ON THE GREEN'S FUNCTION OF THE KINETIC EQUATIONS OF A CASCADE

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The cascade considered consists of several types of particles which, moving in an inhomogeneous medium which may also vary with the time, collide with the particles of the medium and may be scattered or absorbed, and can also produce new cascade particles. Functions are introduced to characterize the distributions of the particles of each type in coordinate space and velocity space at any instant of time. Integral equations are obtained for these functions. For the case of a homogeneous medium not changing with time approximate solutions of these equations which are good for large values of the time, are obtained. Better approximations are obtained by iteration.

We consider a cascade consisting of n types A_j ($j = 1, 2, \ldots, n$) of particles K which move in a given medium and collide with its particles, so that there are scatterings, absorptions, and productions of new particles of the various types. We suppose that the properties of the medium in which the cascade develops are functions of the coordinates and the time. We also assume that the density of the cascade is small, so that its particles have practically no collisions with each other, and so that motion of each particle between collisions, and also its acts of collision with the particles of the medium, are not affected by the motion of the other particles of the cascade. It is further assumed that the motion of the particles between collisions takes place according to the laws of dynamics, being determinate for given initial positions and velocities, and that distributions as to type and velocities of the particles emerging from collisions is a stochastic one. Examples of such cascades are first of all electron-photon cosmic-ray showers, and also nucleon-meson and nucleon-photon cascades.[1-5] Cascades of this type are also studied in the theory of the multiple scattering of particles,[6,7] in the theory of chemical chain reactions,[8,9] in the theory of gas discharges, and so on.

In the theory of cosmic-ray showers one usually considers only the energy distribution of the particles of the different types as function of the depth, and their scattering and distribution as to direction are found separately.[1] Sometimes also all these quantities are considered simultaneously, but the time is still not introduced into the discussion. But in such treatments the postulate that the process be of the Markov type, which is necessary for the setting up of equations for the distribution function, is only approximately fulfilled.[10] In the researches of Janossy[3] the development of a shower is traced through in great detail. The particles are characterized not only by the magnitude, but also by the direction of the velocity, so that the Markov property of the process is assured. The results obtained make it possible to follow the motion of each particle, but with the development of the cascade the number of variables increases without limit, and the expressions become complicated. In general the discussion of cascade processes is always carried out under strong simplifying assumptions regarding the form of the functions giving the distribution of the particles in the cascade, and also of the form of the functions characterizing the elementary processes of scattering, of absorption, and of production of new particles. Therefore it has seemed to us worth while to find equations for the functions that characterize at each given moment the distribution of the particles as to type, position, and velocity, without interesting ourselves in the history of each individual particle. This seems expedient because, on one hand, in such a treatment the Markov character of the process is assured, and on the other hand the number of variables does not increase with the development of the cascade, so that the equations are not very complicated. By the use of suitable substitutions one can solve these equations, though indeed only approximately, under very general assumptions about the functions characterizing the elementary processes of motion and transmutation of the particles, and, as we shall see in two subsequent papers, by means of the resulting distribution functions one can solve a number of more complicated problems.

Let A_j, $\mathbf{r}$, and $\mathbf{v}$ be the type, radius vector, and velocity of a single cascade particle. We assume that between collisions this particle moves according to the law

$$d\mathbf{r}/dt = \mathbf{v}, \quad d\mathbf{v}/dt = \mathbf{F}_j(t, \mathbf{r}, \mathbf{v}), \tag{1}$$

where $\mathbf{F}_j$ is the force acting on the particle divided by its mass. We denote by

$$\varphi_j^s = \varphi_j(t, \mathbf{r}, \mathbf{v}, s), \qquad \psi_j^s = \psi_j(t, \mathbf{r}, \mathbf{v}, s) \tag{2}$$

the solution of the system (1), i.e., the values of the radius vector and the velocity of the particle at the time s, if at the time t it has the radius vector $\mathbf{r}$ and the velocity $\mathbf{v}$.

In a given collision the incident particle may be scattered or absorbed. In either case new particles may be produced. Since we are interested not in the individual paths of the separate particles, but only in their distribution as to type, position, and velocity, for the sake of unifying the notation we shall regard scattering as absorption of the incident particle and production of a new particle of the same type at the same position, but with a different velocity. We shall also regard processes of spontaneous decay and production of new particles conventionally as collisions, since both kinds of processes are characterized by transition probabilities of the same type, and we shall not concern ourselves with the mechanisms of elementary processes.

$$Q_{jk_i}^m(t, \mathbf{r}, \mathbf{v}, \mathbf{w}_i)\, dt\, d\mathbf{w}_i \qquad (i = 1, 2, \ldots m) \tag{3}$$

denote the probability that the particle of type A_j with radius vector $\mathbf{r}$ and velocity $\mathbf{v}$ undergoes a collision in the time interval t, t + dt and that among the particles obtained from this process there are m particles (m = 0, 1, ...) of type A_{k_i} with velocities between $\mathbf{w}_i$ and $\mathbf{w}_i + d\mathbf{w}_i$. Here $d\mathbf{w}_i$ means the product $d\mathbf{w}_1 d\mathbf{w}_2 \ldots d\mathbf{w}_m$, k_i (i = 1, 2, ...) and m are integers between 1 and m, and the $\mathbf{w}_i$ are m arbitrary velocities. We shall take the probabilities (3) to be given.

The value of (3) for m = 0

$$p_j(t, \mathbf{r}, \mathbf{v})\, dt = Q_j^0(t, \mathbf{r}, \mathbf{v})\, dt \tag{4}$$

gives the probability that the particle K undergoes a collision of arbitrary type in the time interval dt, and the value of (3) for m = 1

$$Q'_{jk}(t, \mathbf{r}, \mathbf{v}, \mathbf{w})\, dt\, d\mathbf{w} \tag{5}$$

gives the probability that this particle undergoes a collision in the time interval dt and among the particles produced there is a particle A_k with velocity between $\mathbf{w}$ and $\mathbf{w} + d\mathbf{w}$.

Out of all possible collisions we single out a certain class of collisions — let us call it the class C — by using any chosen criterion as to the time and place of the collision, and also as to the types and velocities of the incident particle and the emerging particles. For example, we can include in the class C all collisions in which the incident particle has an energy larger than a given value E and the sum of the energies of the particles emerging is less than another given value E′. The class C can be empty, and also can include all collisions. But by the definition given we cannot assign to the class C those collisions, for example, in which the incident particle has appeared as the result of another collision with a given number of emerging particles. In particular, we can take as the class C those collisions in which the incident particle is absorbed without producing any other particles — the class C_1 — or those collisions in which there are particles of the type of the incident particle among those emerging — the class C_2. We designate all the particles of the same type as the particle K produced from it in collisions of the class C as the swarm of particle K. If the particle K itself does not belong to the swarm of another particle, we call it the primary particle of the swarm.

