BEHAVIOR OF PARTICLES OF SMALL EFFECTIVE MASS IN SUPERFLUID HELIUM

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The kinetic equation for the interaction of electrons with thermal excitations in He II is solved by the Fokker-Planck method. The electric current density is determined for fields of various strengths.

THE study of the behavior of foreign particles in liquid He II is of considerable interest toward understanding the nature of their interaction with the elementary excitations. The only substance soluble in He II is He^3, the behavior of which in solution has already been investigated both experimentally[1] and theoretically.[2]

A. I. Shal'nikov has called attention to the possibility of studying the behavior in He II of light particles such as electrons, obtained through ionization of the helium by some type of source. If the effective mass m of the electron in solution is of the order of the value for the free electron, the average velocity of the electrons in a state of thermal equilibrium will be substantially greater than that of sound over virtually the entire temperature interval down to $T \sim 10^{-2}$. In such a case the interaction of the electrons with the phonons will take place primarily by a single-phonon mechanism, i.e., the electron can emit and absorb phonons, thereby altering its direction through some angle without appreciably changing its energy (we shall perform the corresponding calculations below). In its interaction with a roton the electron is scattered by the latter, again changing its energy only very slightly in the process.

This circumstance makes it possible to apply the Fokker-Planck method, in the form developed by Davydov,[3] to the solution of the kinetic equation.

It is possible to determine experimentally the mobility of charged particles in electric fields. Estimates show that for various reasonable fields and mean free paths the values to be expected for the electron drift velocity are much greater than the velocity of the normal component. In such a case the excitation gas may be regarded as at rest and in thermal equilibrium.

We shall seek a distribution function for the electrons in the form of a series in Legendre polynomials, limiting ourselves to the zero and first terms of the expansion (the convergence of the series is demonstrated in Ref. 3):

$$f = f_0 + f_1 \cos\vartheta \ldots +$$

Here ϑ is the angle made by the electron momentum with the direction of the electric field.

As Davydov has shown, the equations for f_0 and f_1 can be represented in the form

$$\frac{1}{3p^2}\frac{\partial}{\partial p}(p^2 eEf_1) = \frac{1}{p^2}\frac{\partial}{\partial p}p^2\left(A_2\frac{\partial f_0}{\partial p} + A_1 f_0\right); \quad (1)$$

$$eE\,\partial f_0/\partial p = -Bf_1; \quad (2)$$

$$B = \int (1-\cos\psi)\,dw, \quad A_2 = \frac{1}{2}\int (\delta p)^2\,dw, \quad (3)$$

dw is the probability for scattering through an angle ψ, and δp is the corresponding change in the absolute value of the momentum.

The collision integral in Eq. (1) has, as it should, the form of a divergence in the space of absolute values of the momenta of a certain current

$$s = A_2\,\partial f_0/\partial p + A_1 f_0.$$

The coefficient A_1 may be found from the requirement that s fall identically to zero for the Maxwell distribution $f_0 \sim \exp(-p^2/2mkT)$, and turns out to be $A_1 = A_2 p/mkT$.

From (1) and (2) we obtain the equation for f_0:

$$\frac{1}{p^2}\frac{\partial}{\partial p} p^2\left[\left(\frac{e^2E^2}{3B} + A_2\right)\frac{\partial f_0}{\partial p} + A_2 \frac{p}{mkT} f_0\right] = 0.$$

The first integral corresponds to constant current s. The constant of integration is to be set equal to zero; in the alternative case the current goes to ∞ at $p = 0$.

From the equation obtained for f_0 the following quadrature is found:

$$f_0 = C \exp\left\{-\int_0^\infty \frac{A_2 p\,dp}{mkT\,(A_2 + e^2E^2/3B)}\right\}. \tag{4}$$

The constant here is determined from the normalization condition. The function f_1 is readily expressed in terms of f_0 with the aid of (2) and (4):

$$f_1 = \frac{eE}{B}\,\frac{A_2 p}{mkT\,(e^2E^2/3B + A_2)} f_0.$$

Knowing f, it is simple to obtain the electric current density

$$j = \int e\frac{p\cos\vartheta}{m} f p^2 dp\, do = \frac{4\pi}{3}\frac{e^2E}{m^2kT}\int_0^\infty \frac{f_0 A_2 p^4\,dp}{B\,(A_2 + e^2E^2/3B)}, \tag{5}$$

from which it is possible to determine the electrical conductivity and the mobility of the particles $u = j/ne^2E$.

Equations (4) and (5) completely solve the problem, provided concrete expressions are known for A_2 and B; we now turn to the computation of these.

