THE BEHAVIOR OF A COMPLETELY IONIZED PLASMA IN A STRONG MAGNETIC FIELD*

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In the first part of this work certain paradoxes are considered which arise when the transfer equations are applied to a quasi-stationary, completely ionized plasma in the case in which the frequency of collisions between particles is much smaller than their frequency of rotation in the magnetic field. In the second part a cylindrical pinched plasma is considered in which the plasma pressure is balanced by the electrodynamic forces produced by the current which flows through the pinched plasma. Several solutions of the plasma equations are found for stationary and non-stationary cases.

1. CERTAIN PARADOXES

THE nature of the particle motion in a plasma depends critically on the ratio between the collision frequency $1/\tau$ and the frequency of rotation in the magnetic field ω. When $\omega\tau \ll 1$ the particles move in virtually straight lines between collisions and the influence of the magnetic field on kinetic effect (diffusion coefficient, heat conductivity, etc.) is small. In the second case, when $\omega\tau \gg 1$, the magnetic field has a strong effect on the plasma kinetics. In this case, if one compares the results obtained by a direct application of the transfer equations with those which are to be expected on the basis of the motion of individual particles in a magnetic field, there are seeming contradictions and paradoxes. Certain of these paradoxical situations have been considered by Alfvén and Spitzer;[1,2] a number of paradoxes are considered by us below.

We shall limit ourselves to the case in which the gradients of all quantities and the electric field are perpendicular to the magnetic field. The transfer equations from Ref. 3 will be used. We will assume that the processes involved are so slow that all quantities change only slightly in the time between collisions. It is easy to show that in this case, we can neglect the terms $d\mathbf{v}/dt$ and $\operatorname{div} \sigma$ (inertia and viscosity) in the projections of the transfer equations on the plane perpendicular to the magnetic field. Using these equations it is possible to obtain explicit expressions for the components of electron velocity $\mathbf{v}_e$ and ion velocity $\mathbf{v}_i$ perpendicular to the magnetic field in terms of the gradients

$$\begin{aligned} \mathbf{v}_e &= \frac{c}{H^2}[\mathbf{E}\times\mathbf{H}] - \frac{c}{eH^2n}[\mathbf{H}\times\nabla nT_e] - \frac{mc^2}{e^2H^2\tau}\left\{(T_e+T_i)\frac{\nabla n}{n} + \nabla T_i - \frac{1}{2}\Delta T_e\right\}, \\ \mathbf{v}_i &= \frac{c}{H^2}[\mathbf{E}\times\mathbf{H}] + \frac{c}{eH^2n}[\mathbf{H}\times\nabla nT_i] - \frac{mc^2}{e^2H^2\tau}\left\{(T_e+T_i)\frac{\nabla n}{n} + \nabla T_i - \frac{1}{2}\nabla T_e\right\}. \end{aligned} \tag{1}$$

where m is the mass of the electron, c is the velocity of light, $1/\tau$ is the frequency of electron collisions with ions, T_e and T_i are the electron and ion temperatures. Here we carry out an expansion in powers of $1/H$ and discard terms of order higher than $1/H^2$. The plasma is assumed to be comprised of electrons and singly-charged ions with charge e. The number of particles of each type, as is generally assumed for a plasma, is taken to be the same, $n_i = n_e = n$. The velocity components along the magnetic field cannot be expressed directly in terms of the gradients; we will assume that these are all zero. Using Eq. (1) we find that the electric current depends only on the pressure gradient $p = n(T_e + T_i)$ and is perpendicular to this quantity

$$\mathbf{j} = en(\mathbf{v}_i - \mathbf{v}_e) = cH^{-2}[\mathbf{H}\times\nabla p]. \tag{2}$$

It is of interest to compare Eq. (1) with the picture based on the motion of individual particles in the magnetic field in order to interpret conveniently the various terms in these expressions.

*This work was performed in 1952.