In the following arguments the concept of the class C is not necessary. It is very suitable, however, in obtaining greater precision in finding the desired functions that give the distribution of the cascade particles. As we shall see, this classification makes it possible to consider collisions that lead only to scattering of the particles separately from those that lead to the production of new particles; the latter collisions may be small in number in comparison with the former, but are quite essential for the development of the cascade. For similar reasons the classification is useful in cases in which the processes causing the development of the cascade at high and low energies are different. This case occurs, for example, in cosmic rays, for which ionization, Compton scattering, and pair production become prominent at different energies.

The quantities (3) — (5) which we have defined on the basis of all possible collisions can also be defined with the restriction to collisions of class C only. We denote the quantity (5) for $k = j$, calculated only for collisions of class C, by

$$P_j(t, \mathbf{r}, \mathbf{v}, \mathbf{w})\, dt\, d\mathbf{w}. \tag{6}$$

Obviously this is the probability for a collision of class C of one particle of type A_j during the time dt, accompanied by the emission of a particle of its swarm with velocity between **w** and **w** + d**w**. Furthermore we denote by

$$P_{jk} dt\, d\mathbf{w} = (Q'_{jk} - \delta_{jk} P_k)\, dt\, d\mathbf{w} \tag{7}$$

the probability of a collision of the particle K in the time dt accompanied by the emission of one primary particle of type A_k with velocity between **w** and **w** + d**w**.

Suppose that at the time s there is a particle K_0 of type A_i with radius vector **q** and with velocity **u**. We call it the initial particle of the cascade. We denote by

$$W_i(s, \mathbf{q}, \mathbf{u}, t, \mathbf{r}, \mathbf{v})\, d\mathbf{r}\, d\mathbf{v} \tag{8}$$

the probability that at the time t a particle belonging to the swarm of K_0 has radius vector between **r** and **r** + d**r** and velocity between **v** and **v** + d**v**. Since according to the assumptions that have been made the development of the cascade is caused by the action of the medium, and the state of the medium does not depend on the propagation of the cascade, it is clear that the development of the cascade will have the Markov property.[10] Because of this the functions W_i will satisfy the equation[11] (integration with respect to vector quantities is always taken over the entire space):

$$\begin{aligned} W_i(s, \mathbf{q}, \mathbf{u}, t, \mathbf{r}, \mathbf{v}) &= \delta(\varphi_i^s - \mathbf{q})\, \delta(\psi_i^s - \mathbf{u})\, \partial(\varphi_i^s, \psi_i^s)/\partial(\mathbf{r}, \mathbf{v}) \\ &+ \iint_s^t W_i(s, \mathbf{q}, \mathbf{u}, \tau, \varphi_i^\tau, \mathbf{w})\, P_i(\tau, \varphi_i^\tau, \mathbf{w}, \psi_i^\tau)\, M(t,\mathbf{r},\mathbf{v},\tau)\, [\partial(\varphi_i^\tau, \psi_i^\tau)/\partial(\mathbf{r}, \mathbf{v})]\, d\tau\, d\mathbf{w}. \end{aligned} \tag{9}$$

Here the functions φ_i^s and ψ_i^s are given by Eq. (2), and

$$M_i(t, \mathbf{r}, \mathbf{v}, s) = \exp\left(- \int_{\tau=s}^{t} p_i(\tau, \varphi_i^\tau, \psi_i^\tau)\, d\tau\right) \tag{10}$$

gives the probability that a particle of type A_i that at the time t has radius vector **r** and velocity **v** has not undergone any collisions in the time interval s, t.[11]

Let $V_j(t, \mathbf{r}, \mathbf{v})\, d\mathbf{r}\, d\mathbf{v}$ be the probability for finding at the time t a cascade particle of type A_j with radius vector between **r** and **r** + d**r** and velocity between **v** and **v** + d**v**, and let $U_j(t, \mathbf{r}, \mathbf{v})$ be the probability for production of one primary particle in the time interval dt with radius vector between **r** and **r** + d**r** and velocity between **v** and **v** + d**v**. Let

$$V_{ij}(s, \mathbf{q}, \mathbf{u}, t, \mathbf{r}, \mathbf{v})\, d\mathbf{r}\, d\mathbf{v} \quad \text{and} \quad U_{ij}(s, \mathbf{q}, \mathbf{u}, t, \mathbf{r}, \mathbf{v})\, dt\, d\mathbf{r}\, d\mathbf{v} \tag{11}$$

be these same probabilities, but on the hypothesis that the cascade is produced from a single particle K_0 of type A_i that has appeared at the time s at the point **q** with the velocity **u**.

Making use of the Markov property of the process, we find that the functions U_j and V_j are connected by the relations

$$V_j(t, \mathbf{r}, \mathbf{v}) = \iiint_{-\infty}^{t} U_j(s, \mathbf{q}, \mathbf{u})\, W_j(s, \mathbf{q}, \mathbf{u}, t, \mathbf{r}, \mathbf{v})\, ds\, d\mathbf{q}\, d\mathbf{u}, \tag{12}$$

$$U_j(t, \mathbf{r}, \mathbf{v}) = U_j^0(t, \mathbf{r}, \mathbf{v}) + \sum_i \int V_i(t, \mathbf{r}, \mathbf{w})\, P_{ij}(t, \mathbf{r}, \mathbf{w}, \mathbf{v})\, d\mathbf{w}, \tag{13}$$

where U_j^0 is the density of sources of initial particles that occur not as the result of collisions of the cascade particles, but as the consequence of external processes.

By elimination of U_j we get

$$V_j(t, \mathbf{r}, \mathbf{v}) = V_j^0(t, \mathbf{r}, \mathbf{v}) + \sum_i \int_{-\infty}^{t} \iiint V_i(s, \mathbf{q}, \mathbf{u}) P_{ij}(s, \mathbf{q}, \mathbf{u}, \mathbf{w}) W_j(s, \mathbf{q}, \mathbf{w}, t, \mathbf{r}, \mathbf{v})\, ds\, d\mathbf{q}\, d\mathbf{u}\, d\mathbf{w}. \tag{14}$$

Here for brevity we have denoted by

$$V_j^0(t, \mathbf{r}, \mathbf{v}) = \int_{-\infty}^{t} \iint U_j^0(s, \mathbf{q}, \mathbf{u}) W_j(s, \mathbf{q}, \mathbf{u}, t, \mathbf{r}, \mathbf{v})\, ds\, d\mathbf{q}\, d\mathbf{u}$$

the density of particles of type A_j belonging to the swarms of the particles produced by the external sources.

