Hamiltonian of particles and radiation field expressed by Eq. (39). In this case the theory is nonrelativistic as regards the momenta of the particles.

Let us consider a system of interacting electrons and positrons in an external field. For the study of the stationary states of the system we shall not include the interaction of the particles with the radiation field, which, as a perturbation, causes transitions between the stationary states of the system. According to the Fock method[4] the change from the space of second quantization to the configuration space is accomplished by writing the state vector of the system as a column whose elements are wave functions in the configuration space with definite numbers of particles. In all the states of occupation the difference of the numbers of electrons and positrons has a single fixed value. This is a consequence of the law of conservation of charge. The action of the quantum operators on this column must be consistent with the commutation relations.

Let us write the configuration wave function of a system of n electrons and m positrons in the form of a product of determinants of one-electron and one-positron functions, satisfying the requirements of orthogonality and normalization:

$$\psi_{\alpha_1\cdots\alpha_n,\,\beta_1\cdots\beta_m}(\mathbf{r}_1\cdots\mathbf{r}_n,\,\mathbf{s}_1\cdots\mathbf{s}_m)=(n!\,m!)^{-1/2}\|\varphi_i(\mathbf{r}_j,\alpha_j)\|_{i,\,j=1\cdots n}\|\chi_i(\mathbf{s}_j,\beta_j)\|_{i,\,j=1\cdots m} \tag{40}$$

Then the variation principle $\delta W = 0$, where W is the mathematical expectation of the operator $H_0 + H_1$ in a state having n electrons and m positrons, leads, under the conditions of orthogonality and normalization of the one-electron and one-positron functions, to the Fock equations of the self-consistent field for the system of electrons and positrons in the external field, with the Hamiltonian operator expressed in the form of a power series in $1/m$ (Ref. 6).

$$\begin{gathered}\Big[\mathscr{H}_e(x)+e^2\sum_{j=1}^{n}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\varphi_j^*(x')\varphi_j(x')\,dx'-e^2\sum_{j=1}^{m}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\chi_j^*(x')\chi_j(x')\,dx'\Big]\varphi_i(x)\\ -e^2\sum_{j=1}^{n}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\varphi_j^*(x')\varphi_i(x')\,dx'\,\varphi_j(x)=\sum_{j=1}^{n}\lambda_{ij}\varphi_j(x),\qquad i=1\ldots n;\\ \Big[\mathscr{H}_p(x)+e^2\sum_{j=1}^{n}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\chi_j^*(x')\chi_j(x')\,dx'=e^2\sum_{j=1}^{n}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\varphi_j^*(x')\varphi_j(x')\,dx'\Big]\chi_i(x)\\ -e^2\sum_{j=1}^{m}\int\frac{1}{|\mathbf{r}-r'|}\chi_j^*(x')\chi_i(x')\,dx'\,\chi_j(x)=\sum_{j=1}^{m}\mu_{ij}\chi_j(x),\qquad i=1\ldots m.\end{gathered} \tag{41}$$

The state of a system of electrons and positrons subject to an external field and interacting with the radiation field is described by a state vector Ω satisfying the equation

$$(H_0+H_1+H')\,\Omega=id\Omega/dt.$$

According to perturbation theory the transition probability per unit time is given by the formula

$$w=2\pi\,|(\Omega_n,\ H'\Omega_0)|^2\rho_n, \tag{42}$$

where Ω_0 and Ω_n are the state vectors describing the initial and final states respectively. For a second-order effect

$$(\Omega_n,H'\Omega_0)=\sum_I\frac{(\Omega_n,\ H'\Omega_I)(\Omega_I,\ H'\Omega_0)}{E_0-E_I}, \tag{43}$$

where Ω_I is the state vector describing the intermediate state.

The effect of the external field on the radiative processes comes about in two ways. Since the expression for the probability of the process involves the wave function of the system, which is a solution of the equation for the stationary states, in which the potentials of the external field appear, the external field has an effect on the radiative processes through the dependence of the wave functions of the system on the potentials of the external field. The potentials of the external field also appear explicitly in the expression (18) for the interaction Hamiltonian. For radiative processes in which the photon energy is comparable with the rest energy of the electron, the potentials of the external field do not appear in the interaction Hamiltonian as terms of zeroth order in p/m [Eq. (39)]. But the effect of the Coulomb interaction of the particles comes into the radiative processes through the wave function of the system determined by the Fock equations (41).

