[5] Iu. L. Klimontovich and V. P. Silin, J. Exptl. Theoret. Phys. (U.S.S.R.) 23, 151 (1952).
[6] N. N. Bogoliubov, Проблемы динамической теории в статистической физике (Problems of Dynamical Theory in Statistical Physics) Moscow-Leningrad, Gostekhizdat, 1946.
[7] N. N. Bogoliubov and K. P. Gurov, JETP (U.S.S.R.), 17, 614 (1947); J. Phys. (U.S.S.R.) 11, No. 5 (1947).
[8] A. H. Wilson, The Theory of Metals, Cambridge, 1953 (Russ. Transl.).

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*TRANSPORT PHENOMENA IN A COMPLETELY IONIZED TWO-TEMPERATURE PLASMA**

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A system of transport equations has been obtained for a plasma consisting of electrons and one kind of positive ions placed in an electric and magnetic field. The system includes the continuity equations, equations of motion, and the equation of heat transport for electrons and ions. The electron and ion temperatures are considered to be different. The case of arbitrary ratio of the particle collision frequency to the Larmor frequency is considered. The derivation of the transport equations from the kinetic equations is similar to that of Chapman and Cowling.

1. THE TRANSPORT EQUATIONS

THE state of a completely ionized plasma can be characterized by the electron and ion distribution functions $f_\alpha(t, \mathbf{r}, \mathbf{v})$. In the presence of electric and magnetic fields **E** and **H** these distribution functions satisfy the following system of kinetic equations (see, for example, Chapman and Cowling[1])

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}\nabla f_\alpha + \frac{e_\alpha}{m_\alpha}\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}\times\mathbf{H}]\right)\nabla_v f_\alpha + \sum_\beta S_{\alpha\beta}(f_\alpha f_\beta) = 0, \tag{1.1}$$

where α denotes the type of particle (1 — electrons, 2 — ions).

Following Landau,[2] we take the collision integrals $S_{\alpha\beta}(f_\alpha f_\beta)$, which give the change in the distribution of particles of type α as a result of their collisions with particles of type β, to be of the form:

$$S_{\alpha\beta}(f_\alpha f_\beta) = \frac{2\pi\lambda e_\alpha^2 e_\beta^2}{m_\alpha}\frac{\partial}{\partial v_i}\int\left\{\frac{f_\alpha}{m_\beta}\frac{\partial f'_\beta}{\partial v'_k} - \frac{f'_\beta}{m_\alpha}\frac{\partial f_\alpha}{\partial v_k}\right\}U_{ik}d\mathbf{v}', \tag{1.2}$$

where

$$U_{ik} = (u^2\delta_{ik} - u_i u_k)/u^3, \quad u_i = v_i - v'_i.$$

The "Coulomb logarithm" λ appearing in (1.2) is equal to the logarithm of the ratio of the maximum and minimum impact parameters $\lambda = \ln(p_{max}/p_{min})$. For the smaller impact parameter one should substi-

* Work performed in 1952.

tute into the above $p_{min} \approx e^2/mv^2 \approx e^2/T$. The maximum impact parameter is determined by the fact that the Coulomb field of particles in a plasma is screened at distances of the order of a Debye length $p_{max} \approx \delta_D = (T/4\pi e^2 n)^{1/2}$. At high speeds when $e^2/hv < 1$ (or $v/c > 1/137$) a smaller value should be taken for the maximum impact parameter, i.e., one for which the scattering angle becomes of the same order as its quantum-theoretical indeterminacy $p_{max} = \delta_D e^2/hv$. The influence of the magnetic field on the collision event itself is not taken into account in (1.2), which is permissible only for not too strong fields when the radius of curvature of the particle trajectory is large compared to a Debye length.

The method of obtaining the transport equations, starting from the kinetic equations, is given in detail in Chapman and Cowling's monograph,[1] where expressions are also obtained for the heat flow and for the stress tensor for a single-component ionized gas in a magnetic field, and also for the electrical conductivity of a plasma in a magnetic field. The electrical conductivity of plasma (in particular, of a multicomponent one) in a magnetic field is investigated in greater detail by Cowling[3] by the same method. The method described by Chpman and Cowling[1] assumes that the temperatures of all the components of the plasma are the same. A calculation of the electrical conductivity and of the electronic heat conductivity of the plasma was carried out by Landshoff[4] without assuming that the temperatures of the ions and the electrons are equal for not very strong magnetic fields (the ratio of the Larmor frequency to the collision frequency lying between zero and six). The transport equations for a plasma in a magnetic field of arbitrary intensity were obtained by Fradkin[5] for the case of equal electron and ion temperatures.

In this paper the method given by Chapman and Cowling[1] is somewhat modified to obtain a separate system of transport equations for each plasma component. The microscopic parameters (density, average velocity, and temperature) for the ions and the electrons are given by the following expressions:

$$n_\alpha(t, \mathbf{r}) = \int f_\alpha(t, \mathbf{r}, \mathbf{v})\, d\mathbf{v}, \quad \mathbf{v}^0_\alpha(t, \mathbf{r}) = \frac{1}{n_\alpha}\int \mathbf{v} f_\alpha(t, \mathbf{r}, \mathbf{v})\, d\mathbf{v}, \quad \frac{3}{2} T_\alpha(t, \mathbf{r}) = \frac{1}{n_\alpha}\int \frac{m_\alpha}{2}(\mathbf{v} - \mathbf{v}^0_\alpha)^2 f_\alpha(t, \mathbf{r}, \mathbf{v})\, d\mathbf{v}. \tag{1.3}$$

The temperatures of the ions and of the electrons are, generally speaking, taken to be different. This turns out to be possible because of the large difference in the masses of the ions and the electrons. Thus, plasma is regarded as consisting of two mutually interpenetrating fluids — the electron and the ion fluids — similarly to the way in which this has already been done in the less refined investigation of Schlüter.[6]

For subsequent developments it is convenient to introduce into (1.1) the change of variables $\mathbf{v}_\alpha = \mathbf{v} - \mathbf{v}^0_\alpha(t, \mathbf{r}_\alpha)$. We then obtain for $f_\alpha(t, \mathbf{r}, \mathbf{v}_\alpha)$ the equation

$$\frac{d_\alpha f_\alpha}{dt} + \mathbf{v}\nabla f_\alpha + \left(\frac{e_\alpha}{m_\alpha}\mathbf{E} + \frac{e_\alpha}{m_\alpha c}[\mathbf{v}^0_\alpha \times \mathbf{H}] - \frac{d_\alpha \mathbf{v}^0_\alpha}{dt}\right)\nabla_v f_\alpha - \frac{\partial v^0_{\alpha i}}{\partial x_k} v_k \frac{\partial f_\alpha}{\partial v_i} + \frac{e_\alpha}{m_\alpha c}[\mathbf{v}\times\mathbf{H}]\nabla_v f_\alpha + \sum_\beta S_{\alpha\beta}(f_\alpha f_\beta) = 0. \tag{1.4}$$

Here $d_\alpha/dt = \partial/\partial t + \mathbf{v}^0_\alpha \nabla$, and the index α has been omitted from $\mathbf{v}_\alpha$ for the safe of brevity.

