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ON MAGNETOHYDRODYNAMICAL EQUILIBRIUM CONFIGURATIONS

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The equilibrium conditions for bounded systems of a conducting gas in a magnetic field are investigated. We have obtained the equilibrium conditions for a thin ring with a helical current (a) taking into account gravitational forces; (b) assuming the ring to be surrounded by gas; and (c) in an external magnetic field. We have formulated a theorem about the correspondence between magnetohydrodynamical equilibrium systems and hydrodynamical vortices. Using this theorem we reduce the problem of the equilibrium conditions for magnetohydrodynamical configurations to the theory of stationary flow of an incompressible fluid. We consider, for the case of axial symmetry, general equilibrium conditions for distributed currents.

INTRODUCTION

In recent years there have appeared a number of papers [1-9] investigating the equilibrium conditions and the stability of equilibrium configurations of currents in a conducting medium. The interest in these problems is partly due to attempts to explain the existence of magnetic fields in cosmic space, the magnetism of stars, and of the earth, and partly because of experimental investigations of high-current gas discharges. The current configurations investigated in the papers mentioned had axial symmetry and were unbounded along the axis of symmetry. Such systems are idealizations of real systems, except for the case where they are bounded at the ends (for instance, by electrodes in the case of gas discharges). It is therefore undoubtedly of interest to find the equilibrium conditions of bounded configurations occurring under the influence of electrodynamic forces. Recently Bostick[10] has found experimentally plasma equilibrium configurations, which he calls plasmoids, for which magnetic fields are apparently essential. Apart from the possible applications mentioned above, an investigation of such configurations may be of interest for the theory of ball lightning. In this case one often makes the natural assumption[11] that ball lightning is a closed current produced at the moment of the thunder discharge. If we forget for a moment the problem of the source of energy of ball lightning,[12] then the question whether or not the above hypothesis is possible depends on whether such a dynamic stable configuration can exist.

In the present paper we obtain the equilibrium conditions for a few systems described by the magnetohydrodynamical equations

$$-\nabla p + \frac{1}{c}[\mathbf{j}\times\mathbf{H}] + \rho\nabla\Phi = 0, \quad \text{curl}\,\mathbf{H} = \frac{4\pi}{c}\mathbf{j}, \qquad \nabla^2\Phi = -4\pi G\rho, \quad \text{div}\,\mathbf{H} = 0. \tag{1}$$

In these equations ρ and p are the density and pressure of the conducting gas, Φ the gravitational potential, G the gravitational constant, and **j** and **H** the current density and the magnetic field. These equations have been investigated before by a number of authors.[4-8] Lüst and Schlüter[5] obtained a bounded configuration in weak fields ($\mathbf{j}\times\mathbf{H} = 0$). Prendergast[8] found a solution for the case where the whole of the magnetic field was concentrated inside a gravitating sphere.

Some general considerations about the necessary conditions for a stable equilibrium can be obtained from the virial theorem given by Chandrasekhar and Fermi[2] for the case of a magnetohydrodynamical system. This theorem has the following form for the case of a closed system in equilibrium,

$$3(\gamma-1)U + \mathfrak{M} + \Omega = 0, \qquad U = \frac{1}{\gamma-1}\int_V p\,d\mathbf{r}, \quad \Omega = -\frac{1}{2}\int\rho\Phi\,d\mathbf{r}, \quad \mathfrak{M} = \int\frac{H^2}{8\pi}\,d\mathbf{r}. \tag{2}$$

where U is the internal energy of the gas, $\mathfrak{M}$ the energy of the magnetic field, and Ω the gravitational energy.

If $\Omega = 0$ there is no equilibrium possible of a closed system. For an open system and $\Omega = 0$ one can easily obtain instead of (2)

$$\int_V \left(3p + \frac{H^2}{8\pi}\right) d\mathbf{r} = \oint \left\{\left(p + \frac{H^2}{8\pi}\right)\mathbf{r} - \frac{1}{4\pi}(\mathbf{rH})\,\mathbf{H}\right\} ds. \quad (3)$$

Let the current be concentrated in a restricted region of space. Choosing the volume of integration sufficiently large we get from (3)

$$\int_V \left(3p + \frac{H^2}{8\pi}\right) d\mathbf{r} = \oint p\mathbf{r}ds = 3p_e V. \quad (4)$$

It is therefore obvious that an equilibrium is possible provided that the external pressure p_e exceeds the average pressure in the configuration.