Let us consider by Fock's method the two-photon annihilation of a positron in a many-electron system. We assume that in the initial state there are n electrons and one positron. In the intermediate state the numbers of electrons and positrons remain the same as in the initial state, while there has been emitted a photon with wave vector $\mathbf{k}_1$ and polarization vector $\mathbf{e}_1$ (denoted as state I_1) or else $\mathbf{k}_2$, $\mathbf{e}_2$ (denoted as state II). In the final state there are two photons $\mathbf{k}_1$, $\mathbf{e}_1$ and $\mathbf{k}_2$, $\mathbf{e}_2$ and $n-1$ electrons. The configuration wave functions of the initial, intermediate, and final electron-positron states are denoted by $\psi_{\alpha_1\ldots\alpha,\beta_1}(\mathbf{r}_1\ldots\mathbf{r}_n,\mathbf{s}_1)$, $\psi''_{\alpha_1\ldots\alpha_n,\beta_1}(\mathbf{r}_1\ldots\mathbf{r}_n,\mathbf{s}_1)$, and $\psi'_{\alpha_1\ldots\alpha_{n-1}}(\mathbf{r}_1\ldots\mathbf{r}_{n-1})$, respectively. Using the rules for the action of the matrix elements of quantized operators on the configuration wave functions, determined in the same way as in Ref. 4, we get[6]

$$(\Omega_{I_1}, H'\Omega_0) = -e\sqrt{\frac{2\pi}{k_1}}\int\psi''^{*}_{\alpha_1\ldots\alpha_n,\beta_1}(\mathbf{r}_1\ldots\mathbf{r}_n,\mathbf{s}_1)\,e^{-i(\mathbf{k}_1\mathbf{r})}\sum_{i=1}^{n}\delta_{\alpha\alpha_i}(\mathbf{r}_1-\mathbf{r}_1)\left(-\frac{ic_2^{(\infty)}}{2m}(\sigma[\mathbf{k}_1\mathbf{e}_1])\right)_{\alpha\beta}$$

$$\times\psi_{\alpha_1\ldots\alpha_{i-1}\beta\alpha_{i+1}\ldots\alpha_n,\beta_1}(\mathbf{r}_1\ldots\mathbf{r}_{i-1}\mathbf{r}\mathbf{r}_{i+1}\ldots\mathbf{r}_n,\mathbf{s}_1)\,d\mathbf{r}_1\ldots\ldots d\mathbf{r}_n\,d\mathbf{r}\,d\mathbf{s}_1 \tag{44}$$

$$+(-e)\sqrt{\frac{2\pi}{k_1}}\int\psi''^{*}_{\alpha_1\ldots\alpha_n,\beta_1}(\mathbf{r}_1\ldots\mathbf{r}_n,\mathbf{s}_1)\,e^{-i(\mathbf{k}_1\mathbf{r})}\delta_{\alpha\beta_1}\delta(\mathbf{r}-\mathbf{s}_1)\left(\frac{ic_2^{(\infty)}}{2m}(\sigma[k_1\mathbf{e}_1])\right)_{\alpha\beta}\psi_{\alpha_1\ldots\alpha_n,\beta}(\mathbf{r}_1\ldots\mathbf{r}_n,\mathbf{r})\,d\mathbf{r}_1\ldots d\mathbf{r}_n\,d\mathbf{s}_1\,d\mathbf{r},$$

$$(\Omega_n, H'\Omega_{I_1}) = (-e)\sqrt{\frac{2\pi}{k_2}}\sqrt{n}\int\psi'^{*}_{\alpha_1\ldots\alpha_{n-1}}(\mathbf{r}_1\ldots\mathbf{r}_{n-1})\,e^{-i(\mathbf{k}_2\mathbf{r})}\delta(\mathbf{r}-\mathbf{r}')\,(c_1^{(\infty)}\varepsilon(\sigma\mathbf{e}_2))_{\alpha\beta}$$

$$\times\psi''_{\alpha_1\ldots\alpha_{n-1}\beta,\alpha}(\mathbf{r}_1\ldots\mathbf{r}_{n-1}\mathbf{r},\mathbf{r}')\,d\mathbf{r}_1\ldots\ldots d\mathbf{r}_{n-1}\,d\mathbf{r}\,d\mathbf{r}'. \tag{45}$$

