A QUANTUM MECHANICAL TRANSPORT EQUATION FOR ELECTRONS IN METALS

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Submitted to JETP editor February 12, 1957

J. Exptl. Theoret. Phys. (U.S.S.R.) 33, 451-458 (August, 1957)

We derive a quantum mechanical transport equation for electrons in a metal in the field of an electromagnetic wave. In contrast to the usual treatment, our equation is not based on the assumption that the photon energy is small compared to kT.

DURING the last decade there has been developed a theory of the skin-effect in metals* based on the following transport equation for the electron distribution function $f(\mathbf{p}, \mathbf{r}, t)$

$$\partial f/\partial t + (\mathbf{p}/m)\,\partial f/\partial \mathbf{r} + e\,\mathbf{E}\,\partial f/\partial \mathbf{p} = I(f), \tag{1}$$

where $I(f)$ is the collision integral. In this equation the electrical field $\mathbf{E}$ plays the role of a classical force. However, if the frequency ω of the electromagnetic field is sufficiently high, one must take quantum effects into account, even if the threshold for the internal photoeffect, i.e., for a transition of an electron into another band, is not yet reached. From the quantum mechanical point of view the interaction of an electromagnetic wave with the electrons takes place through the absorption and emission of photons (with a simultaneous absorption and emission of phonons in the lattice). If in these processes the change in the electron energy $\hbar\omega$ is small compared to the breadth of the tail of the Fermi distribution kT, one can consider the gain of energy by the electron to be practically continuous.

As long as the condition

$$\hbar\omega \ll kT \tag{2}$$

is satisfied, one can thus use classical methods and use the transport equation (1). We notice that condition (2) is extremely far reaching. At room temperatures ($T \cong 300°$) it involves $\lambda \gg 40\mu$ and for $T = 10°$ we have $\lambda \gg 0.2$ cm, which means that quantum effects can be of importance even in the radio range.

If $\hbar\omega \gtrsim kT$, it is necessary to treat the problem quantum mechanically. This is, in particular, true for the infrared region of the spectrum, where usually inequality (2) holds in reverse. Although this fact has been known for a long time (see, for instance, Sec. 53 of Ref. 2) up to the present nobody has carried out anything like the complete investigation of this case. The only attention paid to this problem was in a paper in 1954.[3] Holstein has calculated the volume absorption in a metal using perturbation theory. In this case he assumes that the electron distribution and the field do not depend on the position coordinates. Holstein's results are therefore, generally speaking, only applicable in the case of the normal skin-effect. On the other hand, if we exclude the region of very high temperatures, the skin-effect in the infrared region of the spectrum is usually anomalous. It is thus necessary to construct a method which enables us to consider simultaneously quantum effects and the anomalous character of the skin-effect. The present paper is devoted to obtaining such an equation.

For the time being, to fix our ideas, we shall consider a system of N electrons in a macroscopic volume V.

Such a system can in quantum mechanics conveniently be described through a density operator $\hat{F}$ which satisfies the equation[4]

$$\partial \hat{F}/\partial t = [\hat{H}, \hat{F}] \equiv (1/ih)(\hat{H}\hat{F} - \hat{F}\hat{H}) \tag{3}$$

and the normalization condition $\mathrm{Tr}\ \hat{F} = 1$; in Eq. (3) $\hat{H}$ is the Hamiltonian of the system.

The operator $\hat{F}$ gives a complete quantum mechanical description of the system since the average value of any dynamical variable R can be calculated from the formula

$$R_{\mathrm{Av}} = \mathrm{Sp}\,(\hat{R}\hat{F}). \tag{4}$$

*See, for instance, the survey article by Ginzburg and Motulevich.[1]

In quantum mechanics one normally uses the density matrix $\rho(x, x')$ which is the x-representation of the operator $\hat{F}$.

We shall use the so-called mixed representation of the density matrix which is determined in the following way[5]

$$f(p, x, t) = (2\pi\hbar)^{-3N} \int d\tau \rho\left(x - \frac{\tau}{2}, x + \frac{\tau}{2}\right) e^{-i\tau p/\hbar}.$$

The "quantum distribution function" which is introduced in this way does not have the meaning of a probability density for the state with coordinates p and x, as can be seen, in particular, from the fact that the function f(p, x, t), as follows from its definition, though essentially positive is not always so without fail. However, if we use it, we obtain the most complete analogy with the classical case. In particular, Eq. (4) for the averages now becomes

$$\langle \hat{R}(\hat{p}, \hat{x}) \rangle_{Av} = \int dp\, dx\, R(p, x) f(p, x, t).$$