As is well known, in a strong magnetic field a particle moves along the magnetic field lines, rotating about these lines with a frequency $\omega = eH/mc$. The presence of the electric field and a small inhomogeneity in the magnetic field means that the center of the circle moves across the magnetic field with a velocity $\mathbf{V}$

$$\mathbf{V} = \frac{c}{H}[\mathbf{E}\times\mathbf{h}] - \frac{v_\perp^2}{2\omega}\left[\frac{\nabla H}{H}\times\mathbf{h}\right] - \frac{\overline{v_\parallel^2}}{\omega}[(\mathbf{h}\cdot\nabla)\,\mathbf{h}\times\mathbf{h}]. \tag{3}$$

Here $\mathbf{h} = \mathbf{H}/H$ is a unit vector in the direction of the magnetic field. The first term in Eq. (3) is usually called the electric-drift term, the second the magnetic-drift term and the third the centrifugal-drift term. If we average Eq. (3) over a Maxwellian distribution we find an expression for the flow of circle centers (the charge of the particles is assumed to be positive)

$$\bar{\mathbf{V}} = \frac{c}{H}[\mathbf{E}\times\mathbf{h}] - \frac{cT}{e}\left[\frac{\nabla H}{H} + (\mathbf{h}\cdot\nabla)\,\mathbf{h}\times\mathbf{h}\right]. \tag{4}$$

Using the condition $\mathbf{H}\cdot\operatorname{curl}\mathbf{H} = 0$ we can transform this expression to the form

$$\bar{\mathbf{V}} = \frac{c}{H}[\mathbf{E}\times\mathbf{h}] + \frac{cT}{e}\operatorname{curl}\frac{\mathbf{h}}{H}. \tag{4'}$$

Thus, the first term in Eq. (1) may be interpreted easily—it is the electric-drift term.

The second term in Eq. (1), which we will call the Larmor flux, is related to the fact that particles which intersect any area, moving from right to left and from left to right, move between regions with different densities and temperatures, as a result of which the fluxes do not cancel. The particles travel a distance of order $\rho = mvc/eH$ and "carry" a flux nv, hence, the resultant flux is proportional to $\nabla n\bar{v}^2 \sim \nabla nT$.

At first glance, it seems to be paradoxical that there are no terms in Eq. (1) which correspond to the magnetic and centrifugal drift terms and which contain explicitly spatial derivatives of the magnetic field. Actually, however, the absence of these terms is completely reasonable since the magnetic field, whether uniform or not, does not disturb the Maxwellian distribution. Hence, if the density and temperature of the particles are independent of coordinates, the particle flux should not be coordinate-dependent (cf. for example, Ref. 2). Under these conditions the magnetic and centrifugal drift terms appear as edge effects, giving rise to "by-pass" currents at the boundaries of the region characterized by constant density and temperature. One is easily convinced of this by a simple example. If we re-write the terms in Eq. (1) of order $1/H$ in the form

$$n\mathbf{v} = \frac{cn}{H}[\mathbf{E}\times\mathbf{h}] + \frac{cnT}{e}\operatorname{curl}\frac{\mathbf{h}}{H} - \operatorname{curl}\left(\frac{cnT}{eH}\mathbf{h}\right) = n\bar{\mathbf{V}} - \operatorname{curl}\left(\frac{cnT}{eH}\mathbf{h}\right),$$

it becomes immediately obvious that the difference between the flux of "centers" and the flux of particles through any area is completely determined by the values of the quantities at the boundaries of this area. This difference is due to the fact that some particles pass close to the edges of the area and the "centers" of the circles associated with these do not pass through the area. The flux created by these particles is cnT/eH (per unit length of edge along $\mathbf{h}$) where the opposite edges of the area are traversed in opposite directions.