Here we have used the well known expressions for the matrix elements of the operator A (see, for example, Ref. 7). The first term in Eq. (44) consists in turn of a sum of n terms, in which the i-th term corresponds to the emission of the photon $\mathbf{k}_1$, $\mathbf{e}_1$ by the i-th electron. The second term in Eq. (44) corresponds to the emission of the photon $\mathbf{k}_1$, $\mathbf{e}_1$ by the positron. As intermediate states we must take all possible function $\psi''_{\alpha_1\ldots\alpha_n,\beta_1}(\mathbf{r}_1\ldots\mathbf{r}_n,\mathbf{s}_1)$. We assume that the ψ'' form a complete set:

$$\sum\psi''^{*}_{\alpha_1\ldots\alpha_n\beta_1}(\mathbf{r}_1\ldots\mathbf{r}_n,\mathbf{s}_1)\,\psi''_{\alpha'_1\ldots\alpha'_n\beta_1}(\mathbf{r}'_1\ldots\mathbf{r}'_n,\mathbf{s}'_1)=\delta_{\alpha_1\alpha'_1}\ldots\delta_{\alpha_n\alpha'_n}\ldots\delta_{\beta_1\beta'_1}\,\delta(\mathbf{r}_1-\mathbf{r}'_1)\ldots\delta(\mathbf{r}_n-\mathbf{r}'_n)\,\delta(\mathbf{s}_1-\mathbf{s}'_1), \tag{46}$$

where the sum is taken over all possible states ψ''. We assume that initially all electrons are in bound states, and that the positron is either in a bound state or in a quasifree state with energy close to m (a "slow" positron). The matrix element (44) is appreciably different from zero only when in the intermediate state the particle that has emitted the photon $\mathbf{k}_1$ is in a quasifree state with energy close to $(m^2+k_1^2)^{1/2}$ and each of the other particles has an energy close to m. This is so owing to the fact that in Eq. (44) the factor $e^{-i(\mathbf{kr})}$ is a very rapidly oscillating function. Consequently we take for the difference of the energies of the initial and intermediate states simply:

$$\Delta E_1 = E_0 - E_{I_1} = -\sqrt{m^2+k_1^2}-k_1+m,\quad \Delta E_2 = E_0 - E_{I_2} = -\sqrt{m_1^2+k_2^2}-k_2+m. \tag{47}$$

Using Eq. (46), we get

$$(\Omega_n, H'\Omega_0)=\sum_{I_1}\frac{(\Omega_n, H'\Omega_{I_1})(\Omega_{I_1}, H'\Omega_0)}{\Delta E_1}+\sum_{I_2}\frac{(\Omega_n, H'\Omega_{I_2})(\Omega_{I_2}, H'\Omega_0)}{\Delta E_2}=e^2\frac{2\pi}{\sqrt{k_1k_2}}\frac{\sqrt{n}}{\Delta E_1}\int\psi'^{*}_{\alpha_1\ldots\alpha_{n-1}}(\mathbf{r}_1\ldots\mathbf{r}_{n-1})$$

$$\times\Big[\sum_{i=1}^{n}e^{-i(\mathbf{k}_2\mathbf{r}_n)}e^{-i(\mathbf{k}_1\mathbf{r}_i)}\delta(\mathbf{r}_n-\mathbf{s}_1)(c_1^{(\infty)}\varepsilon(\sigma\mathbf{e}_2))_{\beta_1\alpha_n}\left(-\frac{ic_2^{(\infty)}}{2m}(\sigma[\mathbf{k}_1\mathbf{e}_1])\right)_{\alpha_i\beta}\psi_{\alpha_1\ldots\alpha_{i-1}\beta\alpha_{i+1}\ldots\alpha_n,\beta_1}(\mathbf{r}_1\ldots\mathbf{r}_n,\mathbf{s}_1)$$