For the sake of simplicity we shall consider one-particle distribution functions. If an external electromagnetic field is present this function can be determined as follows

$$f(\mathbf{P}, \mathbf{r}, t) = (2\pi\hbar)^{-3} \int d\tau \rho\left(\mathbf{r} - \frac{\tau}{2}, \mathbf{r} + \frac{\tau}{2}\right) \exp\left\{-\frac{i}{\hbar}\tau\left(\mathbf{P} + \frac{e}{c}\mathbf{A}\right)\right\},$$

where $\mathbf{P}$ is the generalized momentum and $\mathbf{A}$ the vector potential of the field. For physical reasons it is convenient, however, to deal with distribution functions of the ordinary momenta. We change at the same time the normalization of the distribution function in such a way that it gives the average relative occupation number with respect to the momenta. We make thus the following substitution

$$N(2\pi\hbar)^3 f(\mathbf{P}, \mathbf{r}, t) = f'(\mathbf{p}, \mathbf{r}, t); \quad \mathbf{P} = \mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{r}, t). \tag{5}$$

Since the time is involved in the transformation we have

$$N(2\pi\hbar)^3 \frac{\partial}{\partial t} f(\mathbf{P}, \mathbf{r}, t) = \frac{\partial}{\partial t} f'(\mathbf{p}, \mathbf{r}, t) + \frac{e}{c}\mathbf{A}(\mathbf{r}, t) \frac{\partial}{\partial \mathbf{p}} f'(\mathbf{p}, \mathbf{r}, t).$$

The primes will be omitted in the following.

It can easily be shown that the electrical current density can now be written in the usual way

$$\mathbf{j}(\mathbf{r}, t) = \frac{2e}{(2\pi\hbar)^3} \int \frac{\mathbf{p}}{m} f(\mathbf{p}, \mathbf{r}, t)\, d\mathbf{p}.$$

Finally one obtains easily the relation

$$F_{\mathbf{p}, \mathbf{p}'} = \frac{1}{N(2\pi\hbar)^3} \int d\mathbf{r} f\left(\frac{\mathbf{p} + \mathbf{p}'}{2}, \mathbf{r}, t\right) e^{i\mathbf{r}(\mathbf{p} - \mathbf{p}')/\hbar}. \tag{6}$$

We consider a system consisting of electrons, photons and lattice phonons. We shall use methods obtained from the transport equations developed by Bogoliubov and Gurov.[6,7]

The conclusions below are based on perturbation theory; this means that we assume the interaction of the electrons with the lattice vibrations and the electromagnetic field to be small. In metal optics the occupation numbers of photons are always large compared to unity. The electromagnetic field can thus be considered to be an external, classical field, satisfying Maxwell's equations. Under our assumptions, the density operator of the system $\hat{F}$ will depend on the coordinates of N electrons $(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$, on the momenta of S phonons $(\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_S)$, and on the time t.

We write the Hamiltonian of the system in the form

$$\hat{H}(1, \ldots N; \mathbf{q}_1, \ldots, \mathbf{q}_S; t) = \sum_{k=1}^{N} \hat{H}(k) + \sum_{k=1}^{S} H(\mathbf{q}_k) + \sum_{i=1}^{N} \sum_{k=1}^{S} \hat{U}(i, k);$$

$$\hat{H}(k) = \frac{1}{2m}\left(-i\hbar \frac{\partial}{\partial \mathbf{r}_k} + \frac{e}{c}\mathbf{A}(\mathbf{r}_k, t)\right)^2, \quad \hat{H}(\mathbf{q}) = \frac{1}{2}\hbar\nu_q (\hat{b}_{\mathbf{q}}^{+}\hat{b}_{\mathbf{q}} + \hat{b}_{\mathbf{q}}\hat{b}_{\mathbf{q}}^{+}),$$

where $\hat{b}_{\mathbf{q}}^{+}$ and $\hat{b}_{\mathbf{q}}$ are the creation and annihilation operators of phonons of momentum $\mathbf{q}$,

$$(N_{\mathbf{q}} + 1 \mid \hat{b}_{\mathbf{q}}^{+} \mid N_{\mathbf{q}}) = \sqrt{N_{\mathbf{q}} + 1}, \quad (N_{\mathbf{q}} - 1 \mid \hat{b}_{\mathbf{q}} \mid N_{\mathbf{q}}) = \sqrt{N_{\mathbf{q}}}.$$

Finally, the operator of the interaction between electrons and phonons is of the form,

$$\hat{U}(1, \mathbf{q}) = C(\Delta / \mathbf{q} / / V 2Mu)^{1/2} (\hat{b}_{\mathbf{q}}^{+} e^{-i\mathbf{qr}/\hbar} + \hat{b}_{\mathbf{q}} e^{i\mathbf{qr}/\hbar}),$$

where C is a constant of the order of magnitude of the average electron energy in the metal,[8] M the mass of an atom in the lattice, u the velocity of sound, and Δ the volume of an elementary cell in the crystal.