Multiplication of (1.4) in turn by 1, $m_\alpha \mathbf{v}_\alpha m_\alpha v^2_\alpha/2$ and integration over the velocities leads to the following system of equations for the macroscopic parameters of the gas α

$$d_\alpha n_\alpha / dt + n_\alpha \operatorname{div} \mathbf{v}^0_\alpha = 0, \quad m_\alpha n_\alpha \frac{d_\alpha \mathbf{v}^0_\alpha}{dt} = -\operatorname{grad} p_\alpha - \operatorname{div} \pi_\alpha + e_\alpha n_\alpha\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}^0_\alpha \times \mathbf{H}]\right) + \mathbf{R}_\alpha, \quad \frac{3}{2} n_\alpha \frac{d_\alpha T_\alpha}{dt} + p_\alpha \operatorname{div} \mathbf{v}^0_\alpha$$
$$+ \pi_{\alpha ik}\frac{\partial v^0_{\alpha i}}{\partial x_k} + \operatorname{div} \mathbf{q}_\alpha = Q_\alpha. \tag{1.5}$$

The first of these equations is the equation of continuity, the second is the equation of momentum transfer, and the third is the equation of heat transfer. The following average quantities have been introduced:

$$p_\alpha = \int (m_\alpha v^2_\alpha/3) f_\alpha d\mathbf{v} = n_\alpha T_\alpha, \quad \pi_{\alpha ik} = \int m_\alpha\left(v_{\alpha i} v_{\alpha k} - \frac{1}{3}\delta_{ik} v^2_\alpha\right) f_\alpha d\mathbf{v}, \quad \mathbf{q}_\alpha = \int (m_\alpha v^2_\alpha/2)\, \mathbf{v}_\alpha f_\alpha d\mathbf{v},$$

$$\mathbf{R}_\alpha = -\int m_\alpha \mathbf{v}_\alpha S_{\alpha\beta}(f_\alpha f_\beta)\, d\mathbf{v}, \quad Q_\alpha = -\int (m_\alpha v^2_\alpha/2)\, S_{\alpha\beta}(f_\alpha f_\beta)\, d\mathbf{v}.$$

Here p_α is the partial pressure of particles α. We shall denote the symmetric tensor π_{ik} as the viscosity tensor. The vector $\mathbf{q}_\alpha$ is the heat flux for particles of type α; $\mathbf{R}_\alpha$ and Q_α are respectively the average rate of change of the momentum of particles α and the rate of liberation of heat in the gas formed by particles α due to their collisions with particles β.

Equations (1.5) may be used for finding the macroscopic parameters after first finding the local distribution functions and expressing the quantities (1.6) in terms of the macroscopic parameters and their derivatives. The following sections are devoted to this. The expressions obtained as a result of the approximate solution of the kinetic equations are given in (2.3), (2.5), (2.6), (3.16), (3.18), (3.19), (3.20), (4.13), (4.14′), and (4.16) while the necessary coefficients are given in a table (see p. 364).

As in the case of the usual equations of gas dynamics, the condition for the applicability of these expressions is that all these quantities should vary but little over distances on the order of a mean free path and during times on the order of the mean collision time between particles. This holds for not too high values of the ratio of the Larmor frequency to the collision frequency $\omega\tau \lesssim 1$. But in the case of strong magnetic fields when $\omega\tau \gg 1$ the problem becomes more complicated. For the applicability of the results obtained it is necessary that during the time between two collisions the particle be displaced through a distance so small that all quantities undergo only a small change over it. The motion of the particle transversely to the magnetic field is limited by its Larmor radius ρ which is less than the mean free path ℓ by a factor $\omega\tau$. Therefore in a number of cases the conditions of applicability for processes close to stationary ones are relaxed, and we require only that $L_\perp \gg \rho$, $L_\parallel \gg \ell$ ($L_\perp$, $L_\parallel$ are characteristic distances in directions perpendicular and parallel to the magnetic field over which the various quantities vary significantly). However, this is true only in the presence of a sufficient degree of symmetry. In an inhomogeneous magnetic field (and also in the presence of a transverse electric field) in addition to the Larmor rotation in a circle the particles also undergo a drifting motion across the magnetic field.[7] This drift occurs with a velocity of the order of $v\rho/L_\perp$ (v is the thermal velocity) and leads to a displacement of the particle during the time between collisions by a distance of the order of $\ell\rho/L_\perp$. Therefore the condition $L_\perp \gg \rho$ is sufficient only in the case when the system possesses such symmetry that the particle drifts are not directed along gradients of density, temperature, etc. If such a special symmetry does not exist then the stricter condition is obtained $L_\perp \gg \ell\rho/L_\perp$, or $L_\perp \gg \sqrt{\ell\rho}$.

2. METHOD OF APPROXIMATE SOLUTION OF THE KINETIC EQUATIONS

The ion and electron distribution functions satisfy a system of kinetic equations. By taking into account the smallness of the ratio of the electron mass to the ion mass we can simplify the problem and solve these equations separately. This is connected with the fact that the electron velocities are much higher than the ion velocities so that the relative velocity of the electron and the ion nearly coincides with the electron velocity. Therefore, the cross term S_{12} is to a high degree of accuracy independent of the detailed shape of the ion distribution function, but is determined by specifying the average values of n_2, $\mathbf{v}_2^0$, and T_2 for the ions.

The tensor U_{ik} in the electron-ion collision integral may be expanded in powers of the ion velocity and integrated over the ion velocities. Neglecting the viscosity tensor for the ions we obtain:

$$S_{12} = -\frac{2\pi\lambda e_1^2 e_2^2 n_2}{m_1^2}\frac{\partial}{\partial v_l}\left\{V_{lk}\frac{\partial f_1}{\partial v_k} + \frac{m_1}{m_2}\left(\frac{2v_l}{v^3}f_1 + \frac{T_2}{m_1}\frac{3v_l v_k - v^2\delta_{lk}}{v^5}\frac{\partial f_1}{\partial v_k}\right)\right\}, \quad V_{ik} = (v^2\delta_{ik} - v_i v_k)/v^3. \tag{2.1}$$

The electron velocity is here referred to the mean ion velocity. We shall denote the first (principal) term in (2.1) by S'_{12}. It reduces to zero for any spherically symmetric electron distribution function and does not give any change in the electron energy. Later we shall need an expression for the friction force acting on the electrons due to the presence of the ions in the case when the electrons have a Maxwellian distribution (we denote it by $f_1^{(0)}$) displaced with respect to the ion distribution by the amount $\mathbf{u} = \mathbf{v}_1^0 - \mathbf{v}_2^0$. Assuming this shift to be small compared to the electron thermal velocities and neglecting terms $\sim m_1/m_2$ we shall easily obtain with the aid of (2.1)

$$\mathbf{R}_1^{(0)} = -\int m_1\mathbf{v}S'_{12}(f_1^{(0)})\,d\mathbf{v} = -(m_1 n_1/\tau_1)\,\mathbf{u}. \tag{2.2}$$

Here we have introduced the "time between collisions" between electrons and ions

$$\tau_1 = 3\sqrt{m_1}T_1^{3/2}/4\sqrt{2\pi}\,\lambda e_1^2 e_2^2 n_2. \tag{2.3}$$

The ion-electron collision integral can also be simplified by expanding U_{ik} in terms of the ratio of the ion speed to the electron speed. Assuming that the electron distribution function differs but little from a Maxwellian one, with the difference between the mean velocities of the electrons and the ions being small compared to the electron velocities, we obtain:

$$S_{21} = -\frac{m_1 n_1}{m_2 n_2}\frac{1}{\tau_1}\frac{\partial}{\partial v_i}\left(v_i f_2 + \frac{T_1 \partial f_2}{m_2 \partial v_i}\right) + \frac{1}{m_2 n_2}\mathbf{R}_2 \nabla_v f_2. \quad (2.4)$$

The ion velocity is here referred to their average velocity. This collision term, as should have been expected, has the same structure as in the case of brownian particles in a moving medium.