The last terms in Eq. (1) (these are enclosed in curly brackets), in contrast with the first terms, depend on particle collisions, in particular, collisions of electrons with ions. These terms may provisionally be called diffusion terms. They are the same for ions and electrons and depend only on the density gradient and temperature gradient but are independent of the electric field. At first glance both of these results seem paradoxical. Actually, the factor which characterizes the diffusion of particles across the magnetid field is of order $D \sim \rho^2/\tau$ because a particle which does not experience collisions circles about one point and, as a result of collisions, is displaced randomly by an amount of the order of the Larmor radius ρ during the time between impulses τ. In an ion of mass M, both ρ and τ are larger by a factor of $\sqrt{M/m}$ than for electrons, hence it is to be expected that the ion diffusion coefficient should be $\sqrt{M/m}$ times larger than the electron diffusion coefficient. In actuality, however, this is not the case. Suppose, for example, there is an ion-density gradient along the x-axis and the magnetic field is along the z-axis; under these conditions there will be produced a Larmor ion flow along the y-axis with a velocity $v_y = (cT/eHn)\,dn/dx$. In this case it is impossible to apply the formula for ρ^2/τ directly in estimating the diffusion coefficient because the diffusion takes place in a moving medium and in collisions an ion will receive, on the average, some momentum along the y-axis. We now go over to a system of coordinates in which $v_y = 0$. In this system there is an electric field $E_x = (v_y/C)H = (T/e) d \ln x/dx$ in which the ions have a Boltzmann distribution and the ion flux vanishes because the motion cancels the diffusion. Thus collisions between identical particles do not result in diffusion of these particles across the magnetic field. Collisions between electrons and ions, however, lead to "diffusion" because the Larmor flows of ions and electrons are in opposite directions. The resultant flux along the x-axis may be considered as the result of drift produced by the effect of the friction force between electrons and ions R directed along the y-axis. Inasmuch as $R_{ion} = -R_{elec}$, the velocities of both kinds of particles are identical.

We now consider the role of the electric field. Suppose there is an electric field along the z-axis. It produces drift of particles of both signs along the y-axis with a velocity $v_y = -cE/H$. In the system of coordinates in which $v_y = 0$ the electric field $E_x' = 0$ so that no flux due to an electric field is produced along the y-axis.

It is well known that when a magnetic field changes slowly in time the energy associated with the transverse motion of particles $\epsilon_\perp = mv_\perp^2/2$ changes in proportion to the field — the betatron effect. The heat transfer equation does not contain a term proportional to $\partial H/\partial t$. However, it is easy to show from a simple example that the betatron effect is taken into account.

Suppose, for example, that a uniform magnetic field is directed along the z-axis and increases with time. Let the plasma occupy a cylindrical volume which is infinite along the z-axis. We will assume that the temperature and density of the plasma are constant over the volume (thus there will be no heat flow) and, for simplicity, we neglect collisions between electrons and ions. We also neglect the shielding of the external magnetic field by currents in the plasma. Then the induction electrical field is $E = E_\varphi = -\dot{H}r/2c$. The electrical drift leads to a contraction of the plasma with velocity $v_r = -\dot{H}r/2H$. The heat-transfer equation assumes the form

$$\frac{3}{2} n \frac{dT}{dt} = -nT \operatorname{div} \mathbf{v} = -nT(\dot{H}/H).$$

This expression is precisely the betatron effect. Actually, in betatron heating only the energy associated with the transverse motion $d\epsilon = \epsilon_\perp dH/H$ is increased directly. Collisions establish an equal distribution over the various degrees of freedom so that $\epsilon_\perp = (2/3)\epsilon$ and $(3/2)\, d\epsilon/dt = (\epsilon/H)\, dH/dt$. In this case the betatron effect appears as heating due to the adiabatic contraction of the plasma.

We now consider the case in which the ion-electron collisions exactly "balance" the electric drift so that the plasma is at rest. In this case heat fluxes are produced in the plasma, hence, we shall consider the total increase in the energy of the plasma. If the heat-transfer equations for the ions and electrons are added and integrated over the volume of the plasma cylinder (unit height along the z-axis), we obtain

$$\frac{3}{2} \frac{d}{dt} \int n (T_e + T_i)\, 2\pi r\, dr = \int Ej 2\pi r dr.$$

Substituting everywhere $E = -\dot{H}r/2c$, $j = (cH)\, \partial p/\partial r$ and integrating by parts we again find $(3/2)\, d\epsilon/dt = (\epsilon/H)\, dH/dt$. In this case the betatron effect is manifest in the evolution of Joule heat.

The two cases which have been considered differ in one respect. In the first case the ions and electrons are heated in the same way by the contraction. In the second case the heat is generated directly in the electron gas and only then, by collisions, is it transferred to the ions, although it would appear that in betatron heating the ions should obtain as much heat as the electrons. The point is, however, that in the absence of an ion current in the plasma a radial electric field is produced; the magnitude of this field is determined by the ion balance condition $enE_r = -\partial n T_i/\partial r$. This field causes an ion drift in the azimuthal direction against the vortex electric field. It is easy to show that its work (negative) on this drift exactly compensates for the betatron heating of the ions. The electrons also drift in the radial field and acquire as much energy as is lost by the ions.