We introduce now operators that depend only on a fraction of all the particles, defined as follows

$$\hat{F}^{\nu,\sigma}(1 \ldots, \nu; \mathbf{q}_1, \ldots \mathbf{q}_\sigma; t) = \operatorname*{Sp}_{\substack{(\nu+1, \ldots N) \\ (\sigma+1, \ldots, S)}} \hat{F}^{N,S}(1, \ldots N; \mathbf{q}_1, \ldots \mathbf{q}_S; t).$$

Applying this operation to both sides of equation (3) we find

$$\frac{\partial}{\partial t}\hat{F}^{\nu\sigma} = \left[\sum_{k=1}^{\nu} \hat{H}(k) + \sum_{k=1}^{\sigma} \hat{H}(\mathbf{q}_k), \hat{F}^{\nu\sigma}\right] + \sum_{i=1}^{\nu}\sum_{k=1}^{\sigma} [\hat{U}(i, \mathbf{q}_k), \hat{F}^{\nu\sigma}]$$
$$+ (N-\nu)\sum_{k=1}^{\sigma} \operatorname*{Sp}_{(\nu+1)} [\hat{U}(\nu+1, \mathbf{q}_k)\hat{F}^{\nu+1,\sigma}] + \sum_{i=1}^{\nu}\sum_{k=\sigma+1}^{S} \operatorname*{Sp}_{(k)} [\hat{U}(i, \mathbf{q}_k)\hat{F}^{\nu,\sigma+1}].$$

Hence it follows in particular ($N \gg 1$)

$$\frac{\partial \hat{F}(1)}{\partial t} = [\hat{H}(1), \hat{F}(1)] + \sum_{k=1}^{S} \operatorname*{Sp}_{(k)} [\hat{U}(1, \mathbf{q}_k), \hat{F}(1, \mathbf{q}_k)], \quad \frac{\partial \hat{F}(\mathbf{q})}{\partial t} = [\hat{H}(\mathbf{q}), \hat{F}(\mathbf{q})] + N \operatorname*{Sp}_{(1)} [\hat{U}(1, \mathbf{q}), \hat{F}(1, \mathbf{q})],$$

$$\frac{\partial \hat{F}(1, \mathbf{q})}{\partial t} = [\hat{H}(1) + \hat{H}(\mathbf{q}) + \hat{U}(1, \mathbf{q}), \hat{F}(1, \mathbf{q})] + N \operatorname*{Sp}_{(2)} [\hat{U}(2, \mathbf{q})\hat{F}(1, 2; \mathbf{q})] + \sum_{k=2}^{S} \operatorname*{Sp}_{(k)} [\hat{U}(1, \mathbf{q}_k), \hat{F}(1; \mathbf{q}, \mathbf{q}_k)]. \tag{7}$$

If there is no interaction operator, the dependence on the coordinates of several particles is obviously of the form of a product of one-particle operators; in particular,

$$\hat{F}(1, \mathbf{q}) = \hat{F}(1)\hat{F}(\mathbf{q}), \quad \hat{F}(1, 2) = (1 - \hat{P}_{12})\hat{F}(1)\hat{F}(2),$$
$$\hat{F}(1, 2; \mathbf{q}) = \hat{F}(\mathbf{q})(1 - \hat{P}_{12})\hat{F}(1)\hat{F}(2), \quad \hat{F}(1; \mathbf{q}, \mathbf{q}') = \hat{F}(1)\hat{F}(\mathbf{q})\hat{F}(\mathbf{q}'),$$

where $\hat{P}_{12}$ is the "permutation" operator.* If the interaction is not zero we introduce instead of $\hat{F}(1, \mathbf{q})$ the correlation operator

$$\hat{G}(1, \mathbf{q}) = \hat{F}(1, \mathbf{q}) - \hat{F}(1)\hat{F}(\mathbf{q}).$$

Since we have assumed the interaction between electrons and phonons to be small, the operator $\hat{G}$ is obviously also small. For problems connected with the behavior of metals in an electromagnetic field (in particular, for calculating the current) it is sufficient to restrict ourselves in the transport equation to terms quadratic in the magnitude of the interaction between electrons and phonons. Correspondingly one can retain in Eq. (7) only the terms linear in this quantity. In the same approximation we can take for the state operator of the phonons its equilibrium value

$$(N_{\mathbf{q}} / \hat{F}(\mathbf{q}) / N'_{\mathbf{q}}) = \delta(N_{\mathbf{q}}, N'_{\mathbf{q}})\,\delta(N_{\mathbf{q}}, \overline{N}_{\mathbf{q}}), \quad \overline{N}_{\mathbf{q}} = [e^{h\nu_{\mathbf{q}}/kT} - 1]^{-1}, \tag{8}$$

where $h\nu_{\mathbf{q}} = u\,|\mathbf{q}|$ is the energy of a phonon of momentum $\mathbf{q}$. If, at the same time, we take into account the fact that the operator $\hat{U}(1, \mathbf{q})$ is non-diagonal in the occupation numbers of the phonons we obtain easily the following set of equations

$$\partial \hat{F}(1)/\partial t = [\hat{H}(1), \hat{F}(1)] + \sum_{k=1}^{S} \operatorname*{Sp}_{(k)} [\hat{U}(1, \mathbf{q}_k), \hat{G}(1, \mathbf{q}_k)] \tag{9}$$