The transfer of heat from the electrons to the ions by means of collisions can be easily calculated with the aid of (2.4). By neglecting the deviation of the ion distribution function from a Maxwellian one we obtain:

$$Q_2 = \frac{3m_1 n_1}{m_2 \tau_1}(T_1 - T_2). \quad (2.5$$

This expression has been obtained by Landau.[2] The liberation of heat in the electron gas can be most simply computed by making use of the conservation of energy in collisions. Simple calculations yield:

$$Q_1 = -\mathbf{R}_1 \mathbf{u} - Q_2. \quad (2.6)$$

By evaluating the corresponding integrals or by considering directly collisions between particles it may be easily seen that if the energies of the light and of the heavy particles are of the same order of magnitude then the times for the exchange of energy for the light particles (electrons) among themselves, and also for the heavy particles (ions) are much shorter than that between the electrons and the ions. Let the time for energy exchange between electrons be τ_1, the time for energy exchange between ions be τ_2, and the time for energy exchange between electrons and ions be τ_3, then:

$$\tau_1 : \tau_2 : \tau_3 = 1 : \sqrt{m_2/m_1} : m_2/m_1. \quad (2.7)$$

It is this relation which allows us to treat the problem with ions and electrons of different temperatures, since equilibrium is established within each gas more rapidly than that between them. The transfer of momentum from the electrons to the ions, in contrast to the transfer of energy, is not small and occurs during times of the same order of magnitude as the exchange of momentum between electrons. The transfer of the ion momentum to the electrons occurs during the same time as the transfer of energy, and is therefore small compared to the exchange of momentum between the ions.

Thus for the ions the cross collision integral is small compared with the self integral $S_{21} \ll S_{22}$. But for the electrons although the cross integral is not small it may be represented as the sum of two integrals $S_{12} = S'_{12} + S''_{12}$ such that the first of them gives only the transfer of momentum but does not lead to a change in the electron energy, while the second one is small (see 2.1). The kinetic equation for the electrons may be written in the following form:

$$S_{11}(f_1 f_1) + S'_{12}(f_1 f'_2) + [\mathbf{v}\times\boldsymbol{\omega}_1]\nabla_v f_1 = -\left\{\frac{d_1 f_1}{dt} + \mathbf{v}\,\nabla f_1 + \left(\frac{e_1}{m_1}\mathbf{E} + \frac{e_1}{m_1 c}[\mathbf{v}_1^0 \times \mathbf{H}] - \frac{d_1 \mathbf{v}_1^0}{dt}\right)\nabla_v f_1 - \frac{\partial v_{1i}^0}{\partial x_k} v_k \frac{\partial f_1}{\partial v_i}\right.$$
$$\left. + S'_{12}(f_1, f_2 - f'_2) + S''_{12}(f_1 f_2)\right\}. \quad (2.8)$$

The terms on the right hand side of this equation are small when the gradients are small, the time variations are slow and the shift of the mean velocities of the electrons and the ions is small. The magnitude of the vector $\boldsymbol{\omega}_1 = (e_1/mc)\mathbf{H}$ is equal to the Larmor frequency of the electrons (we note that $e_1 < 0$, $\omega_1 < 0$). The term $S'_{12}(f_1 f'_2)$ has been added to and substracted from equation (2.8) in which f'_2 is the ion distribution function "shifted" in such a way that the mean velocity of the ions coincides with that of the electrons. The term $S'_{12}(f_1, f_2 - f'_2)$ which appears in the course of this on the right hand side of the equation is small compared to $S'_{12}(f_1, f'_2)$ provided that the relative macroscopic velocity between the electrons and the ions is small compared to the thermal velocities of the electrons.

The zero-order approximation satisfies the equation without the right hand side. The solution of this equation is arbitrary Maxwellian distribution:

$$f_1^{(0)} = n_1 (2\pi T_1/m_1)^{-3/2} \exp(-m_1 v^2/2T_1). \quad (2.9)$$

The magnetic field evidently has no influence on the Maxwellian distribution.

We shall assume that the parameters of this distribution correspond to the density and the temperature at the given point; the velocity in (2.9) is referred to its mean value $\mathbf{v}_1^0$. In seeking now a correction to this Maxwellian distribution we shall require that it should not change the values of these macroscopic parameters.

Let the electron distribution function have the form $f_1 = f_1^{(0)}(1+\Phi)$ where Φ is a small correction. By substituting this expression into (2.8) and discarding terms of the second order of smallness we shall obtain the equation which determines the correction. We shall replace the time derivatives of n, $\mathbf{v}^0$, and T appearing on the right hand side by their zero-order approximations. Without its right hand side the equation has the solutions $\Phi = 1, v^2$, and therefore in order for the equation to be soluble its right hand side must be orthogonal to these solutions. By multiplying the equation for the correction by 1, $\mathbf{v}$, $mv^2/2$ and by integrating over the velocities, by taking into account the condition $\int \mathbf{v} f_1^0 \Phi\, d\mathbf{v} = 0$ we shall obtain the expression for the zero order approximations for the time derivatives n, $\mathbf{v}^0$, T which must be substituted into the right hand side. The whole procedure is analogous to the one developed by Chapman and Cowling.[1]

As a result we obtain for the first-order correction:

$$I_1(\Phi) + I'_{12}(\Phi) + f_1^{(0)}[\mathbf{v}\times\omega_1]\nabla_v\Phi = -f_1^{(0)}\Big\{\Big(\frac{m_1v^2}{2T_1} - \frac{5}{2}\Big)\mathbf{v}\nabla\ln T_1 + \frac{m_1}{2T_1}\Big(v_iv_k - \frac{v^2}{3}\delta_{ik}\Big)w_{1ik} + \frac{\mathbf{R}_1^{(1)}\mathbf{v}}{n_1T_1}$$
$$+\Big(\frac{3\sqrt{\pi}}{\sqrt{2}}\frac{(T_1/m_1)^{3/2}}{v^3} - 1\Big)\frac{m_1}{T_1}\frac{\mathbf{u}\mathbf{v}}{\tau_1}\Big\}. \qquad (2.10)$$

Here we have introduced the notation:

$$I_1(\Phi) = S_{11}(f_1^{(0)}, f_1^{(0)}\Phi) + S_{11}(f_1^{(0)}\Phi, f_1^{(0)}),\quad I'_{12}(\Phi) = S'_{12}(f_1^{(0)}\Phi,\ f'_2); \qquad (2.11)$$

$$\mathbf{R}_1^{(1)} = -\int m_1\mathbf{v}S'_{12}(f_1^{(0)}\Phi,\ f'_2)\,d\mathbf{v}. \qquad (2.12)$$

In the right hand side of (2.10) the terms $\sim m_1/m_2$ have been omitted. The integral $S'_{12}(f_1^{(0)}, f_2 - f'_2)$ has been expanded in powers of $u/\sqrt{T_1/m_1}$ (this quantity is assumed to be small) and only the first term of the expansion has been retained. In the second term in curly brackets symmetrization has been performed and a symmetric tensor with zero trace has been introduced

$$w_{ik} = \partial v_i^0/\partial x_k + \partial v_k^0/\partial x_i - 2/3\,\delta_{ik}\operatorname{div}\mathbf{v}^0. \qquad (2.13)$$

The kinetic equation for the ions is transformed in a similar way with, however, the one difference that the whole cross collision term [it is taken to be of the form (2.4)] is regarded as small and is transferred to the right-hand side. The zero-order approximation to the ion distribution function which satisfies the equation without its right-hand side is the Maxwellian distribution $f_2^{(0)}$. The ion distribution function is represented in the form $f_2 = f_2^{(0)}(1+\Phi)$ and the following equation is obtained for the small correction Φ:

$$I_2(\Phi) + f_2^{(0)}[\mathbf{v}\times\omega_2]\nabla_v\Phi = -f_2^{(0)}\Big\{\Big(\frac{m_2v^2}{2T_2} - \frac{5}{2}\Big)\mathbf{v}\nabla\ln T_2 + \frac{m_2}{2T_2}\Big(v_iv_k - \frac{v^2}{3}\delta_{ik}\Big)w_{2ik}\Big\}. \qquad (2.14)$$

The terms on the right-hand side associated with collisions with electrons are mutually cancelled. As a result Eq. (2.14) has the same form as for a simple gas (not a mixture). Thus the form of the ion distribution function in the approximation under consideration is determined only by collision of ions with ions. On the contrary the form of the electron distribution function is determined both by self (electron-electron) collisions, and by cross (electron-ion) collisions.