$$+e^{-i((\mathbf{k}_1+\mathbf{k}_2)\mathbf{s}_1)}\delta(\mathbf{r}_n-\mathbf{s}_1)(c_1^{(\infty)}\varepsilon(\sigma\mathbf{e}_2))_{\alpha_n\beta_1}\left(\frac{ic_2^{(\infty)}}{2m}(\sigma[\mathbf{k}_1\mathbf{e}_1])\right)_{\beta_1\beta}\psi_{\alpha_1\ldots\alpha_n,\beta}(\mathbf{r}_1\ldots\mathbf{r}_n,\mathbf{s}_1)\Big]\,d\mathbf{r}_1\ldots d\mathbf{r}_n d\mathbf{s}_1$$

$$\left(\begin{array}{c}\text{corresponding term obtained by interchanging}\\ \Delta E_1,\ \mathbf{k}_1,\text{ and }\mathbf{e}_1\text{ with }\Delta E_2,\ \mathbf{k}_2,\text{ and }\mathbf{e}_2,\text{ respectively}\end{array}\right) \tag{48}$$

The sum $\sum_{i=1}^{n}$ in Eq. (48) consists of two parts, namely the term with $i = n$ and the terms with $i \neq n$.

The terms with $i \neq n$ make a negligibly small contribution to the matrix element because of the smallness of the integrals over $\mathbf{r}_i$ and $\mathbf{r}_n$ of the rapidly oscillating functions $e^{-i(\mathbf{k}_2\mathbf{r}_n)}$ and $e^{-i(\mathbf{k}_1\mathbf{r}_i)}$. Physically this corresponds to the fact that in the transition from the initial state to the intermediate state the i-th electron emits the photon $\mathbf{k}_1$ and the momentum $-\mathbf{k}_1$ is transferred to the nucleus, and in the transition from the intermediate to the final state the n-th electron is annihilated with the positron $(n \neq i)$ with the emission of the photon $\mathbf{k}_2$ and the momentum $-\mathbf{k}_2$ is transferred to the nucleus. Such a process, in which there twice occurs transfer of momentum comparable with m, is extremely improbable. As for the terms with $i = n$, they make an important contribution to the matrix element when $|\mathbf{k}_1 + \mathbf{k}_2|$ is small. When $|\mathbf{k}_1 + \mathbf{k}_2|$ is small we can replace $\mathbf{k}_1$ and $\mathbf{k}_2$ by $\mathbf{k}$, which, by the law of conservation of energy, is equal to m, and ΔE_1 and ΔE_2 by $-(2m)^{1/2}$. Neglecting the terms with $i \neq n$ and setting

$$K = -\frac{i}{2\sqrt{2}\,m}\left[\varepsilon(\sigma\mathbf{e}_2)(\sigma[\mathbf{k}_1\mathbf{e}_1]) + \varepsilon(\sigma\mathbf{e}_1)(\sigma[\mathbf{k}_2\mathbf{e}_2])\right],$$

we get

$$(\Omega_n, H'\Omega_0) = -e^2\frac{\sqrt{2}\pi}{km}\sqrt{n}\int \psi^{*}_{\alpha_1\ldots\alpha_{n-1}}(\mathbf{r}_1\ldots r_{n-1})\,e^{-i((\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}_n)}(K_{\beta,\alpha_n} - K_{\alpha_n\beta_1})\cdot\psi_{\alpha_1\ldots\alpha_{n-1}\alpha_n,\beta_1}(\mathbf{r}_1\ldots\mathbf{r}_{n-1}\mathbf{r}_n,\ \mathbf{r}_n)\,d\mathbf{r}_1\ldots d\mathbf{r}_n. \qquad (49)$$