*The operator $\hat{F}$ is symmetrical with respect to the permutation of identical particles. However, in order to calculate matrix elements we have to deal with antisymmetric combinations of one-electron wave functions, which is inconvenient. It is better to use certain methods to antisymmetrize the operator $\hat{F}$ after which one can take the matrix elements with respect to a product of one-electron ψ-functions. The meaning of the operation P_{12} is clear from the equation

$$(\mathbf{p}_1, \mathbf{p}_2 / \hat{P}_{12}\hat{F}(1)\hat{F}(2) / \mathbf{p}'_1, \mathbf{p}'_2) = F_{\mathbf{p}_1, \mathbf{p}'_2} F_{\mathbf{p}_2 \mathbf{p}'_1}.$$

$$\partial\hat{G}(1,\mathbf{q})/\partial t = [\hat{H}(1)+\hat{H}(\mathbf{q}),\ \hat{G}]+[\hat{U}(1,\mathbf{q}),\hat{F}(1)\hat{F}(\mathbf{q})]-N\operatorname*{Sp}_{(2)}[\hat{U}(2,\mathbf{q}),\hat{P}_{12}\hat{F}(1)\hat{F}(2)\hat{F}(\mathbf{q})], \tag{10}$$

where $\hat{\mathbf{F}}(\mathbf{q})$ is of the form (8).

We go now over to an expansion in terms of the magnitude of the vector potential of the electromagnetic field which we write in the form

$$\mathbf{A}(\mathbf{r},t) = \mathbf{A}_0(\mathbf{r})\,e^{i\omega t} + \mathbf{A}_0^*(\mathbf{r})\,e^{-i\omega t}$$

If there is no field the electron distribution is homogeneous and does not depend on the time; any deviation from this distribution will be assumed to be due to the electromagnetic field. It is therefore natural to represent the quantities we are trying to determine in the following form

$$\hat{F}(1,t) = \hat{F}_0(1) + \hat{F}_1(1,t),\ \ \hat{G}(1,\mathbf{q},t) = \hat{G}_0(1,\mathbf{q}) + \hat{G}_1(1,\mathbf{q},t). \tag{11}$$

In these equations the operators $\hat{\mathbf{F}}_0$ and $\hat{\mathbf{G}}_0$ do not depend on the time and describe the equilibrium state of the system of electrons which are interacting with the phonons; however, the operators $\hat{\mathbf{F}}_1$ and $\hat{\mathbf{G}}_1$ which represent the influence of the electromagnetic field will be proportional to the amplitude of this field, $\mathbf{A}_0$, and be of the form

$$\hat{F}_1(1,t) = \hat{F}_{1\omega}(1)\,e^{i\omega t} + \hat{F}_{1\omega}^{+}(1)\,e^{-i\omega t},\quad \hat{G}_1(1,\mathbf{q},t) = \hat{G}_{1\omega}(1,\mathbf{q})\,e^{i\omega t} + \hat{G}_{1\omega}^{+}(1,\mathbf{q})\,e^{-i\omega t}.$$

Using the notation

$$\hat{B}(1,\mathbf{q},t) = [\hat{U}(1,\mathbf{q}),\hat{F}(1)\hat{F}(\mathbf{q})] - N\operatorname*{Sp}_{(2)}[\hat{U}(2,\mathbf{q}),\hat{P}_{12}\hat{F}(1)\hat{F}(2)\hat{F}(\mathbf{q})],$$

we can write Eq. (10) in the form

$$\partial\hat{G}/\partial t - [\hat{H}(1)+\hat{H}(\mathbf{q}),\ \hat{G}] = \hat{B}. \tag{12}$$

In the same way as in Refs. 6 and 7 we impose on the solution of this equation the condition that the correlation vanish when the particles go to infinity,

$$\lim_{\tau\to\infty}\hat{S}_{t+\tau}\hat{G}(t+\tau)\hat{S}_{t+\tau}^{+} = 0,\ \ \hat{S}_t = \exp\left\{-\frac{1}{i\hbar}\int_0^t dt'\,[\hat{H}(1,t')+\hat{H}(\mathbf{q})]\right\}. \tag{13}$$