Suppose that the cascade is produced by a single particle of type A_i with velocity $\mathbf{u}$ which has appeared at the time t at the point $\mathbf{q}$:

$$U_j^0 = \delta_{ij}\delta(t-s)\delta(\mathbf{r}-\mathbf{q})\delta(\mathbf{v}-\mathbf{u}), \quad V_j^0 = \delta_{ij} W_j(s, \mathbf{q}, \mathbf{u}, t, \mathbf{r}, \mathbf{v}).$$

The corresponding solution $V_{ij}(s, \mathbf{q}, \mathbf{u}, t, \mathbf{r}, \mathbf{v})$ of Eq. (14) is to be regarded as the Green's function of these equations. For it we get the equations

$$V_{ij}(s, \mathbf{q}, \mathbf{u}, t, \mathbf{r}, \mathbf{v}) = \delta_{ij} W_j(s, \mathbf{q}, \mathbf{u}, t, \mathbf{r}, \mathbf{v}) + \sum_k \int_s^t \iiint V_{ik}(s, \mathbf{q}, \mathbf{u}, \tau, \rho, \nu) P_{kj}(\tau, \rho, \nu, \mathbf{w})\, W_j(\tau, \rho, \mathbf{w}, t, \mathbf{r}, \mathbf{v})\, d\tau\, d\rho\, d\nu\, d\mathbf{w}. \tag{15}$$

The functions p_i, P_i, P_{ij}, F_i, and consequently also φ_i^s and ψ_i^s, are known. Then the functions W_i are determined by Eq. (9), and V_{ij} by (15).

Let us suppose that $C \equiv C_1$, and consequently $P_i = 0$. Then from Eq. (9) we get

$$W_i(s, \mathbf{q}, \mathbf{u}, t, \mathbf{r}, \mathbf{v}) = \delta(\varphi_i^s - \mathbf{q})\,\delta(\psi_i^s - \mathbf{u})\, M_i(t, \mathbf{r}, \mathbf{v}, s)\, \partial(\varphi_i^s, \psi_i^s)/\partial(\mathbf{r}, \mathbf{v}), \tag{16}$$

and Eq. (15) takes the form

$$\begin{aligned} V_{ij}(s, \mathbf{q}, \mathbf{u}, t, \mathbf{r}, \mathbf{v}) &= \delta_{ij}\delta(\varphi_j^s - \mathbf{q})\,\delta(\psi_j^s - \mathbf{u})\, M_j(t, \mathbf{r}, \mathbf{v}, s)\, \partial(\varphi_j^s, \psi_j^s)/\partial(\mathbf{r}, \mathbf{v}) \\ &+ \sum_k \int_s^t V_{ik}(s, \mathbf{q}, \mathbf{u}, \tau, \varphi_j^\tau, \mathbf{w}) P_{kj}(\tau, \varphi_j^\tau, \mathbf{w}, \psi_j^\tau)\, M_j(t, \mathbf{r}, \mathbf{v}, \tau)\, [\partial(\varphi_j^\tau, \psi_j^\tau)/\partial(\mathbf{r}, \mathbf{v})]\, d\tau\, d\mathbf{w}. \end{aligned} \tag{17}$$

If the properties of the medium are independent of the coordinates and the time and the forces F_i acting on the particles are equal to zero, we get

$$\begin{aligned} &W_i(s, \mathbf{q}, \mathbf{u}, t, \mathbf{r}, \mathbf{v}) = W_i(t-s, \mathbf{r}-\mathbf{q}, \mathbf{u}, \mathbf{v}), \quad V_{ij}(s, \mathbf{q}, \mathbf{u}, t, \mathbf{r}, \mathbf{v}) = V_{ij}(t-s, \mathbf{r}-\mathbf{q}, \mathbf{u}, \mathbf{v}), \quad P_i(t, \mathbf{r}, \mathbf{v}, \mathbf{w}) = P_i(\mathbf{v}, \mathbf{w}), \\ &P_{ij}(t, \mathbf{r}, \mathbf{v}, \mathbf{w}) = P_{ij}(\mathbf{v}, \mathbf{w}), \quad p_i(t, \mathbf{r}, \mathbf{v}) = p_i(v), \; M_i(t, \mathbf{r}, \mathbf{v}, s) = e^{-p_i(v)|t-s|}, \quad \varphi_i^s = \mathbf{r} - \mathbf{v}(t-s), \; \psi_i^s = \mathbf{v}. \end{aligned} \tag{18}$$

Then, without loss of generality, we can set $s = 0$, $q = 0$, and from Eqs. (9), (11), (17), (18) we find for W_i and V_{ij} with $C \equiv C_1$:

$$W_i(t, \mathbf{r}, \mathbf{u}, \mathbf{v}) = \delta(\mathbf{r} - \mathbf{u}t)\,\delta(\mathbf{v}-\mathbf{u})\, e^{-p_i(v)t}, \tag{19}$$

$$V_{ij}(t, \mathbf{r}, \mathbf{u}, \mathbf{v}) = \delta_{ij}\delta(\mathbf{r}-\mathbf{u}t)\,\delta(\mathbf{v}-\mathbf{u})\, e^{-p_j(v)t} + \sum_k \int_0^t \int V_{ik}(t-\tau, \mathbf{r}-\mathbf{v}\tau, \mathbf{u}, \mathbf{w})\, P_{kj}(\mathbf{w}, \mathbf{v})\, e^{-p_j(v)\tau}\, d\tau\, d\mathbf{w}, \tag{20}$$

and with $C \equiv C_2$ we get

$$W_i(t, \mathbf{r}, \mathbf{u}, \mathbf{v}) = \delta(\mathbf{r}-\mathbf{u}t)\,\delta(\mathbf{v}-\mathbf{u})\, e^{-p_i(v)\tau} + \int_0^t \int W_i(t-\tau, \mathbf{r}-\mathbf{v}\tau, \mathbf{u}, \mathbf{w})\, P_i(\mathbf{w}, \mathbf{v})\, e^{-p_i(v)\tau}\, d\tau\, d\mathbf{w}, \tag{21}$$

$$V_{ij}(t, \mathbf{r}, \mathbf{u}, \mathbf{v}) = \delta_{ij} W_j(t, \mathbf{r}, \mathbf{u}, \mathbf{v}) + \sum_k \int_0^t \iiint V_{ik}(t-\tau, \mathbf{r}-\rho, \mathbf{u}, \nu)\, P_{kj}(\nu, \mathbf{w})\, W_j(\tau, \rho, \mathbf{w}, \mathbf{v})\, d\tau\, \rho\, d\nu\, d\mathbf{w}. \tag{22}$$

All the equations (9), (15), (17), (20) – (22) we have obtained for W_i and V_{ij} are integral equations,[12] and not integro-differential equations such as are usually found in the theory of cosmic-ray showers.[1,4] To each of them three corresponds another integral equation adjoint to it.[12,13]

By well known methods[14,15] one can get from each of these equations two systems of mutually adjoint differential equations for the unknowns W_i and V_{ij}. They are equations of first order with respect to t, but of infinite order with respect to **u**, or to **v**, respectively, and therefore are of no great practical interest.

All the equations (14), (15), (17), (20), and (22) are equations of the type

$$V = A + \Phi V,$$

where A is the inhomogeneous terms and Φ is a corresponding integral operator. By the iteration method an m-th order approximate solution is obtained in the form

$$V^m = A + \Phi A + \Phi^2 A + \ldots + \Phi^{m-1} A + \Phi^m V^0, \tag{23}$$

where V^0 is the initial approximation for V. Whatever V^0 may be, the iteration process converges. If, as is usually done, we take $V^0 = A$, it is not hard to verify[16] that V^m gives a good approximation only for $t - s < m\bar{t}$, where $\bar{t}$ is the average time of free flight of a particle. In order to find a solution giving a good approximation for arbitrary t, we use Eq. (23), but as V^0 we take a function giving a good approximation for V for large t.