The probability per unit time of two-photon annihilation is given by the formula

$$w = (k^2/4\pi)\,|(\Omega_n, H'\Omega_0)|^2. \qquad (50)$$

The dependence of the probability of the process on $|\mathbf{k}_1 + \mathbf{k}_2|$ is clearly shown in Eq. (49). The process is most probable when $|\mathbf{k}_1 + \mathbf{k}_2| = 0$, i.e., the photons emitted must have their directions correlated to be antiparallel. With increase of the departure of the angle between $\mathbf{k}_1$ and $\mathbf{k}_2$ from the value π (for $k_1 = k_2$) the probability decreases rapidly. The process is most probable when each photon has an energy equal to the rest energy of the electron: $k_1 = k_2 = m$. With increase of the difference between the individual photon energies and m, for $k_1 + k_2 = 2m$, the probability decreases rapidly. From this one can get the natural line width of the annihilation radiation. Since the pair being annihilated is bound to the nucleus and the other electrons, the law of conservation of momentum need not be exactly fulfilled, as it must be for free particles, in the sense that the total momentum of the pair annihilated must be equal to the total momentum of the emitted photons. That is, in the center-of-mass system of the electron and positron the emitted photons can have a total momentum $\mathbf{k}_1 + \mathbf{k}_2$ different from zero, and this amount of momentum is transferred to the nucleus. The larger the momentum transferred, the less probable the process.

Let us write ψ and ψ' in the form

$$\psi_{\alpha_1\ldots\alpha_n,\beta_1}(\mathbf{r}_1\ldots\mathbf{r}_n,\ s_1) = \frac{1}{\sqrt{n!}}\,\|\varphi_i(\mathbf{r}_j,\ \alpha_j)\|_{i,j=1\ldots n}\,\chi_1(r_n,\ \beta_1), \qquad (51)$$

$$\psi'_{\alpha_1\ldots\alpha_{n-1}}(\mathbf{r}_1\ldots\mathbf{r}_{n-1}) = \frac{1}{\sqrt{(n-1)!}}\,\|\varphi'_i(\mathbf{r}_j,\ \alpha_j)\|_{i,j=1\ldots n-1}. \qquad (52)$$

Substituting Eqs. (51) and (52) into Eq. (49) and developing the determinant $\|\varphi_i(r_j, \alpha_j)\|$ in terms of the elements $\varphi_i(r_j, \alpha_j)$, we get

$$(\Omega_n, H'\Omega_0) = -e^2\frac{\sqrt{2}\pi}{km}\sqrt{n}\frac{1}{\sqrt{n!}}\frac{1}{\sqrt{(n-1)!}}\left[K_{\beta_1\alpha_n} - K_{\alpha_n^{\bullet}\beta_1}\right]\sum_{l=1}^{n}(-1)^{l+n}$$

$$\times\int\|\varphi_i'^{*}(\mathbf{r}_j,\ \alpha_j)\|_{i,j=1\ldots n-1}\,\|\varphi_k(\mathbf{r}_j,\ \alpha_j)\|_{\substack{k=1\ldots l-1,\ l+1\ldots n\\ j=1\ldots n-1}}\,dx_1\ldots dx_{n-1}\int e^{-i((\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}_n)}\,\varphi_l(\mathbf{r}_n, \alpha_n)\,\chi_1(r_n, \beta_1)\,d\mathbf{r}_n, \qquad (53)$$

where integration over x_i means integration over $\mathbf{r}_i$ and summation over α_i, and repeated indices are summed over. The sum $\sum_{i=1}^{n}$ in Eq. (53) includes the various acts of annihilation and the effect of electronic correlation. Since φ_i and φ'_j relate respectively to the system atom + positron and to the ion that remains after the annihilation, they are not orthogonal to each other. Consequently, the possibility of transitions of electrons accompanying the annihilation is not excluded.

Let us assume that the remaining electrons have made no transitions during the annihilation, i.e., that

φ_i and φ_i' have the same quantum numbers; then in the sum $\sum_{\ell=1}^{n}$ the term with $\ell = n$ makes the principal contribution to the matrix element. The main part of this term is

$$\sum_{i_1 \ldots i_{n-1}} \int \varphi_{i_1}'^{*}(x_1)\,\varphi_{i_1}(x_1)\,dx_1 \int \varphi_{i_2}'^{*}(x_2)\,\varphi_{i_2}(x_2)\,dx_2 \ldots \int \varphi_{i_{n-1}}'^{*}(x_{n-1})\,\varphi_{i_{n-1}}(x_{n-1})\,dx_{n-1}$$

$$= (n-1)!\prod_{i=1}^{n-1} \int \varphi_i'^{*}(x)\,\varphi_i(x_i)\,dx_i, \tag{54}$$