2. CONTRACTION OF THE PLASMA DUE TO HIGH SELF-CURRENT

1. Basic Equations

The magnetic field produced by a large current flowing through the plasma has a very strong effect on the motion of particles in the plasma and on the configuration of the plasma as a whole. The joint effect of the electric field which gives rise to the current and the magnetic field produced by this current is a "drift" of charged particles of both signs within the current-carrying channel; as a result the plasma contracts into a more or less narrow filament. Considering this picture phenomenologically, we can attribute the contraction of the plasma to the mutual attraction between the elementary "filamentary currents" which make up the total current.

As has already been shown in a number of papers (cf. for example, Ref. 6), such a filament is unstable against various deformations; nonetheless it is of interest to study the problem, even if only from the point of view of methodology. The stationary filament has already been considered by Schlüter.[5]

We will consider a plasma filament with cylindrical symmetry, neglecting end effects (electrode effects). The gradients of all quantities are directed along the radius r and the electric current flows along the z-axis while the magnetic field is azimuthal.

It will be assumed that all changes in the plasma occur so slowly that we can neglect "inertia" effects, in which case the momentum balance equations assume the form of equilibrium equations. The magnetic field is assumed large so that $\omega\tau \gg 1$ both for electrons and ions (ω is the Larmor frequency, $1/\tau$ is the collision frequency). The expressions for the kinetic coefficients are expanded in powers of $1/\omega\tau$ and small terms are neglected. It will be assumed that the density of electrons and ions (singly charged) is virtually the same $n_i = n_e = n$. A sufficiently dense plasma is always characterized by this so-called quasi-neutral property. It will also be assumed that the ion and electron temperatures are the same $T_e = T_i = T$.

Under these assumptions the system of basic equations is of the form (cf. Ref. 3):

$$\frac{\partial E}{\partial r} = \frac{1}{c}\frac{\partial H}{\partial t}; \tag{1a}$$

$$\frac{\partial rH}{r\partial r} = \frac{4\pi}{c} j; \tag{1b}$$

$$\frac{\partial n}{\partial t} + \frac{\partial rnv}{r\partial r} = 0; \tag{1c}$$

$$j = \sigma\left(E + \frac{v}{c}H\right) - \frac{3}{2}\frac{cn}{H}\frac{\partial T}{\partial r}; \tag{1d}$$

$$-\frac{\partial P}{\partial r} = -2\frac{\partial n T}{\partial r} = \frac{1}{c} jH; \tag{1e}$$

$$3n\left(\frac{\partial T}{\partial t} - v\frac{\partial T}{\partial r}\right) + 2nT\frac{\partial rv}{r\partial r} + \frac{\partial rq}{r\partial r} = \left(E + \frac{v}{c}H\right)j - \beta n^2\sqrt{T}. \tag{1f}$$

Here Eqs. (1a) and (1b) are Maxwell's equations, E is the axial component of the electric field and H is the magnetic field. Eq. (1c) is the equation of continuity and v is the radial velocity both for ions and electrons. The expression for the current density (1d) is obtained from the electron equilibrium equation if it is projected on the z-axis: $en(E + vH/c) = R$. The force R is made up of friction forces, which depend on the relative velocities between the electrons and ions $(m/e\tau)j$ and on thermal forces which depend on the electron temperature gradient $(3/2)(n/\omega\tau)\,\partial T/\partial r$. The time between ion-electron collisions τ and the coefficient of electrical conductivity σ are expressed as follows: ("the Coulomb logarithm" $\lambda \approx 10$):

$$\tau = 3\sqrt{m}\,T^{3/2}/4\sqrt{2\pi}\,\lambda e^4 n \approx 3.5\cdot 10^4 T_{\mathrm{ev}}^{3/2}/n_{\mathrm{cm}^{-3}}\ (\mathrm{sec}), \tag{2}$$

$$\sigma = e^2 n\tau/m = \sigma_1 T^{3/2}, \quad \sigma_1 = 0.9\cdot 10^{13}\,\mathrm{sec}^{-1}(\mathrm{ev})^{-3/2}. \tag{3}$$