For $\Omega = 0$ and $p_e = 0$ equilibrium is only possible in an external magnetic field. In that case, the surface integral in (3) does not go to zero. If, for instance, the field is uniform at infinity and equal to H_0, we have

$$\int \left(3p + \frac{H^2}{8\pi}\right) d\mathbf{r} = \frac{H_0^2}{8\pi} V. \quad (5)$$

The gas pressure and the pressure due to the magnetic field in the configuration can thus be balanced in the following three cases: (a) gravitational attraction, (b) pressure of an external gas, and (c) pressure of an external magnetic field.

Corresponding to this we shall obtain in Secs. 1 to 3 three cases of the simplest possible bounded equilibrium configurations, all of the form of a thin ring. Of special interest is the ring with a current maintained in equilibrium by the pressure of an external gas (Sec. 2) since such systems are apparently stable.

In Sec. 4 it is shown that equilibrium configurations correspond to hydrodynamical vortices. Based on this analogy we give an example of a spherical configuration. In Sec. 5 we obtain the equations describing the equilibrium conditions for axially symmetric configurations.

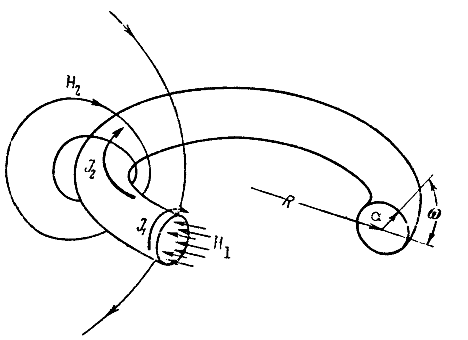

FIG. 1

1. GRAVITATING RING WITH CURRENT

Let us consider the equilibrium condition for a ring of a perfect gas under the following assumptions. The large radius of the ring R is considerably larger than its small radius $a\,(R \gg a)$ (see Fig. 1). The gas density is constant along a cross section and the electric conductivity is infinite. Along the ring there is a helical surface current producing inside the ring a magnetic field,

$$\mathbf{H}_1 = \{0,\ H_\varphi,\ 0\} \quad (6)$$

and outside the ring a field

$$\mathbf{H}_2 = \{H_r,\ 0,\ H_z\}. \quad (7)$$

The field $\mathbf{H}_1$ is related to the current I_1 flowing a round the axis of the ring,

$$H_\varphi = 2I_1 / cr. \quad (8)$$

and $\mathbf{H}_2$ is related to the axial current I_2 (see Fig. 1). The fields $\mathbf{H}_1$ and $\mathbf{H}_2$ are obviously mutually orthogonal.

The equations describing the behavior of the ring can be obtained from the Lagrangian

$$L = T(R,\ a) + \mathfrak{M}(R,\ a,\ I_1,\ I_2) - \Psi(V,\ T) - \Omega(R,\ a). \quad (9)$$

where T is the kinetic energy; if the density is constant, we have

$$T = \frac{1}{2}\int_V \rho v^2 d\mathbf{r} = \frac{M}{2}\left(\dot{R}^2 + \frac{\dot{a}^2}{2}\right), \quad (10)$$

and $\mathfrak{M}$ is the potential function of the currents taken with the opposite sign and is equal to the energy of the magnetic field. Since the fields $\mathbf{H}_1$ and $\mathbf{H}_2$ are orthogonal, $\mathfrak{M}$ will be equal to the sum of the energies of these fields,

$$\mathfrak{M} = (1/2c^2)(L_1 I_1^2 + L_2 I_2^2), \tag{11}$$

where L_1 and L_2 are the coefficients of self-induction of the corresponding currents,

$$L_1 = 4\pi (R - \sqrt{R^2 - a^2}) \approx \frac{2\pi a^2}{R}, \quad L_2 = 4\pi R\left(\ln \frac{8R}{a} - 2\right). \tag{12}$$

The third term in the Lagrangian,

$$\Psi(V, T) = -NT \ln VT^{1/(\gamma-1)} \tag{13}$$

is the free energy of a perfect gas, N is the total number of particles, $V = 2\pi^2 Ra^2$ is the volume of the ring, and T the temperature in energy units. Finally

$$\Omega(R, a) = -(GM^2/2\pi R)[\ln(8R/a) + 1/4] \tag{14}$$

is the gravitational energy of the ring and M its mass. We have, corresponding to the Lagrangian (9) the generalized velocities and coordinates

$$\dot{q} = \dot{a}, \ \dot{R}, \ I_1, \ I_2, \ T; \ q = a, \ R. \tag{15}$$

We must consider I_1, I_2, and T as cyclic velocities (see Ref. 13 for the possibility of considering T as a cyclic velocity).