3. HEAT FLOW AND TRANSFER OF MOMENTUM

The solution of Eq. (2.10) may be sought in the form

$$\Phi(\mathbf{v}) = \Phi_i(v)v_i + \Phi_{ik}(v)\Big(v_iv_k - \frac{v^2}{3}\delta_{ik}\Big). \qquad (3.1)$$

The first term may be represented in the form: $\Phi_i v_i = \Phi_u + \Phi_T$, where

$$\Phi_u = A_u\mathbf{v}\mathbf{u}_\parallel + A'_u\mathbf{v}\mathbf{u}_\perp + A''_u\mathbf{v}[\omega_1\times\mathbf{u}]; \qquad (3.2)$$

$$\Phi_T = A_T\mathbf{v}\nabla_\parallel\ln T_1 + A'_T\mathbf{v}\nabla_\perp\ln T_1 + A''_T\mathbf{v}[\omega_1\times\nabla\ln T_1]. \qquad (3.3)$$

The coefficients A are functions of the absolute value of the velocity. The subscript ‖ denotes the component of the corresponding vector along the magnetic field while ⊥ denotes the perpendicular component. It is evidently sufficient to find A′ and A″; the expression for A is obtained from A′ by setting $\omega_1 = 0$.

Knowing Φ_i one can find the heat flow and the momentum transfer due to collisions (the friction force and the thermal force).

Hereinafter it will be convenient to utilize the expansion of the functions A into a series of Sonine polynomials of order 3/2. The Sonine polynomials have the following orthogonality property:

$$\int_0^\infty e^{-x} L_p^{(3/2)}(x) L_q^{(3/2)}(x) x^{3/2} dx = \frac{\Gamma(p + 5/2)}{p!} \delta_{pq}. \tag{3.4}$$

In place of the velocity **v** we introduce the dimensionless variable

$$\mathbf{s} = (m_1 / 2T_1)^{1/2} \mathbf{v}. \tag{3.5}$$

We write the expansion of A'_u, A''_u in terms of Sonine polynomials in the form:

$$A'_u = -\frac{m_1}{T_1} \sum_{k=1}^{\infty} a'_k L_k^{(3/2)}(s^2), \quad A''_u = -\frac{m_1}{T_1} \sum_{k=1}^{\infty} a''_k L_k^{(3/2)}(s^2). \tag{3.6}$$

The orthogonality relation (3.4) leads to the fact that the heat flux is determined by the first expansion coefficient

$$\mathbf{q}_u = {}^5/_2\, n_1 T_1 (a'_1 \mathbf{u}_\perp + a''_1 [\boldsymbol{\omega}_1 \times \mathbf{u}]). \tag{3.7}$$

The transfer of momentum by collisions (friction force) will be expressed in the following way:

$$\mathbf{R}_u^{(1)} = {}^5/_2 \frac{m_1 n_1}{\tau_1} \sum_{k=1}^{\infty} \alpha'_{0k} (a'_k \mathbf{u}_\perp + a''_k [\boldsymbol{\omega}_1 \times \mathbf{u}]). \tag{3.8}$$

Here the following notation has been introduced for the matrix elements:

$$\alpha'_{0k} = \frac{4\tau_1}{15 n_1} \int s_i I'_{12} (L_k^{(3/2)}(s^2) s_i) d\mathbf{v}, \tag{3.9}$$

where τ_1 is the time between collisions [see (2.3)]. The matrix elements (3.9) are calculated in the Appendix.

In a similar way the heat flux and the transfer of momentum due to a temperature gradient may be expressed in terms of the expansion coefficients. On writing A'_T, A''_T in the form

$$A'_T = \tau_1 \sum_{k=1}^{\infty} a'_k L_k^{(3/2)}(s^2), \quad A''_T = \tau_1 \sum_{k=1}^{\infty} a''_k L_k^{(3/2)}(s^2), \tag{3.10}$$

we obtain:

$$\mathbf{q}_T = -\frac{5}{2} \frac{n_1 T_1 \tau_1}{m_1} (a'_1 \nabla_\perp T_1 + a''_1 [\boldsymbol{\omega}_1 \times \nabla T_1]); \tag{3.11}$$

$$\mathbf{R}_T = -\frac{5}{2} n_1 \sum_{k=1}^{\infty} \alpha'_{0k} (a'_k \nabla_\perp T_1 + a''_k [\boldsymbol{\omega}_1 \times \nabla T_1]). \tag{3.12}$$

Of course the a'_k, a''_k are different from those in (3.7), (3.8), and the corresponding subscripts have been omitted merely to reduce the awkwardness of the notation.

The equation which determines that part of the correction which is due to the transverse component of the temperature gradien has the form:

$$I_1(\Phi) + I'_{12}(\Phi) + f_1^{(0)} [\mathbf{v} \times \boldsymbol{\omega}_1] \nabla_v \Phi = -f_1^{(0)} \left\{ \left(s^2 - \frac{5}{2}\right) \mathbf{v} \nabla_\perp \ln T_1 + \frac{\mathbf{R}_T^{(1)} \mathbf{v}}{n_1 T_1} \right\}. \tag{3.13}$$

We represent the thermal force in the form:

$$\mathbf{R}_T^{(1)} = n_1 T_1 (K' \nabla_\perp \ln T_1 + K'' [\boldsymbol{\omega}_1 \times \nabla \ln T_1]).$$

We introduce the complex quantities $A = A' + i\omega_1 A''$, $K = K' + i\omega_1 K''$. For the function A we obtain

$$I_1(As) + I'_{12}(As) + i\omega_1 f_1^{(0)} As = -f_1^{(0)}\left\{\left(s^2 - \frac{5}{2}\right) + K\right\} s. \tag{3.14}$$

Making use of the orthogonality of Sonine polynomials we obtain an infinite system of equations for the expansion coefficients $a_k = a'_k + i\omega_1 a''_k$

$$\sum_{l=1}^{\infty} (\alpha_{kl} + \alpha'_{kl}) a_l + i\omega_1\tau_1 \frac{\Gamma(k + {}^5/_2)}{k!\,\Gamma({}^7/_2)} a_k = \delta_{1k}. \tag{3.15}$$

The matrix elements α_{ik}, α'_{ik} are evaluated in the Appendix.

For an approximate evaluation of the coefficients a_k one can break off the series (3.10) after the first two terms and also break off in a corresponding manner the infinite system of equations (3.15). As a result one obtains a system of two equations which yield the coefficients a'_1, a'_2, a''_1, a''_2 while (3.11) and (3.12) yield the heat flux and the thermal force. For $|\omega_1|\tau_1 \gg 1$ one can in this way correctly obtain the first two terms in the expansions in powers of $(\omega_1\tau_1)^{-1}$ of the coefficients of interest to us. For $|\omega_1|\tau_1 \lessgtr 1$ there is no such formal parameter which characterizes the accuracy of the calculations, but in spite of this one can hope that the coefficients obtained in this way will give a sufficiently good approximation.