Carrying out the differentiation, we have by virtue of (12)

$$\frac{\partial}{\partial\tau}\hat{S}_{t+\tau}\hat{G}(t+\tau)\hat{S}_{t+\tau}^{+} = \hat{S}_{t+\tau}\left\{\frac{\partial}{\partial\tau}\hat{G}(t+\tau) - [\hat{H}(1,t+\tau)+\hat{H}(\mathbf{q}),\ \hat{G}(t+\tau)]\right\}\hat{S}_{t+\tau}^{+} = \hat{S}_{t+\tau}\hat{B}(t+\tau)\hat{S}_{t+\tau}^{+}.$$

Hence we have after integration

$$\hat{S}_{t+\tau}\hat{G}(t+\tau)\hat{S}_{t+\tau}^{+} - \hat{S}_t\hat{G}(t)\hat{S}_t^{+} = \int_0^{\tau} d\tau'\hat{S}_{t+\tau'}\hat{B}(t+\tau')\hat{S}_{t+\tau'}^{+}.$$

We now go to the limit $\tau\to\infty$; then, by virtue of (13) the first term vanishes and we obtain the solution of (12) in the form

$$\hat{G}(t) = -\int_0^{\infty} d\tau\exp\left\{-\frac{1}{i\hbar}\int_t^{t+\tau} dt'\,[\hat{H}(1,t')+\hat{H}(\mathbf{q})]\right\}\hat{B}(t+\tau)\exp\left\{+\frac{1}{i\hbar}\int_t^{t+\tau} dt'\,[\hat{H}(1,t')+\hat{H}(\mathbf{q})]\right\}.$$

If we restrict ourselves to terms linear in the field we find thus after some simple calculations

$$(\mathbf{P},N_{\mathbf{q}}|\hat{G}_0|\mathbf{P}',N_{\mathbf{q}}') = -2\pi\hbar\delta_+(E_P-E_{P'}+(N_{\mathbf{q}}-N_{\mathbf{q}}')h\nu_q)(\mathbf{P},N_{\mathbf{q}}|\hat{B}_0|\mathbf{P}',N_{\mathbf{q}}'),$$

$$(\mathbf{P},N_{\mathbf{q}}|\hat{G}_{1\omega}|\mathbf{P}',N_{\mathbf{q}}) = -2\pi\hbar\delta_+(E_P-E_{P'}+(N_{\mathbf{q}}-N_{\mathbf{q}}')h\nu_q+\hbar\omega)$$

$$\times(\mathbf{P},N_{\mathbf{q}}|\hat{B}_{1\omega}|\mathbf{P}',N_{\mathbf{q}}') - 2\pi\hbar\frac{e}{mc\hbar\omega}(A_0\cdot\mathbf{P}-\mathbf{P}')[\delta_+(E_P-E_{P'}$$

$$+(N_{\mathbf{q}}-N_{\mathbf{q}}')h\nu_q+\hbar\omega)-\delta_+(E_P-E_{P'}+(N_{\mathbf{q}}-N_{\mathbf{q}}')h\nu_q)](\mathbf{P},N_{\mathbf{q}}|\hat{B}_0|\mathbf{P}',N_{\mathbf{q}}'),$$

where

$$\hat{B}_0(1,\mathbf{q}) = [\hat{U}(1,\mathbf{q}),\ \hat{F}_0(1)\hat{F}(\mathbf{q})] - N\operatorname*{Sp}_{(2)}[\hat{U}(2,\mathbf{q}),\hat{P}_{12}\hat{F}_0(1)\hat{F}_0(2)\hat{F}(\mathbf{q})],$$

$$\hat{B}_{1\omega}(1,\mathbf{q}) = [\hat{U}(1,\mathbf{q}),\hat{F}_{1\omega}(1)\hat{F}(\mathbf{q})] - N\operatorname*{Sp}_{(2)}[\hat{U}(2,\mathbf{q}),\hat{P}_{12}(\hat{F}_{1\omega}(1)\hat{F}_0(2)$$

$$+\hat{F}_0(1)\hat{F}_{1\omega}(2))\ \hat{F}(\mathbf{q})],\ \ \delta_+(x) = {}^1\!/_2\,\delta(x) + i/2\pi x.$$

If we now want to obtain the transport equation it is sufficient to substitute into the right hand side of Eq. (9) the correlation operator $\hat{G}$ given in this way. We restrict ourselves for the moment to the case of a spatially homogeneous distribution. The operator $\hat{F}$ is then diagonal in the p-representation and we have in agreement with Eq. (6)

$$(\mathbf{P} \mid \hat{F} \mid \mathbf{P}') = N^{-1}\,\delta(\mathbf{P} - \mathbf{P}')\,f(\mathbf{p},t).$$