We may note that the electrical conductivity depends only on the temperature. Eq. (1e) is the plasma equilibrium equation projected on the radius; Eq. (1f) is the total heat balance equation. The main role in the transfer of heat across the strong magnetic field is played by ions as a consequence of the fact that the Larmor radius for the ions is so much larger than that for the electrons. In Eq. (1f) we will assume only an ionic flow; hence we have the following expression for the heat conductivity

$$q = -\varkappa\partial T/\partial r, \quad \varkappa = \sqrt{2M/m}\,mc^2 nT/e^2\tau H^2 \tag{4}$$

(M is the mass of the ion, m is the mass of the electron).

The first term on the right side of Eq. (1f) is to be associated with the generation of Joule heat while the second term is the heat loss produced as a consequence of electron bremsstrahlung. The coefficient β can be obtained using the results given by Heitler.[4] Averaging over the Maxwellian electron distribution we have

$$\beta = 32\sqrt{2}\, e^2 / 3\sqrt{\pi}\cdot 137\, m^{3/2} c^2. \tag{5}$$

If inertia effects are taken account in the equation for the radial motion of the plasma, in place of Eq. (1e) we obtain

$$Mn\,(\partial v / \partial t + v \partial v / \partial r) = -\partial p / \partial r - jH / c. \tag{1e$'$}$$

We now compare the inertia term in (1e′) with the pressure force. The first quantity is of order Mnv/t, where t is the characteristic time for radial motion. In the case of uniform motion we may set $t \sim a/v$ where a is the radius of the plasma cylinder. The second term is of order p/a. Thus the ratio of the inertia term to the pressure term is of order Mv^2/T and if the radial velocity is small compared with the thermal velocities of the ions the inertia effects can be neglected. Suppose for example, $a \sim 10$ cm, $t \sim 10^{-5}$ sec; then $Mv^2 \sim 1$ Mev which, in cases of practical interest, is always much smaller than the energy associated with thermal motion.

If, at the starting time, the radial distribution of current density and pressure is such that the pressure and magnetic attraction are not balanced at each point, or if there is a sudden change in the current strength, a rapid redistribution of matter takes place in the plasma; this effect can be accompanied by the production of intense shock waves. In this case the velocities associated with the motion may become "supersonic" and the inertia effect can play an important role (cf. for example, Ref. 7). Below we shall consider only the "continuous" cases in which inertia effects can be neglected.

Substituting the expression for the current from (1d) in (1e) we find that the radial velocity is composed of an electric particle drift toward the axis and a diffusion outwards, resulting from collisions

$$v = -\frac{cE}{H} - \frac{2mc^2T}{e^2\tau H^2}\left(\frac{1}{n}\frac{\partial n}{\partial r} + \frac{1}{4T}\frac{\partial T}{\partial r}\right). \tag{6}$$

The filament is considered in the coordinate system ("laboratory") in which the ion velocity along the axis is equal to zero. Under these conditions, the equilibrium equations, written separately for ions and electrons, assume the form

$$-\partial (nT) / \partial r + enE_r = 0; \tag{7}$$

$$-\partial (nT) / \partial r - enE_r = jH/c. \tag{8}$$

The ions are constrained in the plasma column by the radial electric field E_r which transfers the pressure of the ion to the electrons. If $\partial T/\partial r = 0$, the ions have a Boltzmann distribution in this field.

Determining the current density from (8) and integrating over radius, we have

$$J = \int j 2\pi r dr = -e \int n \left\{\frac{cE_r}{H} + \frac{cnT}{eH}\left(\frac{1}{H}\frac{\partial H}{\partial r} - \frac{1}{r}\right)\right\} 2\pi r dr.$$

This expression shows graphically the mechanism which makes it possible for the electrons to carry current across the magnetic field — the drift of electrons under the effect of the radial electric field and the inhomogeneous magnetic field.