The coefficients a_k depend on two dimensinless parameters: the "magnetization coefficient" $\omega_1\tau_1$ and the effective charge of the ions $Z = n_2 e_2^2 / n e_1^2 \approx e_2/|e_1|$ (the ratio of the matrix elements α_{ik} and α'_{ik} depends on the latter).

Expressions for the thermal force and for the heat flow due to the temperature gradient have the form:

$$\mathbf{R}_{1T} = -n_1\left\{\gamma_{uT}\nabla_{\parallel} T_1 + \frac{1}{\Delta}(\beta'_{uT}x + \gamma'_{uT})\nabla_{\perp} T_1 - \frac{1}{\Delta}(\beta''_{uT}x + \gamma''_{uT})\tau_1[\boldsymbol{\omega}_1 \times \nabla T_1]\right\};$$

$$\mathbf{q}_{1T} = -\frac{n_1 T_1 \tau_1}{m_1}\left\{\gamma_{TT}\nabla_{\parallel} T_1 + \frac{1}{\Delta}(\beta'_{TT}x + \gamma'_{TT})\nabla_{\perp} T_1 - \frac{1}{\Delta}(\beta''_{TT}x + \gamma''_{TT})\tau_1[\boldsymbol{\omega}_1 \times \nabla T_1]\right\}, \tag{3.16}$$

where

$$\Delta = x^2 + \delta_1 x + \delta_0, \quad x = \omega_1^2\tau_1^2. \tag{3.16'}$$

The values of the coefficients for Z = 1, 2, 3, 4 are given in the following table:

	$Z=1$	$Z=2$	$Z=3$	$Z=4$
$\gamma_{TT} = \gamma'_{TT}/\delta_0$	3.1616	4.8901	6.0641	6,9200
$\gamma_{Tu} = \gamma'_{Tu}/\delta = \gamma_{uT}$	0.7110	0.9052	1.0158	1,0904
$\gamma_{uu} = 1 - \gamma'_{uu}/\delta_0$	0.5129	0.4408	0.3965	0,3752
δ_0	3.7703	1.0465	0.5814	0,4106
δ_1	14.79	10.80	9.618	9,055
β'_{TT}	4.664	3.957	3.721	3,604
γ'_{TT}	11.92	5.118	3.525	2,841
β''_{TT}	5/2	5/2	5/2	5/2
γ''_{TT}	21.67	15.37	13,53	12,65
$\beta'_{Tu} = \beta'_{uT}$	5.101	4.450	4.233	4.124
$\gamma'_{Tu} = \gamma'_{uT}$	2.681	0.9473	0.5905	0.4478
$\beta''_{Tu} = \beta''_{uT}$	3/2	3/2	3/2	3/2
$\gamma''_{Tu} = \gamma''_{uT}$	3.053	1,784	1,442	1.285
β'_{uu}	6.416	5.523	5.226	5.077
γ'_{uu}	1.837	0.5956	0.3515	0.2566
β''_{uu}	1.704	1.704	1.704	1.704
γ''_{uu}	0.7796	0.3439	0.2400	0.1957

The correction to the electrón distribution function, necessitated by the exixtence of a relative velocity, is found in a completely analogous manner. The infinite system of algebraic equations for the complex coefficients $a_k = a'_k + i\omega_1 a''_k$ has the form:

$$\sum_{l=1}^{\infty} (\alpha_{kl} + \alpha'_{kl}) a_l + \frac{\Gamma(k + {}^5/_2)}{k!\,\Gamma({}^7/_2)} i\omega_1\tau_1 a_k = \alpha'_{0k}. \tag{3.17}$$

Expressions for the friction force and for the heat flow due to the relative velocity have the following form (in order to obtain the total friction force the term $\mathbf{R}_1^{(0)}$ has also been added):

$$\mathbf{R}_{1u} = \frac{m_1 n_1}{\tau_1}\left\{-\gamma_{uu}\mathbf{u}_{\parallel} - \mathbf{u}_{\perp} + \frac{1}{\Delta}(\beta'_{uu}x + \gamma'_{uu})\mathbf{u}_{\perp} - \frac{1}{\Delta}(\beta''_{uu}x + \gamma''_{uu})\tau_1[\boldsymbol{\omega}_1 \times \mathbf{u}]\right\};$$

$$\mathbf{q}_{1u} = -n_1 T_1\left\{\gamma_{Tu}\mathbf{u}_{\parallel} + \frac{1}{\Delta}(\beta'_{Tu}x + \gamma'_{Tu})\mathbf{u}_{\perp} - \frac{1}{\Delta}(\beta''_{Tu}x + \gamma''_{Tu})\tau_1[\boldsymbol{\omega}_1 \times \mathbf{u}]\right\}. \tag{3.18}$$

The expression for Δ is here the same as in (3.16), The coefficients are given in the table. The coefficients β_{uT}, γ_{uT} coincide with the coefficients β_{Tu}, γ_{Tu}. This coincidence is, of course, not accidental. It amy be shown that it is a consequence of the principle of symmetry of kinetic coefficients (Onsager's principle).

The accuracy of the evaluation of the coefficients given in the table may be checked by calculating some of them in the next approximation. If in expansions (3.6) and (3.10) one keeps not two, but three terms, and if one breaks off in a corresponding manner the infinite systems (3.15) and (3.17), then for the coefficients γ one obtains the values $\gamma_{uT} = 0.7135$, $\gamma_{TT} = 3.1636$, $\gamma_{uu} = 0.5098$ (for $Z = 1$). Making use of the value of the coefficient γ_{uu} from the table one can obtain for the electrical conductivity of the plasma along the magnetic field the expression $\sigma_{\parallel} = 1.950\ n_1 e_1^2 \tau_1 / m_1$. However, if one uses the more accurate value of the coefficient one obtains $\sigma_{\parallel} = 1.962\ n_1 e_1^2 \tau_1 / m_1$. One may consider that the coefficients in the table have an accuracy of the order of 1%.

The equation for the correction to the ion distribution function is solved in the same manner as the one for the electrons. The infinite system of equations for the complex coefficients is analogous to the system (3.15) with only the single difference that cross matrix elements are absent. Therefore the expansion coefficients depend only on the parameter $\omega_2 \tau_2$. The heat flow may be computed from formula (3.11) after replacing the subscript 1 by 2. As a result of this we obtain:

$$\mathbf{q}_2 = -\frac{n_2 T_2 \tau_2}{m_2} \left\{ 3.906\, \nabla_{\parallel} T_2 + \frac{(2\omega_2^2 \tau_2^2 + 2.645)\, \nabla_{\perp} T_2 - \left(\frac{5}{2} \omega_2^2 \tau_2^2 + 4.65 \right) \tau_2\, [\boldsymbol{\omega}_2 \times \nabla T_2]}{\omega_2^4 \tau_2^4 + 2.70\, \omega_2^2 \tau_2^2 + 0.677} \right\}, \tag{3.19}$$

where

$$\tau_2 = 3 \sqrt{m_2}\, T_2^{3/2} / 4 \sqrt{\pi}\, \lambda e_2^4 n_2, \quad \boldsymbol{\omega}_2 = (e_2 / m_2 c)\, \mathbf{H}. \tag{3.20}$$