where $i_1, \ldots, i_{n-1}$ is a permutation of the numbers $1, \ldots, n-1$, and the summation is taken over all permutations $i_1, \ldots, i_{n-1}$. The other part of the term is a correction to the main part (54), for example

$$\int \varphi_1'^{*}(x_1)\,\varphi_2(x_1)\,dx_1 \int \varphi_2'^{*}(x_1)\,\varphi_1(x_2)\,dx_2 \ldots$$

and so on. The other terms in the sum $\sum_{\ell=1}$, with $\ell \neq n$, are corrections to the main part of the term with $\ell = n$ given by Eq. (54), and correspond to annihilations in which the positron is annihilated with the electron that was in the state ℓ $(\ell \neq n)$, but the annihilation is accompanied by the transition of another electron into this state. This is an effect of electronic correlation.

Let us examine the possibility of reducing the problem of the annihilation of a positron in a many-electron system to the corresponding one-electron problem. If we assume that the annihilation of the pair does not change the one-electron functions of the remaining electrons, then ψ' is given by the determinant

$$\| \varphi_i(r_j, \alpha_j) \|_{i,\, j=1 \cdots n-1} / \sqrt{(n-1)!},$$

where φ appears instead of φ'. Then the only nonvanishing contribution in the sum $\sum_{\ell=1}^{n}$ in Eq. (53) is the main part of the term with $\ell = n$, Eq. (54); the effect of electronic correlation disappears, and also no transitions of the remaining electrons are possible. The result is that Eq. (53) reduces to the following:

$$-\, e^2 \frac{\sqrt{2}\,\pi}{km} (K_{\beta_1\alpha_n} - K_{\alpha_n\beta_1}) \int e^{-i((\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}_n)}\,\varphi_n(\mathbf{r}_n, \alpha_n)\,\chi_1(\mathbf{r}_n, \beta_1)\,d\mathbf{r}_n. \tag{55}$$

This is just the result obtained by multiplying the integral

$$\int e^{-i((\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}_n)}\,\varphi_n(\mathbf{r}_n)\,\chi_1(\mathbf{r}_n)\,d\mathbf{r}_n. \tag{56}$$

by the corresponding matrix element for the annihilation of free particles. Consequently, the reduction of the problem of annihilation in a many-electron system to the corresponding one-electron problem is equivalent to the assumption that the wave functions of the remaining electrons are not changed as a result of the annihilation, or the neglect of the possibility of transitions by the remaining electrons and of electronic correlation effects.

As a concrete example let us consider the two-photon annihilation of a positron in helium. The wave function of the system is of the form

$$\psi_{\alpha_1,\,\alpha_2\beta_1}(\mathbf{r}_1\mathbf{r}_2\mathbf{s}_1) = \psi(\mathbf{r}_1\mathbf{r}_1\mathbf{s}_1)\,\Phi(\alpha_1\alpha_2\beta_1). \tag{57}$$

Let us consider separately the cases of orthohelium and parhelium. The Schrödinger function of the system orthohelium + positron has the form

$$\psi_{\text{ortho}}(\mathbf{r}_1\mathbf{r}_2\mathbf{s}_1) = (\varphi_1(\mathbf{r}_1)\,\varphi_2(\mathbf{r}_2) - \varphi_2(\mathbf{r}_1)\,\varphi_1(\mathbf{r}_2))\,\chi_1(\mathbf{s}_1)/\sqrt{2}, \tag{58}$$

and that of the system parhelium + positron has the form

$$\psi_{\text{para}}(\mathbf{r}_1\mathbf{r}_2\mathbf{s}_1) = (\varphi_1(\mathbf{r}_1)\,\varphi_2(\mathbf{r}_2) + \varphi_2(\mathbf{r}_1)\,\varphi_1(\mathbf{r}_2))\,\chi_1(\mathbf{s}_1)/\sqrt{2}. \tag{59}$$