Since the operator $\hat{F}_0$ does not depend on the time, its matrix elements can depend only on the generalized momenta,

$$(\mathbf{P} \mid \hat{F}_0 \mid \mathbf{P}') = N^{-1}\delta(\mathbf{P} - \mathbf{P}')\,f_0(P).$$

On the other hand, the distribution function can naturally be written in the form

$$f(\mathbf{p}, t) = f_0(p) + f_{1\omega}(\mathbf{p})\,e^{i\omega t} + f_{1\omega}(\mathbf{p})\,e^{-i\omega t}.$$

Then, in accordance with (11)

$$(\mathbf{P} \mid \hat{F}_{1\omega} \mid \mathbf{P}') = N^{-1}\delta(\mathbf{P} - \mathbf{P}')\,[f_{1\omega}(\mathbf{p}) + \frac{e}{c}\,\mathbf{A}_0 \partial f_0/\partial \mathbf{p}].$$

For the sake of simplicity we shall drop henceforth the index ω. Equation (9) has in the zeroth approximation (with respect to the field) the form

$$\sum_{k=1}^{S} \operatorname*{Sp}_{(k)} [\hat{U}(1, \mathbf{q}_k),\ \hat{G}_0(1, \mathbf{q}_k)] = 0.$$

Going over from a sum to an integral we have in the p-representation

$$V(2\pi\hbar)^{-3} \int_{|\mathbf{q}| \leqslant q_0} d\mathbf{q} \sum_{N_\mathbf{q}=0}^{\infty} (\mathbf{P}, N_\mathbf{q} \mid [\hat{U}(1, \mathbf{q}), \hat{G}_0(1, \mathbf{q})] \mid \mathbf{P}', N_\mathbf{q}) = 0.$$

The limiting phonon momentum $\mathbf{q}_0$ is connected with the Debye temperature Θ of the material in the well-known way: $h\nu_{\mathbf{q}_0} = uq_0 = k\Theta$. After simple, but rather lengthy calculations the equation of the zeroth approximation is of the form

$$I^{(\mathrm{cl})}(f_0(P)) = 0, \tag{14}$$

where

$$I^{(\mathrm{cl})}(f(\mathbf{p})) = \frac{\Delta}{(2\pi\hbar)^3}\,\frac{\pi C^2}{Mu\hbar} \int_{|\mathbf{q}| \leqslant q_0} d\mathbf{q}\, q\, \{\delta(a)\,[\bar{N}_q f(\mathbf{p}+\mathbf{q})(1 - f(\mathbf{p})) - (\bar{N}_\mathbf{q} + 1) f(\mathbf{p})(1 - f(\mathbf{p}+\mathbf{q})]$$

$$+ \delta(b)\,[(\bar{N}_\mathbf{q} + 1) f(\mathbf{p}+\mathbf{q})(1 - f(\mathbf{p})) - \bar{N}_q f(\mathbf{p})(1 - f(\mathbf{p}+\mathbf{q})]\},\quad a = E_p - E_{\mathbf{q}+\mathbf{p}} - h\nu_q,\ b = E_p - E_{\mathbf{p}+\mathbf{q}} + h\nu_q$$

is the usual collision integral for electron-phonon interaction. It is well known that this equation is satisfied by the Fermi distribution function

$$f_0(p) = [\exp(E_p - E_0)/kT + 1]^{-1}.$$

In the case of a spatially homogeneous distribution we have $[\hat{H}(1), \hat{F}(1)] = 0$. From Eq. (9) it follows thus in first approximation that

$$-i\omega \hat{F}_1(1) = \sum_{k=1}^{S} \operatorname*{Sp}_{(k)} [\hat{U}(1, \mathbf{q}_k), \hat{G}_1(1, \mathbf{q}_k)].$$

Calculations which in this case are even more cumbersome than in the zeroth approximation lead to the following result

$$i\omega f_1(\mathbf{p}) + i\omega \frac{e}{c}\,\mathbf{A}_0 \frac{\partial f_0}{\partial \mathbf{p}} = \hat{I}\left(f_1(\mathbf{p}) + \frac{e}{c}\,\mathbf{A}_0 \frac{\partial f_0}{\partial \mathbf{p}}\right) + \frac{e}{c}\,(\mathbf{A}_0 \mathbf{p})\,D(p). \tag{15}$$

where $\hat{I}$ is the quantum mechanical generalization of the usual collision integral,

$$\hat{I} f_1(\mathbf{p}) = \frac{\Delta}{(2\pi\hbar)^3}\,\frac{\pi C^2}{Mu\hbar} \int_{|\mathbf{q}| \leqslant q_0} d\mathbf{q} \cdot q \left\{\frac{1}{2}\,[\delta(a + \hbar\omega) + \delta(a - \hbar\omega)]\,[-(\bar{N}_\mathbf{q} + 1) f_1(\mathbf{p}) + \bar{N}_q f_1(\mathbf{p}+\mathbf{q}) + f_0(\mathbf{p}) f_1(\mathbf{p}+\mathbf{q}) + \right.$$