4. THE VISCOSITY TENSOR

In addition to the term proportional to the temperature gradient the right hand side of equation (2.14) also contains a term which depends on the space derivatives of the hydrodynamic velocity. The correction to the ion distribution function must also contain a corresponding term. The equation for this correction term has the form:

$$I(\Phi) + f^{(0)} [\mathbf{v} \boldsymbol{\omega}]\, \nabla_v \Phi = -f^{(0)} \left(s_i s_k - \frac{s^2}{3} \delta_{ik} \right) w_{ik}, \tag{4.1}$$

The subscript 2 (ions) has been omitted in equation (4.11) for the sake of brevity. The dimensionless variable $\mathbf{s}$ has been introduced in place of the velocity. The following abreviation is used below

$$s_{ik} = s_i s_k - (s^2/3) \delta_{ik}. \tag{4.2}$$

We choose the z axis along the magnetic field. We decompose the tensor w_{ik} into a sum of three tensors $w_{ik} = w_{(0)ik} + w'_{(1)ik} + w'_{(2)ik}$ which in this system of coordinates have the components:

$$w_{(0)} = \begin{pmatrix} \tfrac{1}{2}(w_{xx} + w_{yy}) & 0 & 0 \\ 0 & \tfrac{1}{2}(w_{xx} + w_{yy}) & 0 \\ 0 & 0 & w_{zz} \end{pmatrix}; \tag{4.3}$$

$$w'_{(1)} = \begin{pmatrix} \tfrac{1}{2}(w_{xx} - w_{yy}) & w_{xy} & 0 \\ w_{yx} & -\tfrac{1}{2}(w_{xx} - w_{yy}) & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad w'_{(2)} = \begin{pmatrix} 0 & 0 & w_{xz} \\ 0 & 0 & w_{yz} \\ w_{zx} & w_{zy} & 0 \end{pmatrix}. \tag{4.4}$$

We also introduce tensors with the following components:

$$w''_{(1)} = \omega \begin{pmatrix} -2w_{xy} & w_{xx} - w_{yy} & 0 \\ w_{xx} - w_{yy} & 2w_{xy} & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad w''_{(2)} = \omega \begin{pmatrix} 0 & 0 & -w_{yz} \\ 0 & 0 & w_{xz} \\ -w_{zy} & w_{zx} & 0 \end{pmatrix}. \tag{4.5}$$

These tensors have been chosen in such a way that their traces vanish $w_{ii} = 0$, and the application of the operator $(\mathbf{v} \times \boldsymbol{\omega})\, \nabla_v$ transforms terms of the form $w'_{(1)ik} s_{ik}$, $w'_{(2)ik} s_{ik}$ into corresponding terms with double primes and conversely, and makes the term $w_{(0)ik} s_{ik}$ vanish.

The solution (4.1) may be sought in the form:

$$\Phi = \Phi_{(0)} + \Phi_{(1)} + \Phi_{(2)},$$

where

$$\Phi_{(0)} = -B_{(0)}(s^2)\, w_{(0)ik} s_{ik}, \quad \Phi_{(1)} = -(B'_{(1)} w'_{(1)ik} + B''_{(1)} w''_{(1)ik})\, s_{ik}, \quad \Phi_{(2)} = -(B'_{(2)} w'_{(2)ik} + B''_{(2)} w''_{(2)ik})\, s_{ik}. \tag{4.6}$$

All these terms may be calculated independently of each other.

It is convenient to expand the functions $B(s^2)$ into a series of Sonine polynomials of order $5/2$

$$B(s^2) = \tau \sum_{k=0}^{\infty} b_k L_k^{(5/2)}(s^2). \tag{4.7}$$

The contribution of the term $\Phi_{(0)}$ to the viscosity tensor is equal to

$$\pi_{(0)ik} = -nT\tau b_0 w_{(0)ik}. \tag{4.8}$$

Thus, in order to compute the viscosity it is sufficient to know only the zero order expansion coefficient, and the contribution to the viscosity tensor of $\Phi_{(1)}$ and $\Phi_{(2)}$ may be computed in a completely analogous manner. If one substitutes into (4.1) the expression (4.6) for $\Phi_{(0)}$ then one obtains for the function $B_{(0)}$ the equation

$$I(B_{(0)} s_{ik}) = f^{(0)} s_{ik}. \tag{4.9}$$

For the complex quantities $B_{(1)} = B'_{(1)} + 2i\omega B''_{(1)}$, $B_{(2)} = B'_{(2)} + i\omega B''_{(2)}$ the following equations are obtained:

$$I(B_{(1)} s_{ik}) + 2i\omega f^{(0)} B_{(1)} s_{ik} = f^{(0)} s_{ik}; \tag{4.10}$$

$$I(B_{(2)} s_{ik}) + i\omega f^{(0)} B_{(2)} s_{ik} = f^{(0)} s_{ik}. \tag{4.11}$$

Of Eqs. (4.9), (4.10), (4.11) it is sufficient to solve (4.11). Then by setting $\omega = 0$ we shall obtain the solution of (4.9), and by replacing ω by 2ω we shall obtain the solution of (4.10).

The method of solving (4.11) is the same as the method of solving the corresponding equations of Sec. 3. For the coefficients $b_{(2)}$ we obtain the infinite system of equations

$$\sum_{l=0}^{\infty} \beta_{kl} b_l + \frac{\Gamma(k + 7/2)}{k!\,\Gamma(7/2)} i\omega\tau b_k = \delta_{0k}. \tag{4.12}$$

The matrix elements β_{kl} are calculated in the Appendix. The coefficient b_0 may be obtained by breaking off after the first two terms the expansion (4.7) and corresponding to this also the system (4.12)

$$b_{(2)0} = b' + i\omega\tau b'' = \frac{(6/5\,\omega^2\tau^2 + 2.23) - i\omega\tau(\omega^2\tau^2 + 2.38)}{\omega^4\tau^4 + 4.03\,\omega^2\tau^2 + 2.33}. \tag{4.13}$$

Collecting all the terms we obtain for the components of the viscosity tensor for the ions the expressions:

$$\begin{aligned}
\pi_{zz} &= -0.96\, n_2 T_2 \tau_2 w_{zz},\\
\pi_{xx} &= n_2 T_2 \tau_2 \left\{0.48\, w_{zz} - \frac{1}{2} b'(2\omega)(w_{xx} - w_{yy}) - 2\omega_2\tau_2 b''(2\omega)\, w_{xy}\right\},\\
\pi_{yy} &= n_2 T_2 \tau_2 \left\{0.48\, w_{zz} + \frac{1}{2} b'(2\omega)(w_{xx} - w_{yy}) + 2\omega_2\tau_2 b''(2\omega)\, w_{xy}\right\},\\
\pi_{xy} &= \pi_{yx} = -n_2 T_2 \tau_2 \{b'(2\omega)\, w_{xy} - \omega_2\tau_2 b''(2\omega)(w_{xx} - w_{yy})\},\\
\pi_{xz} &= \pi_{zx} = -n_2 T_2 \tau_2 \{b'(\omega) w_{xy} + w_2\tau_2 b''(\omega) w_{yz}\},\\
\pi_{yz} &= \pi_{zy} = -n_2 T_2 \tau_2 \{b'(\omega) w_{yz} - \omega_2\tau_2 b''(\omega) w_{xz}\}.
\end{aligned} \tag{4.14}$$

Expressions (4.14) differ from the corresponding results given (without proof) by Chapman and Cowling[1] in having more accurate coefficients. The equation which determines the electronic viscosity is analogous to the equation for the ions, and is solved in the same way. The system of equations for the complex coefficients b_k has the form:

$$\sum_{l=0}^{\infty} (\beta_{kl} + \beta'_{kl})\, b_l + \frac{\Gamma(k + 7/2)}{k!\,\Gamma(7/2)} i\omega_1\tau_1 b_k = \delta_{0k}. \tag{4.15}$$

The matrix elements for electron-electron and electron-ion collisions are calculated in the Appendix. The coefficients b depend on the effective charge of the ions Z. For $Z = 1$

$$b' + i\omega_1\tau_1 b'' = \frac{(2.05\omega_1^2\tau_1^2 + 8.50) - i\omega_1\tau_1(\omega_1^2\tau_1^2 + 7.91)}{\omega_1^4\tau_1^4 + 13.8\omega_1^2\tau_1^2 + 11.6}. \quad (4.16)$$

The viscosity tensor for the electrons as well as for the ions has the form (4.14), where one needs only to replace the subscript 2 by 1 and to take the values of the coefficients in accordance with (4.16).