Neglecting inertia effects it is possible to obtain a simple relation between the total current flowing in the plasma and the mean particle temperatures. If the expression for the current (1e) is substituted in (1b), multiplied by $c^2Hr^2/2$, and integrated from $r = 0$ to the edge of the plasma cylinder $r = a$ and if it is assumed that $H(a) = 2J/ca$, where J is the total current, we find

$$J^2 = -2\pi c^2 \int_0^a r^2 \frac{\partial p}{\partial r} dr = 2c^2 \left(\int_0^a p 2\pi r dr - \pi a^2 p(a)\right)$$

When the pressure at the edge can be neglected as compared with the internal pressure, it follows that

$$J^2 = 4c^2 N\overline{T},\quad N = \int_0^a n 2\pi r dr,\quad \overline{T} = N^{-1}\int_0^a nT 2\pi r dr. \tag{9}$$

Here N is the number of particles per unit length of filament, $\overline{T}$ is the mean value of the temperature. The relation given in (9) has already been obtained by Schlüter[9] for the case in which the current density and temperature are constant over the radius.

We consider a numerical example. If $N = 10^{18}$, the plasma temperature is $T = 1$ kev at a current of $J = 800$ kiloamperes.

We now estimate the difference between the ion temperature and the electron temperature. For this purpose we compare the Joule heat which has been obtained j^2/σ with the heat transfer due to collisions between electrons and ions. $\Delta Q = (3mn/M\tau)(T_e - T_i)$. If we take Eq. (9) into account the comparison yields $\Delta Q/j^2/\sigma \sim \Pi(T_e - T_i)/T$ where

$$\Pi = e^2N/Mc^2. \tag{10}$$

If $\Pi \gg 1$, the assumption made above that $T_e = T_i = T$ is valid. This dimensionless criterion implies that the ion Larmor radius ρ_i is small compared with the radius of the filament. A simple estimate, taking Eq. (9) into account, yields $\rho_i/a \sim \Pi^{-1/2}$. The dimensionless quantity Π is an important similitude criterion in plasma theory, particularly, in pinched plasmas. A high numerical value of Π is a necessary condition for the applicability of the system of magneto-hydrodynamic equations used above. We shall assume that this condition is satisfied.

2. Stationary Mode

The system of equations given in (1) becomes considerably simpler if all the quantities are time independent. In this case Eq. (1a) and (1c) yield $E = \text{const}$ and $v = 0$.

The temperature change over the radius may be assumed small. If we compare the order of magnitude of the Joule heat j^2/σ with the divergence of the heat flux, which is of order $\kappa(dT/dr)/a$ we find $(a/T)\,dT/dr \sim \sqrt{m/M}$. We can obtain an approximate solution by assuming the temperature constant. In this case the thermal forces vanish and the electric conductivity is constant. Hence, the current density is constant, and the remaining quantities are easily determined

$$H = (2J/ca)\,r/a, \quad n = (2N/\pi a^2)(1 - r^2/a^2), \quad T = \text{const}. \tag{11}$$

This solution has already been obtained by Schlüter.[5] In accordance with Eq. (9), the temperature is completely determined by the total current which flows through the filament. The electric field decreases with increasing current

$$E = J/\sigma\pi a^2 = 8c^3N^{3/2}/\sigma_1\pi a^2J^2. \tag{12}$$

Naturally, the stationary mode can be realized only if the heat generated in the filament is removed. For example, the plasma filament could have its outer surface in contact with the walls of a cylindrical discharge chamber, thereby transferring heat to the walls by conduction. Another possibility is the energy loss from the filament via radiation. In the first case the radius of the filament must be equal to the radius of the discharge chamber a_0; in the second case the radius of the filament can be arbitrary (but must be smaller than the radius of the chamber).

Strictly speaking, in a filament which is in contact with the walls the temperature cannot be constant over the entire cross section since there must be a drop in the region of the walls. Nonetheless, the solution given in (11) is still a good approximation. Numerical integration of the system given in (1) in conjunction with (3), (4) (for $\partial/\partial t = 0$ and without radiation) indicates that the behavior of all quantities over the larger portion of the radius is approximately the same as that given in (11) while close to the edge, where the density and heat conductivity are much lower, the temperature falls off sharply and then rapidly approaches zero.