$$+ f_0(\mathbf{p}+\mathbf{q}) f_1(\mathbf{p})] + \frac{1}{2}[\delta(b+\hbar\omega) + \delta(b-\hbar\omega)]\,[(\overline{N}_q + 1) f_1(\mathbf{p}+\mathbf{q}) - \overline{N}_q f_1(\mathbf{p}) - f_1(\mathbf{p}+\mathbf{q}) f_0(p) - f_1(\mathbf{p}) f_0(\mathbf{p}+\mathbf{q})]\Big\};$$

$$D(p) = + \frac{\Delta}{(2\pi\hbar)^3} \frac{\pi C^2}{2mMu\hbar p_0^2} \frac{1}{\hbar\omega} (e^{\hbar\omega/kT} - 1) f_0(p) e^{(E_p - E_0)/kT} \int\limits_{|\mathbf{q}|\leqslant q_0} d\mathbf{q}\, q\,(\mathbf{pq})\, \overline{N}_q f_0(\mathbf{p}+\mathbf{q})\,[\delta(a+\hbar\omega)$$

$$+ e^{-\hbar\omega/kT}\delta(a-\hbar\omega) + e^{\hbar v q/kT}\delta(b+\hbar\omega) + e^{\hbar v q - \hbar\omega/kT}\delta(b-\hbar\omega)].$$

For $\hbar\omega \to 0$, $\hat{I}(f_1)$ goes over into $I^{(cl)}(f_1)$. In order to understand the meaning of the rest of the terms on the right hand side of Eq. (15) we notice that the last of them can be written in the form

$$\frac{e}{c}(\mathbf{A}_0\mathbf{p})\, D(\mathbf{p}) = \frac{\Delta}{(2\pi\hbar)^3} \frac{\pi C^2}{Mu\hbar} \frac{e}{mc} \int\limits_{|\mathbf{q}|\leqslant q_0} d\mathbf{q}\cdot q\,(\mathbf{A}_0\mathbf{q}) \Big\{\frac{1}{2\hbar\omega}\,[\delta(a+\hbar\omega) - \delta(a-\hbar\omega)]\,[-(\overline{N_q} + 1) f_0(p)(1 - f_0(\mathbf{p}+\mathbf{q}))$$

$$+ \overline{N}_q f_0(\mathbf{p}+\mathbf{q})(1 - f_0(p))] + \frac{1}{2\hbar\omega}\,[\delta(b+\hbar\omega) - \delta(b-\hbar\omega)]\,[-N_q f_0(p)(1 - f_0(\mathbf{p}+\mathbf{q})) + (\overline{N}_q + f) f_0(\mathbf{p}+\mathbf{q})(1 - f_0(p))]\Big\}.$$

Let us imagine further that in the equation $I^{(cl)}(f_0(\mathbf{p})) \equiv 0$ we have substituted $\mathbf{p} = \mathbf{P} + e\mathbf{A}_0/c$ and have carried out an expansion in terms of $\mathbf{A}_0$ accurate up to terms of the first order of magnitude. Then $I^{(cl)}(f_0(\mathbf{p}))$ can be written as the sum of three expressions; the first of these is the same as expression (14) and is equal to zero, the second arises from expanding expressions of the kind $f_0(\mathbf{P} + e\mathbf{A}/c)$ and the third one, finally, arises from the expansion of the δ-function. It is clear that the terms

$$\hat{I}\frac{e}{c}\mathbf{A}_0\frac{\partial f_0}{\partial \mathbf{p}} \text{ and } \frac{e}{c}(\mathbf{A}_0.\mathbf{p})D(\mathbf{p})$$

are the quantum mechanical generalizations of the last two expressions and cancel one another as $\hbar\omega \to 0$.

In the case of a spatially inhomogeneous distribution the method which we have just applied to obtain the transport equation will, generally speaking, not be suitable, since now in the transformation (5) the vector potential depends essentially on the coordinates, and, in particular, from Eq. (6) there does not follow a simple connection between $F_{\mathbf{P},\mathbf{P}'}$ and the function $f(\mathbf{p}, \mathbf{r}, t)$. However, it is necessary to take into account the fact that the distance over which the field or the electron distribution changes materially will always be very large compared to the de Broglie wavelength of an electron near the Fermi surface. In other words, if we write, for instance, $f_1(\mathbf{p}, \mathbf{r}, t)$ as a Fourier integral of the coordinates,

$$f_1(\mathbf{p}, \mathbf{r}, t) = \int f_{1\mathbf{k}}(\mathbf{p}, t)\, e^{i\mathbf{kr}/\hbar}\, d\mathbf{k},$$

$f_{1\mathbf{k}}$ will be noticeably different from zero only, if

$$|\mathbf{k}| \ll p_0. \tag{16}$$

For this reason all terms of Eq. (15) retain their form with the proviso that now $\mathbf{A}_0$ and $\mathbf{f}_1$ will be functions of the coordinate $\mathbf{r}$. We only have to take into account the term $[\hat{H}(1), \hat{F}(1)]$ of Eq. (9) which previously vanished.