The spin functions $\Phi(\alpha_1\alpha_2\beta_1)$ can be obtained by combining the spin functions of orthohelium and parhelium, $\chi(\alpha_1\alpha_2)$, with the spin functions $S(\beta_1)$ of the positron, and the resulting functions must be eigenfunctions of the total spin of the system and of its z component, and of the total spin of the two electrons and the spin of the positron. As a result we get six spin states for the system orthohelium + positron:

$$\Phi^{3/2}_{3/2} = \chi_{11}S_{1/2},\quad \Phi^{-3/2}_{3/2} = \chi_{1\,-1}S_{-1/2},\quad \Phi^{1/2}_{3/2} = \sqrt{\tfrac{1}{3}}\,\chi_{11}S_{-1/2} + \sqrt{\tfrac{2}{3}}\,\chi_{10}S_{1/2},\quad \Phi^{-1/2}_{3/2} = \sqrt{\tfrac{2}{3}}\,\chi_{10}S_{-1/2} + \sqrt{\tfrac{1}{3}}\,\chi_{1\,-1}S_{1/2},$$

$$\Phi^{1/2}_{1/2} = \sqrt{\tfrac{2}{3}}\,\chi_{11}S_{-1/2} - \sqrt{\tfrac{1}{3}}\,\chi_{10}S_{1/2},\quad \Phi^{-1/2}_{1/2} = \sqrt{\tfrac{1}{3}}\,\chi_{10}S_{-1/2} - \sqrt{\tfrac{2}{3}}\,\chi_{1\,-1}S_{1/2}; \tag{60}$$

and two spin states for the system parhelium + positron:

$$\Phi^{1/2}_{1/2} = \chi_{00}S_{1/2},\quad \Phi^{-1/2}_{1/2} = \chi_{00}S_{-1/2}. \tag{61}$$

The lower index on the Φ denotes the total spin of the system, and the upper index gives its z component; the indices on χ are the total spin of the two electrons and its z component; that on the S is the spin component of the positron.

Let us substitute Eqs. (58) to (61) into Eq. (53), average over all spin states of the initial state, and sum over the polarizations of the emitted photons; then we get the probability per unit time of two-photon annihilation of a slow or bound positron with the remaining electron of arbitrary spin in a state described by the wave function φ':

$$w_{\text{ortho}} = e^4\frac{\pi}{2m^2}\Big[\int\varphi'^*(\mathbf{r}_1)\,\varphi_1(\mathbf{r}_1)\,d\mathbf{r}_1\int e^{-i((\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}_2)}\,\varphi_2(\mathbf{r}_2)\,\chi_1(\mathbf{r}_2)\,d\mathbf{r}_2 - \int\varphi'^*(\mathbf{r}_1)\,\varphi_2(\mathbf{r}_1)\,d\mathbf{r}_1\int e^{-i((\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}_2)}\,\varphi_1(\mathbf{r}_2)\,\chi_1(\mathbf{r}_2)\,d\mathbf{r}_2\Big]^2 \tag{62}$$

for the system orthohelium + positron, and

$$w_{\text{para}} = e^4\frac{\pi}{2m^2}\Big[\int\varphi'^*(\mathbf{r}_1)\,\varphi_1(\mathbf{r}_1)\,d\mathbf{r}_1\int e^{-i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}_2}\,\varphi_2(\mathbf{r}_2)\,\chi_1(\mathbf{r}_2)\,d\mathbf{r}_2 + \int\varphi'^*(\mathbf{r}_1)\,\varphi_2(\mathbf{r}_1)\,d\mathbf{r}_1\int e^{-i((\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}_2)}\,\varphi_1(\mathbf{r}_2)\,\chi_1(\mathbf{r}_2)\,d\mathbf{r}_2\Big]^2 \tag{63}$$

for the system parhelium + positron.

When the positron is in a bound state (bound to the atom), one must sum up the probabilities of annihilation of the positron with the electrons in the various states, summing over the Schrödinger functions and spin functions of the final state, and also over **k**. The reciprocal of the resulting sum gives the lifetime of the positron, independent of the density of the medium. When the positron is in a quasifree state, one must take a wave function normalized to unit volume. The reciprocal of the resulting probability, obtained by summing over all final states, specified by Schrödinger and spin functions and by the wave vector **k**, and multiplying by the number of atoms per unit volume, gives the lifetime of the quasifree positron in the medium.

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