In conclusion I wish to express my deep gratitude to B. I. Davydov for suggesting the problem and for his considerable aid in selecting the method for its solution. I also express my thanks to M. A. Leontovich and to G. I. Budker for numerous useful discussions.

APPENDIX

Calculation of Matrix Elements

In the course of calculations in Secs. 3 and 4 it is necessary to evaluate integrals of the form:

$$M_{pq}^{\alpha\beta} = \int \Psi_p S_{\alpha\beta}(f_\alpha^{(0)}\Phi_q, f_\beta^{(0)})\,dv, \quad N_{pq}^{\alpha\beta} = \int \Psi_p S_{\alpha\beta}(f_\alpha^{(0)}, f_\beta^{(0)}\Phi_q)\,dv. \quad (A.1)$$

The role of Ψ_p and Φ_q is played by functions of the form

$$L_p^{(^3/_2)}\left(\frac{mv^2}{2T}\right)v_i, \quad L_p^{(^5/_2)}\left(\frac{mv^2}{2T}\right)\left(v_i v_k - \frac{v^2}{3}\delta_{ik}\right),$$

where $L_p^{(m)}$ are Sonine polynomials (generalized Laguerre polynomials). These polynomials have the following generating functions:

$$(1-\xi)^{-^5/_2}\exp\left(-\frac{t\xi}{1-\xi}\right) = \sum_{p=0}^{\infty}\xi^p L_p^{(^3/_2)}(t), \quad (1-\xi)^{-^7/_2}\exp\left(-\frac{t\xi}{1-\xi}\right) = \sum_{p=0}^{\infty}\xi^p L_p^{(^5/_2)}(t). \quad (A.2)$$

We shall call integrals of the type (A.1) matrix elements. Instead of calculating these matrix elements individually for each pair of values of p, q, it is convenient to calculate the matrix element of the generating functions (A.2). Then by expanding it in powers of ξ, η we shall obtain the desired matrix elements in the form of coefficients of $\xi^p\eta^q$. For example, for polynomials of order $^3/_2$:

$$M^{12}\left(\frac{3}{2}\right) = \sum_{p,\,q=0}^{\infty}\xi^p\eta^q M_{pq}^{12}\left(\frac{3}{2}\right). \quad (A.3)$$

All the required integrals are calculated by the same method, by integration by parts and change of variables $v_i = c_i + \gamma u_i$, $v'_i = c_i + \gamma' u_i$ where the coefficients γ, γ' are so chosen that the exponent contains no terms of the form $c_i u_i$ and that $v_i - v'_i = u_i$. As a result we obtain

$$M^{12}\left(\frac{3}{2}\right) = \frac{8\sqrt{\pi}\lambda e_1^2 e_2^2 n_1 n_2}{m_1^2}(1-\xi\eta)^{-^5/_2}(1-\xi)^{-1}(1-\eta)^{-1}\left(\frac{\beta_1\beta_2(1+x+y)}{B}\right)^{^1/_2}\left\{1 - \frac{(x+y)\beta_1}{B} + \frac{5xy\beta_1^2}{B^2} + \frac{2xy\beta_1\beta_2}{B^2(1+x+y)}\right.$$

$$\left. - \beta_1\left(1-\frac{T_1}{T_2}\right)\left(\frac{1}{B} - \frac{5x\beta_1}{B^2} - \frac{2x\beta_2}{B^2(1+x+y)}\right)\right\}, \quad N^{12}\left(\frac{3}{2}\right) = -\frac{8\sqrt{\pi}\lambda e_1^2 e_2^2 n_1 n_2}{m_1 m_2}(1-\xi)^{-1}(1-\eta)^{-1}\left(\frac{\beta_1\beta_2(1+x)(1+y)}{A}\right)^{^1/_2}$$

$$\times\left\{1 - \frac{x\beta_1 + y\beta_2}{A} + \frac{3xy\beta_1\beta_2}{A^2} + \beta_2\left(\frac{T_2}{T_1} - 1\right)\left(\frac{1}{A} - \frac{3x\beta_1}{A^2}\right)\right\}. \quad (A.4)$$

Here we have introduced the notation

$$\beta_1 = m_1/2T_1, \quad \beta_2 = m_2/2T_2; \quad (A.5)$$

$$x = \frac{\xi}{1-\xi}; \quad y = \frac{\eta}{1-\eta}, \quad A = \beta_1(1+x) + \beta_2(1+y), \quad B = \beta_1(1+x+y) + \beta_2. \quad (A.6)$$

With the aid of (A.4) it is possible to obtain expressions for all the required matrix elements. For collisions between electrons and ions we have, taking into account that $\beta_1 \ll \beta_2$:

$$\sum_{p,\,q=0}^{\infty}\xi^p\eta^q M_{pq}^{12}\left(\frac{3}{2}\right) = \frac{3n_1}{\tau_1}\frac{T_1}{m_1}(1-\xi\eta)^{-1}(1-\xi)^{-^3/_2}(1-\eta)^{-^3/_2}, \quad \sum_{p,\,q=0}^{\infty}\xi^p\eta^q N_{pq}^{12}\left(\frac{3}{2}\right) = -\frac{3n_1}{\tau_1}\frac{T_2}{m_2}(1-\xi)^{-^3/_2}. \quad (A.7)$$

For collisions between ions and electrons we have after interchanging subscripts 1 and 2 and taking into account that $\beta_1 \ll \beta_2$

$$\sum_{p,\,q=0}^{\infty} \xi^p \eta^q M^{21}_{pq}\left(\frac{3}{2}\right) = \frac{3n_1}{\tau_1}\frac{T_2}{m_1}(1-\xi\eta)^{-7/2}\left\{1 - 5\xi\left(\frac{T_1}{T_2}-1\right) + 4\xi\eta\right\}, \quad \sum_{p,\,q=0}^{\infty} \xi^p \eta^q N^{21}_{pq}\left(\frac{3}{2}\right) = -\frac{3n_1}{\tau_1}\frac{T_1}{m_2}(1-\eta)^{-5/2}. \tag{A.8}$$