The probability for phonon emission is found from the usual perturbation theory formulas, where the interaction energy is to be set equal to

$$V = (\partial\Delta_e/\partial\rho)\tilde{\rho};$$

$\tilde{\rho}$ is the variation in the density due to the presence of the phonon, and $\partial\Delta_e/\partial\rho$ corresponds to a dependence $\epsilon = \Delta_e + p^2/2m$ of the impurity (electron) energy upon the density.

Calculations similar to those performed by Zharkov and Khalatnikov[2] lead to the following formula for the probability of phonon emission:

$$dw_{\mathrm{e\,ph}} = \frac{s}{\lambda}\frac{n+1}{8\pi m^3 s^2}\,\delta(\mathbf{p} - \mathbf{p}' - \mathbf{q})\,\delta(\varepsilon - \varepsilon' - sq)\,d\mathbf{p}'\cdot q\,dq, \qquad n = (e^{-sq/kT} - 1)^{-1},$$

n is the Planck distribution function for the phonons, which we take to be in thermal equilibrium, s is the velocity of sound, and the notation $\lambda = \pi\rho h^4/m^3(\rho\partial\Delta_e/\partial\rho)^2$ is introduced for the sake of brevity.

We integrate the probability expression over $d\mathbf{q}$ and over the absolute value of the momentum p', obtaining as a result

$$dw_{\mathrm{e\,ph}} = \frac{s}{\lambda}\frac{n+1}{2m^2s^2} q p' \frac{do}{4\pi} \Big/ \left|\frac{\partial}{\partial\varepsilon'}(\varepsilon - \varepsilon' - sq)\right|. \tag{6}$$

The probability for absorption of a phonon is obtained from (6) by replacing n + 1 by n:

$$dw_{\mathrm{a\,ph}} = \frac{s}{\lambda}\frac{n}{2m^2s^2} q p' \frac{do}{4\pi} \Big/ \left|\frac{\partial}{\partial\varepsilon}(\varepsilon - \varepsilon' + sq)\right|. \tag{7}$$

From the energy and momentum conservation laws for the phonon emission process (these results are sufficiently accurate for the case of phonon absorption as well):

$$p^2/2m = p'^2/2m + sq, \quad \mathbf{p} - \mathbf{p}' = \mathbf{q}$$

we find

$$q^2 \simeq 2p^2(1 - \cos\psi), \quad (\delta p)^2 \equiv (p' - p)^2 \simeq 2m^2s^2(1 - \cos\psi), \tag{8}$$

where ψ is the angle between **p** and **p′**.

Evidently $\delta p/p \sim ms/p \sim (ms^2/kT)^{1/2}$ which justifies the application of the Fokker-Planck method. We can therefore in Eqs. (6) and (7) set $p' \simeq p$ and the quantity

$$\left|\frac{\partial}{\partial\varepsilon'}(\varepsilon-\varepsilon'\pm sq)\right|\approx 1, \quad \text{since} \quad sq\sim sp\ll\frac{p^2}{2m}.$$

No difficulty is now encountered in determining $A_2^{ph} = \frac{1}{2}\int(\delta p)^2(dw_{e\,ph}+dw_{a\,ph})$ with the aid of (8):

$$A_2^{ph}=m^2s^2\int(1-\cos\psi)\frac{s}{\lambda}\frac{2n+1}{2m^2s^2}qp\frac{do}{4\pi}=\frac{s}{8\lambda p^3}\left(\frac{kT}{s}\right)^5\int_0^{2sp/kT}x^4\coth\frac{x}{2}dx, \quad x\equiv sq/kT. \tag{9}$$

In accordance with (8)

$$B^{ph}=A_2^{ph}/m^2s^2. \tag{10}$$

In considering the scattering of an electron by a roton it is reasonable to take their energy of interaction to be a δ-function of the distance between them. This leads, obviously, to isotropic scattering, the cross-section for which is independent of the electron and roton velocities; there is, moreover, no necessity for introducing a relative velocity for the electron and roton, since the velocity of the latter is always very small. The probability for scattering of the electron through some arbitrary angle can thus be represented in the form

$$dw=(v/l)\,do/4\pi, \quad 1/l=\sigma N_r, \tag{11}$$

σ is some unknown cross-section, and N_r is the number of rotons per unit volume.