Let us consider Eq. (20). As the initial approximation we set

$$\begin{aligned} V^0_{ij}(t, \mathbf{r}, \mathbf{u}, \mathbf{v}) = \{(1 + \varepsilon' e^{-\delta' t} + \varepsilon'' e^{-\delta'' t} + \ldots)\, \varepsilon_j \gamma_j^3 / \pi^3 \alpha_j^3 \beta_j^3\, (1 - \varkappa^2{}_j)^{3/2} \\ \times (\gamma_j t - 2\lambda_j)^{3/2}\} \exp\left\{- \delta_j t - \frac{(\mathbf{v} - \varkappa_j \mathbf{u})^2}{\alpha_j^2 (1 - \varkappa_j^2)} - \frac{(\gamma_j \mathbf{r} - \lambda_j (\mathbf{u} + \mathbf{v}))^2}{\beta_j^2 (\gamma_j t - 2\lambda_j)}\right\}, \end{aligned} \tag{24}$$

where

$$\varkappa_j = \exp\{-\gamma_i t\}, \ \lambda_j = (1 - \exp\{-\gamma_j t\} / (1 + \exp\{-\gamma_j t\}). \tag{25}$$

Here the quantities

$$\alpha_j, \beta_j, \gamma_j, \delta_j, \varepsilon_j, \delta', \varepsilon', \delta'', \varepsilon'' \ldots \tag{26}$$

depend on the variables i and **u** that characterize the initial particle. For brevity we have not written the arguments i and **u**. Without loss of genrality we shall suppose that

$$0 < \delta' < \delta'' < \ldots . \tag{27}$$

The quantities (26) must be determined in such a way that the function (24) approximate V_{ij} as well as possible for large t. The expression (24) represents a certain generalization of the Chandrasekhar function that gives the analogous probability in the diffusion of particles in Brownian motion. This function is obtained from (24) for n = 1 if

$$\alpha = \beta / \sqrt{2} = c, \ \gamma = 6\pi a\eta / m, \ \delta = 0, \ \varepsilon = 1, \ \varepsilon' = \varepsilon'' = \ldots = 0, \tag{28}$$

where m, a, and c are respectively the mass, radius, and most probable speed of the particle, and η is the viscosity of the medium.[13] We shall not use the relations (28), since the cascade particles, unlike particles in Brownian motion, are produced and absorbed and do not come into thermal equilibrium with the medium. The formula (24) is a generalization of the expression that we used in Ref. 16 in the solution of an analogous problem of the diffusion of neutrons.

To determine the constants (26), we substitute Eq. (24) into Eq. (20). We find that the error to within which V^0_{ij} satisfies Eq. (20) can be written in the form $V^0_{ij} R^0_{ij}$, where

$$\begin{aligned} R_{ij}(t, \mathbf{r}, \mathbf{u}, \mathbf{v}) = 1 - \delta_{ij} \delta(\mathbf{r} - \mathbf{u}t)\, \delta(\mathbf{v} - \mathbf{u})\, e^{-p_j(v)t} / V^0_{ij}(t, \mathbf{r}, \mathbf{u}, \mathbf{v}) \\ - \sum_k \iint_0 [V^0_{ik}(t - \tau, \mathbf{r} - \mathbf{v}\tau, \mathbf{u}, \mathbf{w}) / V^0_{ij}(t, \mathbf{r}, \mathbf{u}, \mathbf{v})]\, P_{kj}(\mathbf{w}, \mathbf{v})\, e^{-p_j(v)\tau} d\tau\, d\mathbf{w} \end{aligned} \tag{29}$$

is the fractional error. We try to determine the quantities (26) in such a way that the error R_{ij} goes to zero as rapidly as possible with increasing t. From Eq. (24) it is clear that the terms of highest order for large t will be of the type $c \exp\{-(\delta_k - \delta_j) t\}$, so that for $\delta_j \neq \delta_k$ in Eq. (24) terms appear that increase exponentially as t increases. Consequently, in order to make the error a minimum, we must set

$$\delta_j = \delta. \tag{30}$$

Let us expand R_{ij} in powers of t^{-1} (and $\exp\{-\gamma_j t\}$, but these latter go to zero faster than any power of t^{-1}, so that they can be neglected) and try to determine the quantities (26) in such a way that as many as possible of the coefficients of this expansion are zero. In the calculation of R_{ij} we can take $\kappa_j = 0$, $\lambda_j = 1$, $\epsilon' = \epsilon'' = \ldots = 0$, drop the second term in Eq. (29), and take the upper limit in the integration over τ to be ∞, since all these steps lead to errors that decrease exponentially. We find

$$\varepsilon_i R_{ij} = \varepsilon_j - \sum_k \varepsilon_k \frac{\alpha_j^3 \beta_j^3}{\alpha_k^3 \beta_k^3} \iint_0^\infty \frac{(\gamma_j t - 2)^{3/2}}{(\gamma_k (t-\tau) - 2)^{3/2}} e^{-(p_j(v)-\delta)\tau} P_{kj}(\mathbf{w}, \mathbf{v})$$
$$\times \exp\left\{\frac{v^2}{\alpha_j^2} - \frac{w^2}{\alpha_k^2} + \frac{(\gamma_j \mathbf{r} - \mathbf{u} - \mathbf{v})^2}{\beta_j^2 (\gamma_j t - 2)} - \frac{(\gamma_k \mathbf{r} - \mathbf{u} - \mathbf{w})^2}{\beta_k^2 (\gamma_k (t-\tau) - 2)}\right\} d\tau\, d\mathbf{w}. \tag{31}$$

As can be seen, we cannot make more than two coefficients in the expansion of R_{ij} equal to zero, so that we confine ourselves to terms of order t^{-1}. We get:

$$\varepsilon_j R_{ij} = \varepsilon_j - \sum_k \varepsilon_k \frac{\alpha_j^3 \beta_j^3}{\alpha_k^3 \beta_k^3} \iint_0^\infty \left\{ \exp\left\{\frac{v^2}{\alpha_j^2} - \frac{w^2}{\alpha_k^2} - (p_j(v) - \delta)\tau\right\}\right.$$
$$\left.\times P_{kj}(\mathbf{w}, \mathbf{v}) \left\{1 + \frac{1}{t}\left[\frac{3}{2}\tau + \frac{(\gamma_j \mathbf{r} - \mathbf{u} - \mathbf{v})^2}{\beta_j^2 \gamma_j} - \frac{(\gamma_k \mathbf{r} - \mathbf{u} - \mathbf{w})^2}{\beta_k^2 \gamma_k}\right]\right\}\right\} d\tau d\mathbf{w}.$$