There is another remark which must be made concerning the central portion of the filament. A detailed investigation shows that close to the axis of the filament, over a region of width of order $a\Pi^{1/4}$, the difference between the ion temperature and electron temperature is of order $\Pi^{-1/2}$ (not Π^{-1}). In this central region the thermal force is of importance; as a consequence the surrent density is larger than in the remaining portion of the filament. Calculations carried out using a separate heat balance equation for electrons (assuming that $\omega\tau \gg 1$) indicate that the ratio of the current density at the axis to that outside the central region is 1.93.

The energy balance relation in the filament (per unit length) is of the form

$$EJ = Q_{\text{heat}} + Q_{\text{rad}}, \quad Q_{\text{heat}} = 2\pi a q(a), \quad Q_{\text{rad}} = \beta n^{-2}T^{1/2} = \frac{4}{3}\beta T^{1/2}\frac{N^2}{\pi a^2},$$

where the first term is the flux to the wall while the second is the energy loss due to bremsstrahlung. Using Eqs. (9) and (12) it is easy to show that the ratio of radiation to Joule heat depends only on the magnitude of the current in the filament $Q_{\text{rad}}/EJ = J^2/J^2_{\text{rad}}$, where

$$J_{\text{rad}} = (12\,c^4/\beta\sigma_1)^{1/2} = (3\pi\cdot 137\lambda/2)^{1/2} mc^3/e \approx 1{,}4\cdot 10^6\,\text{A}. \tag{13}$$

If the current is smaller then J_{rad}, part of the Joule heat is radiated and the remaining part is removed by conductivity. When $J = J_{rad}$ (9) and (12) yield

$$T = J_{\text{rad}}^2/4c^2N \approx 3\cdot 10^{21}/N\ \text{ev};\ a = (2\beta/3\pi c)^{1/2} N^{3/4}/E^{1/2}. \tag{14}$$

It is impossible to have a current larger than that given in (13) in the stationary mode since the filament cannot be heated beyond the temperature given in (14).* An increase in the electric field in the filament leads to further compression and increased radiation because the temperature and current do not change. The volt-ampere characteristic of the stationary plasma filament thus consists of two parts: a branch with negative slope (12) and the vertical line $J = J_{rad}$. Finally, the value of the current given in (13) is the limiting value for the stationary mode only. In the non-stationary mode the current can exceed the value given in (13).

3. Non-Stationary Modes

In the non-stationary case (1) becomes a system of nonlinear partial differential equations, the solution of which is, in general, extremely difficult. We shall consider only a certain class of solutions in which all quantities change with time in such a way that the distribution over cross section maintains similitude as a function of time. Solutions of this type are usually called "self-similar" solutions. The idea of looking for self-similar solutions was suggested by A. D. Sakharov.

In place of the radius we introduce the variable $x = r/a(t)$. The variables t and x are separable and we perform the integration of usual equations. The equation of continuity indicates that the velocity in the self-similar solution changes in accordance with the relation $v = \dot{a}x$. As in the stationary mode, it may be assumed with reasonable accuracy that the temperature is constant over the radius. In the self-similar solution the magnetic field is expressed in the form $H = (2J/ca)h(x)$ where $h(0) = 0$ and $h(1) = 1$. Using Eqs. (1a) (1b) and (1d) we obtain the following equation for $h(x)$

$$\frac{d}{dx}\left(\frac{1}{x}\frac{dxh}{dx}\right) = k^2 h, \tag{15}$$

where

$$k^2 = (4\pi a^2\sigma/c^2 J)\,dJ/dt = (\sigma_1\pi a^2 J^2/2c^5 N^{3/2})\,dJ/dt. \tag{16}$$

In what follows we shall consider for simplicity only the case in which $dJ/dt > 0$. The solution of Eq. (15), and then all the remaining quantities, are easily expressed in terms of Bessel functions of imaginary argument

$$H = (2J/ca)\,I_1(kx)/I_1(k); \tag{17a}$$

$$n = (N/\pi a^2)\,[I_0^2(k) - I_0^2(kx)]/I_1^2(k); \tag{17b}$$

$$E + \frac{v}{c}H = \frac{2}{c^2}\frac{dJ}{dt}\frac{I_0(kx)}{kI_1(k)}. \tag{17c}$$

This solution applies if the quantity k remains constant in time. The quantity k gives the ratio of the filament radius to the thickness of the skin-layer and characterize the degree of non-stationarity of the process.