From this we obtain directly

$$B^r=\int(1-\cos\psi)(v/l)\,do/4\pi=v/l.$$

To compute A_2^r it is necessary to find the value of $\langle(\delta p)^2\rangle_r$, averaged over the roton distribution. The energy conservation law yields (with P the roton momentum):

$$-p\delta p/m=[P'^2-P^2-2P_0(P'-P)]/2\mu. \tag{12}$$

P′ can be found from the momentum conservation law

$$\mathbf{P}'=\mathbf{P}-\Delta\mathbf{p}; \quad \Delta\mathbf{p}\equiv\mathbf{p}-\mathbf{p}'; \quad \Delta p^2\approx 2p^2(1-\cos\psi);$$

setting $\Delta p \ll P \sim P_0$, we obtain

$$P'=\sqrt{\mathbf{P}^2-2\mathbf{P}\Delta\mathbf{p}+\Delta\mathbf{p}^2}=P-\mathbf{P}\Delta\mathbf{p}/P_0+\Delta\mathbf{p}^2/2P_0-(\mathbf{P}\Delta\mathbf{p})^2/2P_0^3.$$

Setting this value in (12) and squaring, we find

$$\frac{(\delta p)^2}{2}=\frac{1}{2p^2}\left(\frac{m}{\mu}\right)^2\left[\frac{P-P_0}{P_0}\left(\frac{\Delta\mathbf{p}^2}{2}-\mathbf{P}\Delta\mathbf{p}\right)+\frac{(\mathbf{P}\Delta\mathbf{p})^2}{2P_0^2}\right]^2.$$

We must now average this expression over the roton distribution. The average values of $P-P_0$ and $\mathbf{P}\Delta\mathbf{p}$ fall to zero in this process. Indicating by χ the angle between **P** and **Δp**, we obtain:

$$\frac{1}{2}\langle(\delta p)^2\rangle_r=\frac{1}{2p^2}\left(\frac{m}{\mu}\right)^2\left\{p^4(1-\cos\psi)^2\left[\frac{\langle(P-P_0)^2\rangle_p}{P_0^2}-\langle\cos^4\chi\rangle_r\right]+2p^2(1-\cos\psi)\langle(p-p_0)^2\rangle_r\langle\cos^2\chi\rangle_r\right\}.$$

It is easy to obtain

$$\langle(P-P_0)^2\rangle_r=\mu kT; \quad \langle\cos^2\chi\rangle_r=1/3; \quad \langle\cos^4\chi\rangle_r=1/5.$$

Integrating over the angle ψ we obtain finally* (in view of the inequality $\mu kT \ll P_0^2$):

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*A similar computation for the collision of electrons with the ions of a plasma yields

$$A_2^i=\frac{v}{l}\left(\frac{m}{\mu}\right)^2\left(MkT+\frac{2p^2}{3}\right).$$

The first term was found by Davydov, and is the dominant one; the second becomes important only for extremely strong electric fields.

$$A_2^{\mathrm{r}} = \frac{1}{2}\int \langle(\delta p)^2\rangle_{\mathrm{r}} \frac{v}{l}\frac{do}{4\pi} = \frac{v}{l}\left(\frac{m}{\mu}\right)^2\left(\frac{\mu kT}{3} + \frac{2p^2}{15}\right). \tag{13}$$

Substituting in (4) and (5) the values

$$A_2 = A_2^{\mathrm{ph}} + A_2^{\mathrm{r}}, \quad B = B^{\mathrm{ph}} + B^{\mathrm{r}},$$

we obtain the solution of the problem under consideration.

For the sake of brevity, we shall discuss here only a few simple limiting formulas, for pure phonon and pure roton gases.

For the phonon gas in extremely weak fields, for which $eE\lambda \ll (kT)^{3/2}/(ms^2)^{1/2}$, it is possible to take $e^2E^2 \ll A_2B$. The usual Maxwellian distribution is then obtained for f_0. Computing B^{ph} from Eqs. (9) and (10), we can take $sp/kT \ll 1$, and obtain the value

$$B^{\mathrm{ph}} = pkT / \lambda m^2 s^2, \tag{14}$$

with the aid of which we obtain from Eq. (5) the current density

$$j = {}^2/_3 \sqrt{2/\pi}\, ne^2 s^2 m^{1/2} \lambda E\, (kT)^{-3/2}, \tag{15}$$

in this case depending linearly upon the electric field intensity.