From Eq. (24) it is clear that, at least for large t,

$$m_j(v)\, d\mathbf{v} = \frac{1}{\pi^{3/2} \alpha_j^3} \exp\{-v^2/\alpha_j^2\}\, d\mathbf{v} \tag{32}$$

will be the probability that a cascade particle of type A_j has its velocity between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$. If we set

$$\int m_k(w) P_{kj}(\mathbf{w}, \mathbf{v})\, d\mathbf{w} = a_{kj}(v), \quad \int \mathbf{w} m_k(w) P_{kj}(\mathbf{w}, \mathbf{v})\, d\mathbf{w} = \mathbf{v} b_{kj}(v), \quad \int w^2 m_k(w) P_{kj}(\mathbf{w}, \mathbf{v})\, d\mathbf{w} = v^2 c_{kj}(v), \tag{33}$$

the integration over $\mathbf{w}$ and then over τ can be carried out simply. We find

$$\varepsilon_j R_{ij} = \varepsilon_j - \sum_k \varepsilon_k \pi^{3/2} \alpha_j^3 \frac{\beta_j^3}{\beta_k^3} \exp\{v^2/\alpha_j^2\} \left\{\frac{a_{kj}(v)}{p_j(v) - \delta} + \frac{1}{t}\left[\frac{3}{2}\frac{1}{(p_j(v) - \delta)^2} + \frac{(\gamma_j \mathbf{r} - \mathbf{u} - \mathbf{v})^2 a_{kj}(v)}{\beta_j^2 \gamma_j (p_j(v) - \delta)}\right.\right.$$
$$\left.\left. - \frac{(\gamma_k \mathbf{r} - \mathbf{u})^2 a_{kj}(v) - 2(\gamma_k \mathbf{r} - \mathbf{u}) \mathbf{v} b_{kj}(v) + v^2 c_{kj}(v)}{\beta_k^2 \gamma_k (p_j(v) - \delta)}\right]\right\}.$$

We cannot equate to zero the constant term and the coefficient of t^{-1} in this expression, since they involve the variables $\mathbf{r}$ and $\mathbf{v}$, on which the quantities (26) cannot depend. Therefore, as in Ref. 13, we confine ourselves to setting their average values with respect to $\mathbf{v}$ equal to zero. Multiplying by (32), integrating with respect to $\mathbf{v}$, and using certain considerations of symmetry, we get

$$\varepsilon_j - \sum_k \varepsilon_k \frac{\beta_j^3}{\beta_k^3} \int \left\{\frac{a_{kj}(v)}{p_j(v) - \delta} + \frac{1}{t}\left[\frac{3}{2}\frac{a_{kj}(v)}{(p_j(v) - \delta)^2} + \frac{(\gamma_j \mathbf{r} - \mathbf{u})^2 a_{kj}(v) + v^2 a_{kj}(v)}{\beta_j^2 \gamma_j (p_j(v) - \delta)} - \frac{(\gamma_k \mathbf{r} - \mathbf{u})^2 a_{kj}(v) + v^2 c_{kj}(v)}{\beta_k^2 \gamma_k (p_j(v) - \delta)}\right]\right\} d\mathbf{v} = 0. \tag{34}$$

This equation must hold for all $\mathbf{r}$ and t. Therefore it falls apart into five, or more precisely into $5n^2$, equations, since it must be satisfied for all values of the index j and of the omitted index i. Thus we get a number of equations that exceeds by n^2 the number of unknowns involved in Eq. (34). But all of these equations are satisfied if we set

$$\beta_j = \beta, \quad \gamma_j = \gamma. \tag{35}$$

Then Eq. (34) separates into

$$\varepsilon_j - \sum_k \varepsilon_k \int \frac{a_{kj}(v)}{p_j(v) - \delta}\, d\mathbf{v} = 0, \tag{36}$$

$$\sum_k \varepsilon_k \int \left[\frac{3}{2}\frac{a_{kj}(v)}{(p_j(v) - \delta)^2} + \frac{v^2}{\beta^2 \gamma}\frac{a_{kj}(v) - c_{kj}(v)}{p_j(v) - \delta}\right] d\mathbf{r} = 0. \tag{37}$$

From Eq. (36) it can be seen that ϵ_j is determined apart from a certain factor,

$$\varepsilon_j = \varepsilon e_j, \tag{38}$$

where ϵ is a function of i and **u** still to be determined, and the quantities e_j are normalized by

$$\sum e_j = 1. \tag{39}$$

We cannot find the equations that are still lacking in order to determine all the unknowns (26) by equating to zero the successive coefficients in the expansion of R_{ij} as averaged over **v** because, as can easily be verified, this gives an infinite set of new equations. Therefore we must bring in new considerations in order to get the required equations.

From Eqs. (32), (33), and (39) we find that, at least for large t,

$$n_j(v)\,d\mathbf{v} = \sum_i e_i a_{ij}(v)\,d\mathbf{v} \Big/ \sum_{ij} e_i \int a_{ij}(v)\,d\mathbf{v} \tag{40}$$

will be the probability for the production of a primary particle of type A_j with velocity between **v** and **v** + d**v**.

Let $\bar{\mathbf{r}}_i(\mathbf{u})$ be the average radius vector of all particles for large t, if the initial particle of the cascade was of type A_i and had velocity **u**; then

$$R = \sum_i \int n_i(u) |\bar{\mathbf{r}}_i(\mathbf{u})| d\mathbf{u} \tag{41}$$

is the absolute value of this radius vector, averaged over i and **u** according to Eq. (40). Let $\bar{r}^2(t)$ be the dispersion of the displacements of the particles at the time t, averaged over the type and the velocity of the initial particle, so that the quantity

$$D = \lim_{t=\infty} \frac{1}{6t} \bar{r}^2(t) \tag{42}$$

can be called the diffusion coefficient of the cascade particles. The quantities R and D can be found by two methods — from Eq. (24), or by statistical calculations, for example by the method of Yang.[18] As in Ref. 13, by equating these values we find two more equations.

From Eqs. (24), (41), and (42) we find

$$R = \frac{1}{\gamma} \sum_i \int n_i(u)\, u\, d\mathbf{u}, \quad D = \frac{1}{4} \frac{\beta^2}{\gamma}. \tag{43}$$

Let us introduce the concept of the chain C of a given particle K at the time t. This is the succession of particles, each converted into the next, by which K is obtained from K_0. We call the particle K a particle of the m-th generation if its chain consists of m particles. Let $V_{ij}^m(\mathbf{u}, \mathbf{v})\,d\mathbf{v}$ be the probability that one particle among the particles of the m-th generation produced from a single particle of type A_i with velocity **u** is a particle of type A_j with velocity between **v** and **v** + d**v**.