For collisions of identical particles, for example of ions with ions, we have to calculate matrix elements of the form: $H^{22}_{pq}(3/2) = M^{22}_{pq}(3/2) + N^{22}_{pq}(3/2)$,

$$\sum_{p,\,q=0}^{\infty} \xi^p \eta^q H^{22}_{pq}\left(\frac{3}{2}\right) = \frac{3n_2}{\tau_2}\frac{2T_2}{m_2}\xi\eta(1-\xi\eta)^{-2}\left(1-\frac{\xi+\eta}{2}\right)^{5/2}\left\{1 - \frac{\xi+\eta}{2} - \frac{\xi\eta}{8} + \frac{\xi\eta(\xi+\eta)}{4} + \frac{3}{8}\xi^2\eta^2\right\}, \tag{A.9}$$

where $\tau_2 \equiv \tau_{22} = 3\sqrt{m_2}\,T_2^{3/2}/4\sqrt{\pi}\,\lambda e_2^4 n_2$. The corresponding expression for the electrons may be obtained from (A.9) by replacing the subscript 2 by 1, with τ_2 being replaced by $\tau_{11} = \sqrt{2}\,Z\tau_1$. The values of the coefficients α_{pq} used in Sec. 3 may be obtained from (A.9). For the ions:

$$\alpha_{pq} = \frac{4}{5}\begin{pmatrix} 0 & 0 & 0 & 0 & . \\ 0 & 1 & \frac{3}{4} & \frac{15}{32} & . \\ 0 & \frac{3}{4} & \frac{45}{16} & \frac{309}{128} & . \\ 0 & \frac{15}{32} & \frac{309}{128} & \frac{6425}{1024} & . \\ . & . & . & . & . \end{pmatrix}. \tag{A.10}$$

The coefficients α_{pq} for the electrons are smaller by a factor $\sqrt{2}\,Z$.

The values of the coefficients α'_{pq} may be obtained from (A.7)

$$\alpha_{pq} = \frac{2}{5}\begin{pmatrix} 1 & \frac{3}{2} & \frac{15}{8} & \frac{35}{16} & . \\ \frac{3}{2} & \frac{13}{4} & \frac{69}{16} & \frac{165}{32} & . \\ \frac{15}{8} & \frac{69}{16} & \frac{433}{64} & \frac{1077}{128} & . \\ \frac{35}{16} & \frac{165}{32} & \frac{1077}{128} & \frac{2954}{256} & . \\ . & . & . & . & . \end{pmatrix} \tag{A.11}$$

Matrix elements with Sonine polynomials of order $5/2$ which are required for calculating the viscosity tensor may be found by the same method which was used in the case of polynomials of order $3/2$. We require only

$$H^{22}_{pq}\left(\frac{5}{2}\right) = \int L_p^{(5/2)}(s^2)\, s_{ik} I_2\left(L_q^{(5/2)}(s^2)\, s_{ik}\right) d\mathbf{v},\ H^{11}_{pq}\left(\frac{5}{2}\right),\ M^{12}_{pq}\left(\frac{5}{2}\right) = \int L_p^{(5/2)}(s^2)\, s_{ik} I'_{12}\left(L_q^{(5/2)}(s^2)\, s_{ik}\right) d\mathbf{v}. \tag{A.12}$$

They are equal to

$$\sum_{p,\,q=0}^{\infty} \xi^p \eta^q M^{12}_{pq}\left(\frac{5}{2}\right) = \frac{3n_1}{\tau_1}(1-\xi)^{-3/2}(1-\eta)^{-3/2}(1-\xi\eta)^2, \quad \sum_{p,\,q=0}^{\infty} \xi^p \eta^q H^{2}_{pq}\left(\frac{5}{2}\right) = \frac{3n_2}{\tau_2}\left(1-\frac{\xi+\eta}{2}\right)^{-5/2}(1-\xi\eta)^{-1}\left\{1 - \frac{\xi+\eta}{2}\right.$$

$$\left. + \xi\eta\left[\frac{7}{3} + \frac{2\xi\eta - (\xi+\eta)/2 - \xi\eta(\xi+\eta)/12}{1-\xi\eta} + \frac{\xi\eta}{3(1-\xi\eta)^2} + \frac{5}{16}\frac{\xi\eta}{1-(\xi+\eta)/2}\right]\right\}. \tag{A.13}$$

The expressions for H^{11}_{pq} are obtained from H^{22}_{pq} by replacing 2 by 1 and τ_2 by $\tau_{11} = \sqrt{2}\,Z\tau_1$. The coefficients β_{pq}, β'_{pq} are equal to:

$$\beta_{pq} = \frac{6}{5}\begin{pmatrix} 1 & \frac{3}{4} & . \\ \frac{3}{4} & \frac{205}{48} & . \\ . & . & . \end{pmatrix}, \quad \beta'_{pq} = \frac{6}{5}\begin{pmatrix} 1 & \frac{3}{2} & . \\ \frac{3}{2} & \frac{17}{4} & . \\ . & . & . \end{pmatrix} \tag{A.14}$$

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[1] S. Chapman and T. G. Cowling, The Mathematical Theory of Non-Uniform Gases, Cambridge (1939).

[2] L. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) 7, 203 (1937).

[3] T. G. Cowling, Proc. Roy. Soc. A183, 453 (1945).

[4] R. Landshoff, Phys. Rev. 76, 904 (1949).

[5]E. S. Fradkin, J. Exptl. Theoret. Phys. (U.S.S.R.) **32**, 1176 (1957), Soviet Phys. JETP **5**, 956 (1957).
[6]A. Schlüter, Z. F. Naturforsch. **5a**, 72 (1950).
[7]H. Alfvén, Cosmical Electrodynamics, Oxford (1950).

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ANOMALOUS EQUATIONS FOR SPIN 1/2 PARTICLES

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Irreducible relativistic wave equations different from the Dirac equations are derived for spin $\frac{1}{2}$ particles. The particles described by these equations may have one or more proper masses and the corresponding fields have a positive definite charge density. The existence of such equations is not incompatible with the well known proof of uniqueness of the Dirac equations.

ATTEMPTS to construct wave equations of the type

$$(\beta^k \partial_k - i\varkappa)\varphi = 0 \tag{1}$$

but different from the Dirac equations have been unsuccessful. The only exception is the set of equations due to Petras.[1] But these equations are not investigated in details in his paper, and it is therefore not clear whether they meet all of the physical requirements.

The known proofs by Wild[2] and by Gel'fand and Iaglom[3] on the uniqueness of the Dirac equations seemed to imply that anomalous equations for spin $\frac{1}{2}$ particles do not exist, and that the equations of Petras[1] are physically unacceptable.

The proofs of the uniqueness of the Dirac equation are, however, not general. They rely on the assumption that the equation for spin $\frac{1}{2}$ particles can be derived only with the representation of the total Lorentz group for the maximum value of spin $\frac{1}{2}$.

In this paper we start from the representation of D_R for the maximum value of spin $\frac{3}{2}$ and we show that there exist anomalous equations for spin $\frac{1}{2}$ particles, the simplest of them being the Petras equations.

1. RELATIVISTIC FORM OF ANOMALOUS WAVE EQUATIONS

If the equations (1) are covariant, the matrices β_k have to satisfy the known relationships

$$[\beta_k I_{lm}] = g_{kl}\beta_m - g_{km}\beta_l; \tag{2a}$$

$$[I_{kl} I_{mn}] = -g_{km} I_{ln} + g_{lm} I_{kn} + g_{kn} I_{lm} - g_{ln} I_{km}; \tag{2b}$$

$$\beta_j Z = Z \beta^j, \tag{2c}$$

where $I_{k\ell}$ represent infinitesimal rotations and Z is the matrix of space reflection.

If the representation of the total Lorentz group D_R is known, Eqs. (2) can be used to find the form of the matrix β_0. The remaining matrices β_ν ($\nu = 1, 2, 3$) can be determined from (2a).

It is known that there are three non-equivalent irreducible representations of the operators $I_{k\ell}$ and Z for the maximum value of spin $\frac{3}{2}$, with 12, 8 and 4 rows respectively, which we shall denote by τ_3, τ_2 and τ_1.