When $k \ll 1$ there is no skin effect and the distribution of all quantities over the cross section of the filament is very similar to that which obtains in the stationary mode.

When $k \gg 1$ the skin effect is pronounced and the plasma density is constant almost over the entire cross section; at the edge the density falls off sharply and the current and magnetic field penetrate the filament to a small depth. At the edge the magnetic field forms a sort of magnetic wall; at this wall the electrons experience a sharp deflection, being reflected in a way similar to that in which molecules of a gas are reflected at an ordinary solid wall. The ions in the filament are constrained by the radial electric field which is formed close to the edge of the analogous "electric wall." The "magnetic wall" acts

*A similar result has been obtained by R. S. Pease [Proc. Phys. Soc. B70, 445 (1957)].

something like a piston, compressing the ionized gas and heating it. If the compression takes place slowly so that the energy of the particles can become equalized over all three degrees of freedom by collisions, this heating obeys the usual adiabatic relation $T^{3/2}/n$ = const. Writing Eq. (16) in the form $k^2 = (4\dot{J}\tau/J)\Pi$ it is easy to show that the current can change slightly during the time between collisions for $k^2 \gg 1$, if Π is sufficiently large.

Equations (17) contain $J(t)$ and $a(t)$. The time dependence of the current is determined by the properties of the electrical circuit of which the plasma filament is a part. The time change of the filament radius is determined by the energy balance condition: the heating (cooling) due to compression (expansion) of the filament must be such that at each moment the condition which relates the temperature and current is satisfied. Integrating Eq. (1f) over the cross section we find, assuming that the filament is not in contact with the wall, $q(a) = 0$

$$\frac{3}{T}\frac{dT}{dt}+\frac{4}{a}\frac{da}{dt}=\frac{J^2}{NT\sigma\pi a^2}\frac{k^2}{4}\left(\frac{I_0^2(k)}{I_1^2(k)}-1\right)-\frac{Q_{\mathrm{rad}}}{NT}, \tag{18}$$

where Q_{rad} differs from $(J^2/\sigma\pi a^2)(J^2/J_{\mathrm{rad}})$ by only a factor on the order of unity. If radiation is neglected, as may be done when $J \ll J_{\mathrm{rad}}$, and the relation between the current and temperature and Eq. (16) are used, we find

$$\frac{1}{a}\frac{da}{dt}=-\left(\frac{5}{2}-\frac{I_0^2(k)}{I_1^2(k)}\right)\frac{1}{J}\frac{dJ}{dt}. \tag{19}$$

This equation is easily integrated and, in conjunction with the condition k^2 = const, indicates that the self-similar solution corresponds to the case in which the current in the filament increases in accordance with a power law. Thus, for example, if $k = 1.65$, $I_0^2(k)/I_1^2(k) = 5/2$ so that the radius of the filament does not change in time and the current must increase in proportion to $t^{1/3}$. A linear current increase with time corresponds to $k = 3.08$, in which case $I_0^2(k)/I_1^2(k) = 3/2$ and the radius of the filament is inversely proportional to the time. If the current increases so slowly that $k < 1.65$, the filament expands in the course of time. In the limiting cases $k \ll 1$ and $k \gg 1$ the condition that k is constant no longer obtains. When $k \gg 1$, $I_0^2(k)/I_1^2(k) \approx 1$ this means that we can neglet the Joule heat and Eq. (19) gives $aJ^{3/2}$ = const or $T^{3/2}/n$ = const, i.e., the adiabatic equation. When $k \ll 1$ the skin effect is small. If we neglect the skin effect the change of radius with time is determined by the equation (taking radiation into account)

$$\frac{1}{a}\frac{da}{dt}+\frac{3}{2J}\frac{dJ}{dt}=\frac{c^2}{\pi a^2\sigma}\left(1-\frac{J^2}{J_{\mathrm{rad}}^2}\right)=\frac{8c^5N^{3/2}}{\pi\sigma_1 a^2 J^3}\left(1-\frac{J^2}{J_{\mathrm{rad}}^2}\right). \tag{20}$$

Suppose, for example, that $dJ/dt = 0$, $T = 10^3$ ev, and $a = 10$ cm; in this case the filament radius is doubled in a time of the order of 0.05 sec.

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