Following Ref. 13, we find

$$V_{ij}^1(\mathbf{u}, \mathbf{v}) = \delta_{ij}\,\delta(\mathbf{v} - \mathbf{u}), \quad V_{ij}^{m+1}(\mathbf{u}, \mathbf{v}) = \sum_k \int V_{ik}^m(\mathbf{u}, \mathbf{w}) \frac{P_{kj}(\mathbf{w}, \mathbf{v})}{\sum_j \int P_{kj}(\mathbf{w}, \mathbf{v})\,d\mathbf{v}}\,d\mathbf{w}. \tag{44}$$

Let $\bar{\mathbf{r}}_i^m(\mathbf{u})$ be the average displacement of the particles of the m-th generation, if the particle that produced the cascade was of type A_i and had velocity **u**. We get

$$\bar{\mathbf{r}}_i^m(\mathbf{u}) = \sum_j \int V_{ij}^m(\mathbf{u}, \mathbf{v}) \frac{\mathbf{v}}{p_j(v)}\,d\mathbf{v}, \tag{45}$$

$$\bar{\mathbf{r}}_i(\mathbf{u}) = \sum_{m=1}^{\infty} \bar{\mathbf{r}}_i^m(\mathbf{u}). \tag{46}$$

Then from Eqs. (41), (45), and (46) we find a second expression for R and, equating it to the expression (43), we get

$$\sum_i \int n_i(u)\, u\, d\mathbf{u} = \gamma \sum_i \int n_i(u) \left| \sum_{m=}^{\infty} \bar{\mathbf{r}}_i^m(\mathbf{u}) \right| d\mathbf{u}. \tag{47}$$

Let $\bar{t}_N$ and $\bar{r}_N^2$ be the mean lifetime and the dispersion of the displacements of all the particles of the N-th generation, so that we can set[13,18]

$$D \approx \lim_{N=\infty} \frac{1}{6\bar{t}_N} \bar{r}_N^2. \tag{48}$$

Here, following (18), we have

$$\bar{t}_N = N\bar{t},\ \bar{r}_N^2 = \sum_{m,\,n=1}^{N} \overline{\mathbf{r}^m \mathbf{r}^n} = N\bar{r}'^2 + 2(N-1)\overline{\mathbf{r}'\cdot\mathbf{r}^2} + 2(N-2)\overline{\mathbf{r}'\cdot\mathbf{r}^3} + \ldots, \tag{49}$$

where $\bar{t}$ is the mean lifetime of one particle, and $\mathbf{r}^m$ is the displacement of one particle of the m-th generation. We find that

$$\bar{t} = \sum_i \int \frac{n_i(u)}{p_i(u)} d\mathbf{u}, \tag{50}$$

$$\bar{r}'^2 = 2\sum_i \int \frac{n_j(u)}{p_i^2(u)} u^2 d\mathbf{u}, \tag{51}$$

$$\overline{\mathbf{r}'\cdot\mathbf{r}^m} = \sum_{ij} \iint n_i(u)\, V_{ij}^m(\mathbf{u},\mathbf{v}) \frac{\mathbf{u}\cdot\mathbf{v}}{p_i(u)\,p_j(v)} d\mathbf{u}\, d\mathbf{v}. \tag{52}$$

From Eqs. (48) and (49) we find a second expression for D and, equating to Eq. (43), we get finally

$$\frac{3}{4}\frac{\beta^2}{\gamma}\bar{t} = \frac{1}{2}\bar{r}'^2 + \sum_{m=2}^{\infty} \overline{\mathbf{r}'\cdot\mathbf{r}^m}, \tag{53}$$

where $\bar{t}$, $\bar{r}'^2$, and $\overline{\mathbf{r}'\cdot\mathbf{r}^m}$ are given by Eqs. (50) — (52).

Equations (30), (35) — (37), (39), (47), and (53) are sufficient for the determination of the quantities α_j, β_j, γ_j, δ_j, and e_j. In this connection it must be noted that in all these equations i and u do not occur. Consequently, all these quantities, which we had taken dependent on i and u, without indicating this only for the sake of brevity, turn out to be independent of i and u, and β_j, γ_j, δ_j are even independent of j.

In order to find the remaining unknowns

$$\delta_i'(u),\ \delta_i''(u), \ldots \varepsilon_i(u),\ \varepsilon_i'(u),\ \varepsilon_i''(u), \ldots \tag{54}$$

we integrate (24) with respect to r and v and sum with respect to j. Using Eqs. (38) and (39), we find

$$\varepsilon_i e^{-\delta t} + \varepsilon_i\varepsilon_i' e^{-\left(\delta+\delta_i'\right)t} + \varepsilon_i\varepsilon_i'' e^{-\left(\delta+\delta_i''\right)t} + \cdots = N_i(t,u), \tag{55}$$

where $N_i(t, u)$ is the total number of particles at the time t. From the definition of the functions V_{ij} it is clear that $N_i(0, u) = 1$. We denote by N_i', N_i'', . . . the derivatives of the functions N_i with respect to t for t = 0. Differentiating Eq. (55) and setting t = 0, we find

$$\begin{gathered} \varepsilon_i + \varepsilon_i\varepsilon_i' + \varepsilon_i\varepsilon_i'' + \cdots = 1, \quad -\delta\varepsilon_i - (\delta+\delta_i')\varepsilon_i\varepsilon_i' - (\delta+\delta_i'')\varepsilon_i\varepsilon_i'' - \cdots = N_i', \\ \delta^2\varepsilon_i + (\delta+\delta_i')^2\varepsilon_i\varepsilon_i' + (\delta+\delta_i'')^2\varepsilon_i\varepsilon_i'' + \cdots = N_i'', \quad \ldots\ldots\ldots\ldots\ldots\ldots \end{gathered} \tag{56}$$

We calculate the quantities N_i', N_i'' from Eq. (23) with $V^0 = A$. Then, using the fact that the operator Φ is of the order of t, we see that in the calculation of N_i^k we can without loss of accuracy use Eq. (23) with m = k. Breaking off the series $1 + \epsilon' e^{-\delta' t} \ldots$ in Eq. (24) at a certain term, we can write a sufficient number of equations of the type of Eq. (56) to determine all the quantities (54). In particular, if we use two terms we get for ϵ_i', ϵ_i' and δ_i'

$$\varepsilon_i = \frac{N_i'' - N_i'^2}{N_i'' + 2\delta N_i' + \delta^2}, \quad \varepsilon_i' = \frac{(N_i' + \delta)^2}{N_i'' - N_i'^2}, \quad \delta_i' = -\frac{N_i'' + 2\delta N_i' + \delta^2}{N_i' + \delta}. \tag{57}$$

From Eq. (23), with $V^0 = A$ and m = 2, in virtue of Eq. (20), we get for N_i' and N_i'', after some calculation,

$$\begin{gathered} N_i'(u) = -p_i(u) + \sum_j \int P_{ij}(\mathbf{u},\mathbf{v})\, d\mathbf{v}, \\ N_i''(u) = \frac{1}{2}p_i^2(u) + \frac{1}{2}\sum_{jk}\iint P_{ik}(\mathbf{u},\mathbf{w})\,P_{kj}(\mathbf{w},\mathbf{v})\, d\mathbf{w}\, d\mathbf{v} - \frac{1}{2}\sum_j \int (p_i(u) + p_j(v))\,P_{ij}(\mathbf{u},\mathbf{v})\, d\mathbf{v}. \end{gathered} \tag{58}$$

The expressions (57) and (58) determine the quantities (54). It is clear that the values obtained for them depend on i and **u**. They must satisfy the inequality (27).