In the region for which the electric field intensity obeys the inequalities

$$(kT)^{3/2} / (ms^2)^{1/2} \ll eE\lambda \ll (kT)^{5/2} / (ms^2)^{3/2},$$

we have

$$A_2 B \ll e^2E^2,$$

and Eq. (14) for B^{ph} is still applicable. In this case we obtain for the distribution function the value

$$f_0 \sim \exp\{-3p^4 kT / 4m^3 s^2 (eE\lambda)^2\},$$

and the current density turns out to be proportional to $E^{1/2}$:

$$j = \left(\sqrt{2\pi} / 3^{3/4}\,\Gamma({}^3/_4)\right) ne^{3/2} s^{3/2} m^{1/4} \lambda^{1/2} E^{1/2} / (kT)^{3/4}. \tag{16}$$

If $eE\lambda \gg (kT)^{5/2}/(ms^2)^{3/2}$, then* $sp/kT \gg 1$ and we obtain

$$B^{\mathrm{ph}} = 4p^2 / 5\, \lambda m^2 s.$$

The expression for f_0 for this case includes in the exponent the sixth power of the momentum

$$f_0 \sim \exp\{-8p^6 / 25\, m^3 kT\, (eE\lambda)^2\},$$

and the current density is proportional to $E^{1/3}$:

$$j = \left(5^{1/3}\,\Gamma({}^1/_6) / 6\sqrt{\pi}\right) ne^{4/3} s \lambda^{1/3} E^{1/3} / (kT)^{1/3}. \tag{17}$$

For the case of a pure roton gas we shall also discuss three ranges of magnitudes for the electric field. For fields fulfilling the condition $eE\lambda \ll (m/\mu)^{1/2}kT$ we have $e^2E^2 \ll BA_2$, and the Maxwellian distribution function is again obtained for f_0. Calculation of the current density yields a linear dependence upon field intensity

$$j = {}^2/_3 \sqrt{2/\pi}\, ne^2 E / \sigma N_{\mathrm{r}} (mkT)^{1/2}. \tag{18}$$

In fields for which the inequality

$$(m/\mu)^{1/2} kT \ll eEl \ll (\mu/m)^{1/2} kT,$$

is fulfilled we have $e^2E^2 \gg BA_2^{\mathrm{r}}$, and in A_2^{r} it is sufficient to stop with the first term $A_2^{\mathrm{r}} = mpkT/3\mu l$. Using this value we readily obtain the distribution function

$$f_0 \sim \exp\{-p^4 / 4m\mu\, (eEl)^2\}$$

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*Taking the average value of p over the distribution.

and the electric current density

$$j = \left(\sqrt{2\pi}/3\,\Gamma(^3/_4)\right) ne^{3/2}E^{1/2}/\sigma^{1/2}N_r^{1/2}(m\mu)^{1/4}. \qquad (19)$$

Finally, for $eE\ell \gg (\mu/m)^{1/2}kT$ the second term in the expression for A_2^r predominates, and $A_2^r = 2p^3m/15\ell\mu^2$.

In this case* we obtain for the distribution function the value

$$f_0 \sim \exp\{-p^6/15\,m\mu^2kT\,(eEl)^2\}$$

and for the current density

$$j = \frac{2\Gamma(^1/_3)}{3^{7/6}5^{1/6}\sqrt{\pi}}\,\frac{ne^{5/3}E^{2/3}}{(N_r\sigma)^{2/3}(m\mu^2kT)^{1/6}}\,. \qquad (20)$$

These formulas contain the unknown parameters λ, σ, the effective mass m, and also the number of electrons per unit volume n. If it were possible to perform measurements in all three ranges of electric field dependence, all of these unknown parameters could be determined. It should be remembered, however, that the magnitudes of the parameters can cause overlapping of the phonon and roton regions and thereby complicate somewhat the interpretation of the experimental results. For example, the condition $(\mu/m)^{1/2}kT \ll eE\ell$ corresponds to $eE\ell \gg \Delta$, and the creation of several rotons, as well as phonons, can accompany the scattering of electrons by a roton. In this case the dependence (17) will be observed in place of (20). Experimental investigation of the behavior of particles of small effective mass in He II would be of definite interest.

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STATIONARY CONVECTION IN A PLANE LIQUID LAYER NEAR THE CRITICAL HEAT TRANSFER POINT

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Thermal convection which arises above the limit of stability is investigated for a plane liquid layer. A dependence of the amplitude of motion on a parameter which characterizes the departure from critical conditions has been obtained. Possible symmetry of the flow is discussed.

*An analogous situation can in principle arise for the scattering of electrons by heavy atoms. Here an $\sim E^{2/3}$ dependence for the current is also obtained, but with different numerical coefficients; this is understandable, since A_2 is different for ions. In this case, however, the electrons have sufficient energy for ionization, which must be taken into account.