It can easily be shown that the left hand side of the transport equation can be reduced to the expression

$$\frac{\mathbf{p}}{m}\frac{\partial f_1}{\partial \mathbf{r}} + i\,\frac{e}{mc}\frac{1}{(2\pi\hbar)^3}\int d\tau\, d\mathbf{p}' e^{-i\tau(\mathbf{p}-\mathbf{p}')/\hbar} f_0(p')\left[\left(\tau, \frac{\partial}{\partial \mathbf{r}}\right)(\mathbf{p}.\mathbf{A}_0(\mathbf{r})) - \left(\mathbf{p}'.\mathbf{A}_0\left(\mathbf{r}+\frac{\tau}{2}\right) - \mathbf{A}_0\left(\mathbf{r}-\frac{\tau}{2}\right)\right)\right],$$

in which the integral terms can be neglected by virtue of condition (16).

Introducing the electric field strength $\mathbf{E}(\mathbf{r}) = (i\omega/c)\,\mathbf{A}_0(\mathbf{r})$ and the relaxation time operator $\hat{\tau} = -\hat{I}^{-1}$ we write the transport equation finally in the form

$$i\omega f_1 + \frac{\mathbf{p}}{m}\frac{\partial f_1}{\partial \mathbf{r}} + \hat{\tau}^{-1} f_1 = e\mathbf{E}\left[\frac{\partial f_0}{\partial \mathbf{p}} + \frac{1}{i\omega\hat{\tau}}\frac{\partial f_0}{\partial \mathbf{p}} - \frac{1}{i\omega}\mathbf{p}D(p)\right]. \tag{17}$$

In conclusion the author expresses his deep gratitude to the supervisor of this work, Professor V. L. Ginzburg, and also to V. P. Silin for his constant interest and valuable comments.

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Translated by D. ter Haar
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SOVIET PHYSICS JETP VOLUME 6 (33) NUMBER 2 FEBRUARY, 1958

*TRANSPORT PHENOMENA IN A COMPLETELY IONIZED TWO-TEMPERATURE PLASMA**

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Submitted to JETP editor February 13, 1957

J. Exptl. Theoret. Phys. (U.S.S.R.) 33, 459-472 (August, 1957)

A system of transport equations has been obtained for a plasma consisting of electrons and one kind of positive ions placed in an electric and magnetic field. The system includes the continuity equations, equations of motion, and the equation of heat transport for electrons and ions. The electron and ion temperatures are considered to be different. The case of arbitrary ratio of the particle collision frequency to the Larmor frequency is considered. The derivation of the transport equations from the kinetic equations is similar to that of Chapman and Cowling.

1. THE TRANSPORT EQUATIONS

THE state of a completely ionized plasma can be characterized by the electron and ion distribution functions $f_\alpha(t, \mathbf{r}, \mathbf{v})$. In the presence of electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ these distribution functions satisfy the following system of kinetic equations (see, for example, Chapman and Cowling[1])

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\nabla f_\alpha + \frac{e_\alpha}{m_\alpha}\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}\times\mathbf{H}]\right)\nabla_v f_\alpha + \sum_\beta S_{\alpha\beta}(f_\alpha f_\beta) = 0, \tag{1.1}$$

where α denotes the type of particle (1 — electrons, 2 — ions).

Following Landau,[2] we take the collision integrals $S_{\alpha\beta}(f_\alpha f_\beta)$, which give the change in the distribution of particles of type α as a result of their collisions with particles of type β, to be of the form:

$$S_{\alpha\beta}(f_\alpha f_\beta) = \frac{2\pi\lambda e_\alpha^2 e_\beta^2}{m_\alpha}\frac{\partial}{\partial v_i}\int\left\{\frac{f_\alpha}{m_\beta}\frac{\partial f_\beta'}{\partial v_k'} - \frac{f_\beta'}{m_\alpha}\frac{\partial f_\alpha}{\partial v_k}\right\}U_{ik}d\mathbf{v}', \tag{1.2}$$

where

$$U_{ik} = (u^2\delta_{ik} - u_i u_k)/u^3, \quad u_i = v_i - v_i'.$$

The "Coulomb logarithm" λ appearing in (1.2) is equal to the logarithm of the ratio of the maximum and minimum impact parameters $\lambda = \ln(p_{max}/p_{min})$. For the smaller impact parameter one should substi-

* Work performed in 1952.