Consequently, V_{ij}^0 finally takes the form

$$V_{ij}^0(t, \mathbf{r}, \mathbf{u}, \mathbf{v}) = \varepsilon_i e_j \gamma^3 [(1 + \varepsilon_i' e^{-\delta_i' t} + \varepsilon_i'' e^{-\delta_i'' t} + \ldots) / \pi^3 \alpha_j^3 \beta^3 (1 - \varkappa^2)^{3/2} (\gamma t - 2\lambda)^{3/2}]$$
$$\times \exp\left\{-\delta t - \frac{(\mathbf{v} - \varkappa \mathbf{u})^2}{\alpha_j^2 (1 - \varkappa^2)} - \frac{(\gamma \mathbf{r} - \lambda(\mathbf{u} + \mathbf{v}))^2}{\beta^2 (\gamma t - 2\lambda)}\right\} (\varkappa = e^{-\gamma t},\ \lambda = (1 - e^{-\gamma t}) / (1 + e^{-\gamma t})). \tag{59}$$

The constants α_j, β, γ, δ, and e_j are determined from Eqs. (36), (37), (39), (47), and (53). In the solution of these equations it is expedient first to eliminate β, γ, and e_j and determine δ and α_j. Since the functions (54) are not involved in these equations, they are determined subsequently from Eq. (56) or, in particular, from Eqs. (57) and (58). We shall not consider the problem of the existence and uniqueness of the solution. We note only that if several solutions are found one must choose the solution corresponding to the smallest value of δ, since it gives the main value of V_{ij}^0 for large t.

The method that has been explained for the solution of the equations (20) can be applied to Eq. (21). This is an equation of the type of Eq. (20) for $n = 1$. Consequently, one can obtain its solution by iteration, taking as the initial approximation

$$W_i^0(t, \mathbf{r}, \mathbf{u}, \mathbf{v}) = c_i^3 [(e_i(u) e^{-d_i t} + e_i'(u) e^{-d_i'(u) t} + \ldots) / \pi^3 a_i^3 b_i^3 (1 - k_i^2)^{3/2} (c_i t - 2l_i)^{3/2}]$$
$$\times \exp\left\{-\frac{(\mathbf{v} - k_i \mathbf{u})^2}{a_i^2 (1 - k_i^2)} - \frac{(c_i \mathbf{r} - l_i (\mathbf{u} + \mathbf{v}))^2}{b_i^2 (c_i t - 2l_i)}\right\} (k_i = e^{-c_i t},\quad l_i = (1 - e^{-c_i t}) / (1 + e^{-c_i t})), \tag{60}$$

with the quantities a_i, b_i, c_i, d_i, $e_i(u)$, $d_i'(u)$, $e_i'(u)$ ... determined like α, β, γ, δ, $\epsilon_i(u)$, $\delta + \delta_i'(u)$, $\epsilon_i(u)\,\epsilon_i'(u)$ for $n = 1$ and $P_{11} = p_i$.

We look for the solution of Eq. (22) also in the form (24). The determination of the quantities (26) is carried out in the same way. Here, using Eqs. (30) and (35), we get instead of Eq. (34)

$$\varepsilon_j - \sum_k \frac{\varepsilon_k}{\pi^{3/2} \alpha_k^3} \int_0^\infty \iiiint \exp\left\{\delta\tau - \frac{w^2}{\alpha_k^2}\right\} P_{kj}(\mathbf{w}, \mathbf{\nu}) W_j(\tau, \rho, \mathbf{\nu}, \mathbf{v})$$
$$\times \left\{1 + \frac{1}{t}\left[\frac{3}{2}\tau + \frac{(\gamma \mathbf{r} - \mathbf{u} - \mathbf{v})^2}{\beta^2 \gamma} - \frac{(\gamma(\mathbf{r} - \rho) - \mathbf{u} - \mathbf{w})^2}{\beta^2 \gamma}\right]\right\} d\tau\, d\rho\, d\mathbf{\nu}\, d\mathbf{v}\, d\mathbf{w},$$

where W_j is given by the expression (60). From considerations of symmetry it follows that here, as in Eq. (34) the terms containing **r** and **u** drop out. The integration over ρ and **v** is carried out immediately. After this the integration over τ becomes elementary. The integration over **w** reduces to the substitution of the expressions (33).

After carrying out all these rather cumbersome calculations, we get instead of Eqs. (36) and (37)

$$\varepsilon_j - \sum_k \varepsilon_k \int a_{kj}(\nu) \alpha_j(\nu, \delta) d\nu = 0, \tag{61}$$

$$\sum \varepsilon_k \int a_{kj}(\nu) \left\{\frac{3}{2}\left(\frac{b_j^2}{c_j \beta} - \frac{\beta}{\gamma}\right) \frac{\partial \alpha_j(\nu, \delta)}{\partial \delta} + \left(\frac{3}{2} \frac{a_j^2}{\beta \gamma^2} + \frac{21}{2} \frac{a_j^2}{\beta c_j^2} - \frac{\nu^2}{\beta c_j^2} - 3 \frac{b_j^2}{\beta c_j^2} - \frac{2\nu^2}{\beta \gamma c_j} \frac{b_{kj}(\nu)}{a_{kj}(\nu)} - \frac{\nu^2}{\beta \gamma^2} \frac{c_{kj}(\nu)}{a_{kj}(\nu)}\right) \alpha_j(\nu, \delta)\right.$$
$$+ 2\left(-3 \frac{a_j^2}{\beta c_j^2} + \frac{\nu^2}{\beta c_j^2} + \frac{\nu^2}{\beta c_j \gamma} \frac{b_{kj}(\nu)}{a_{kj}(\nu)}\right) \alpha_j(\nu, \delta - c_j) + \left(\frac{3}{2} \frac{a_j^2}{\beta c_j^2} - \frac{3}{2} \frac{a_j^2}{\beta \gamma^2} + \frac{\nu^2}{\beta \gamma^2} - \frac{\nu^2}{\beta c_j^2}\right) \alpha_j(\nu, \delta - 2c_j)$$
$$\left.+ 6\left(\frac{b_j^2}{\beta c_j^3} - 2 \frac{a_j^2}{\beta c_j^3}\right)\left[e_j(\nu) \beta\left(\frac{-}{c}\right) + e_j'(\nu) \beta\left(\frac{d_j(\nu) - \delta}{c_j}\right) + \ldots\right]\right\} d\nu = 0, \tag{62}$$

with[19]

$$\alpha_j(\nu, x) = \frac{e_j(\nu)}{d_j - x} + \frac{e_j'(\nu)}{d_j'(\nu) - x} + \ldots, \quad \beta(x) = \int_0^\infty \frac{e^{-xt}}{1 + e^{-t}} dt.$$

To derive the equations analogous to (47) and (53) we must first of all change the concept of the generation of a particle. We shall call a particle K a particle of the m-th generation if the particles of its chain belong to m different swarms, i.e., if among them there are m primary particles. By $V_{ij}^m(\mathbf{u}, \mathbf{v})\, d\mathbf{v}$ we shall now mean the probability of finding a particle of type A_j with velocity between **v** and **v** + d**v** among the primary particles of the m-th generation, if the cascade is produced from an initial particle of type A_i with velocity **u**. Instead of Eq. (44) we now get

$$V'_{ij}(\mathbf{u}, \mathbf{v}) = \delta_{ij}\,\delta(\mathbf{v}-\mathbf{u}), \quad V_{ij}^{m+1}(\mathbf{u}, \mathbf{v}) = \sum_k \iint V_{ik}^m(\mathbf{u}, \mathbf{w}) \frac{R_k(\mathbf{w}, \nu)\, P_{kj}(\nu, \mathbf{v})}{\sum_j \iint R_k(\mathbf{w}, \nu)\, P_{kj}(\nu, \mathbf{v})\, d\nu d\mathbf{v}}\, d\mathbf{w}\, d\boldsymbol{\nu}$$

with

$$R_k(\mathbf{u}, \mathbf{v}) = \iint_0^\infty W_k(t, \mathbf{r}, \mathbf{u}, \mathbf{v})\, dt d\mathbf{r}.$$

The mean lifetime and mean displacement of the particles of the swarm will be $t_i(u)$ and $ur_i(u)$, where t_i and r_i are suitable functions of u, which can be found without difficulty by means of W_i. Then Eq. (50) remains valid, with $1/p_i$ replaced by t_i, and Eqs. (46) and (52) with $1/p_i$ replaced by r_i, while instead of Eq. (51) we must write

$$\overline{r'^2} = \sum_i \int_0^\infty \iiint r^2 m_i(u)\, W_i(t, \mathbf{r}, \mathbf{u}, \mathbf{v})\, dt\, d\mathbf{r}\, d\mathbf{u}\, d\mathbf{v}.$$

Equations (47) and (53) remain valid with the indicated changes of the definitions of the quantities appearing in them. As regards the definition of the functions (54), for them Eqs. (56), (57), and (58) remain valid, it being only necessary to replace the quantity P_{ij} in Eq. (58) by Q'_{ij} [Eq. (7)].

Both the proposed methods for finding the functions V_{ij} by solution of Eq. (20) or Eq. (22) are in principle equally applicable. It is clear, however, from the structure of Eqs. (36) and (37), and of (61) and (62), respectively, that if the majority of the particles arising in the collisions are particles of the same type as the incident particle, i.e., if $Q'_{ii} \gg \sum_i^{j\neq i} Q'_{ij}$, then it is more advantageous to take C_2 as the class C. Then one must use the method based on Eq. (22) with $P_{ii} = 0$. If, on the other hand, $Q'_{ii} \lesssim \sum_j^{j\neq i} Q'_{jj}$, it is better to use Eq. (20).

From the expression (59), which is a solution both of Eq. (20) and of Eq. (22), we can draw some general conclusions about the distribution of particles in first approximation. The fact that one gets for the δ values not depending on i, j, and u means that the numbers of particles of the various types A_j fall off by the same exponential laws for any i and u. The fact that the values of β and γ are independent of i, j, and u means that the ratio of the densities of particles of different types is the same at different points of space and at different times. The dependence of the quantities α_j and e_j on j (but not on i and u) means that the velocity distributions and the ratios of the densities of the various types of particles depend on the types of particles, but not on the type and velocity of the initial particle. The type and velocity of the initial particle affect the absolute values of the densities only through the common factor $\epsilon(1 + \epsilon' e^{-\delta' t} + \ldots)$. All of this is readily understandable, since for large t the distribution of the particles as to type, position, and velocity must be determined primarily by the transition probabilities p_i, P_i, and P_{ij}, and not by the choice of the initial particle.

The substitution (24) is fortunately chosen because, on one hand, it provides for a rather rapid decrease of the errors $V_{ij}^0 R_{ij}$ with increase of t, makes it possible to satisfy the equations (39), which exceed in number the disposable unknowns, and leads to values independent of r, and on the other hand the approximate solution (59) obtained on the basis of this substitution gives a qualitatively correct representation of the development of the cascade process. In this connection it must be noted that the substitution (24) is convenient practically, since it can be applied for arbitrary choice of the functions p_i, P_i, and P_{ij}, and a large part of the functions (26) turn out to be constants, not even depending on the indices i and j, which can be calculated by means of the single integrations involved in Eqs. (26), (47) and (61), (62), respectively. The method we have given for finding the functions has advantages in comparison with other methods used in such cases, based on the principle of least squares, because in the latter methods the squaring and averaging of errors involves highly multiple integrals.

An idea of the accuracy of the resulting solution can be obtained by calculations of the type of those presented in Ref. 16.

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CALCULATION OF THE INTERACTION POTENTIAL OF ATOMS

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The sum of the Coulomb interaction between atomic nuclei and the change in electron energy connected with the mutual approach of the nuclei is taken as the interaction potential. The electron energy is computed on the basis of the statistical model.

I. It was shown in a previous article[1] that the energy H_0 of electrons, in the approximation of the Thomas-Fermi statistical model and for the presence of two nuclei, lies between the two values H and H_1, which differ by not more than 5 per cent:

$$\frac{1}{e^2}H = \int\left\{\lambda\left[\frac{3}{5}(\rho_{01}+\rho_{02})^{5/3} - \frac{1}{2}(\rho_{01}^{2/3}+\rho_{02}^{2/3})(\rho_{01}+\rho_{02})\right] - \frac{1}{2}\left(\frac{Z_1}{r_1}+\frac{Z_2}{r_2}\right)(\rho_{01}+\rho_{02})\right\}dv,$$

$$\frac{1}{e^2}H_1 = \int\left\{\lambda\left[\frac{1}{2}(\rho_{01}^{2/3}+\rho_{02}^{2/3})(\rho_{01}+\rho_{02}) - \frac{2}{5}(\rho_{01}^{2/3}+\rho_{02}^{2/3})^{5/2}\right] - \frac{1}{2}\left(\frac{Z_1}{r_1}+\frac{Z_2}{r_2}\right)(\rho_{01}+\rho_{02})\right\}dv, \quad (1)$$

where $\lambda = \frac{1}{2}(3\pi^2)^{2/3}\hbar^2/me^2 = 2.52 \times 10^{-8}$ cm, Z_1e and Z_2e are the nuclear charges, $\rho_{01}(r_1)$ and $\rho_{02}(r_2)$ are the Thomas-Fermi electron densities in the atoms without consideration of the mutual interaction, r_1 is the distance to the nucleus of the first atom, r_2, to the nucleus of the second atom.