The first two Lagrangian equations ($\dot{q} = \dot{a}, \dot{R}$) determine the rate of change of the two radii of the ring,

$$M\ddot{R} = F_R, \ \tfrac{1}{2} M\ddot{a} = F_a. \tag{16}$$

The second and third equations ($\dot{q} = I_1, I_2$) give, as long as the ring is supposed to be perfectly conducting, the conservation law of the flux of magnetic induction,

$$L_1 I_1 = \text{const}, \quad L_2 I_2 = \text{const}. \tag{17}$$

The fourth equation ($\dot{q} = T$) is the adiabatic equation

$$T = \text{const}\, V^{-(\gamma-1)}. \tag{18}$$

Assuming the form of the ring to be invariant we find for the condition that the equilibrium is stable,

$$F_q = (\partial L/\partial q)_{\dot{q}} = 0, \quad \partial F_q/\partial q < 0 \quad (q = a, \ R). \tag{19}$$

The motion in the neighborhood of the equilibrium position is according to (16) and (19) determined by the formulae

$$(\delta\ddot{a}) + \omega_a^2(\delta a) = 0, \quad (\delta\ddot{R}) + \omega_R^2(\delta R) = 0, \tag{20}$$

where δa and δR are the deviations from the equilibrium values of the radii and ω_a^2 and ω_R^2 the frequencies of the radial oscillations,

$$\omega_a^2 = -\frac{2}{M}\left(\frac{\partial F_a}{\partial a}\right); \quad \omega_R^2 = -\frac{1}{M}\left(\frac{\partial F_R}{\partial R}\right). \tag{21}$$

Using (17) and (18) one can easily show that $F_q = -\partial W/\partial q$, where

$$W = \frac{NT}{\gamma - 1} + \frac{1}{2c^2}(L_1 I_1^2 + L_2 I_2^2) - \frac{GM^2}{2\pi R}\left(\ln \frac{8R}{a} + \frac{1}{4}\right) \tag{22}$$

is the total energy of the ring at rest. Thus the condition that the equilibrium be stable corresponds to asking for a minimum of the total energy W, as might have been expected.

We write down the expressions for the forces,

$$F_a = \frac{\partial L}{\partial a} = \frac{2\pi I_1^2}{c^2}\frac{a}{R} - \frac{2\pi I_2^2}{c^2}\frac{R}{a} - \frac{GM^2}{2\pi Ra} + \frac{2NT}{a}, \quad F_R = \frac{\partial L}{\partial R} = -\frac{\pi I_1^2}{c^2}\frac{a^2}{R^2} + \frac{2\pi I_2^2}{c^2}\left(\ln\frac{8R}{a} - 1\right) - \frac{GM^2}{2\pi R^2}\left(\ln\frac{8R}{a} - \frac{3}{4}\right) + \frac{NT}{R}. \tag{23}$$

Assuming $F_a = F_R = 0$ we find the following relations which determine the equilibrium conditions

$$H_{10}^2 = 2H_G^2\left(\ln\frac{8R}{a}-\frac{7}{8}\right)\Big/\left(\ln\frac{8R}{a}-\frac{3}{2}\right) - 8\pi p_i\left(\ln\frac{8R}{a}-\frac{1}{2}\right)\Big/\left(\ln\frac{8R}{a}-\frac{3}{2}\right),$$

$$H_{20}^2 = H_G^2\left(\ln\frac{8R}{a}-\frac{1}{4}\right)\Big/\left(\ln\frac{8R}{a}-\frac{3}{2}\right) - 8\pi p_i\Big/\left(\ln\frac{8R}{a}-\frac{3}{2}\right). \quad (24)$$

We have here introduced the notation

$$H_{10} = 2I_1/cR, \quad H_{20} = 2I_2/ca; \quad H_G^2 = GM^2/\pi^2R^2a^2, \quad p_i = NT/V. \quad (25)$$

The virial theorem (2) can be obtained from (24) by multiplying the first equation of (24) by $V/8\pi$, the second by $(V/4\pi)$ $[\ln(8R/a) - 2]$, and adding them together. One can easily ascertain that the necessary conditions for the stability of the equilibrium, $\partial F_a/\partial a < 0$, $\partial F_R/\partial R < 0$ (stability of the equilibrium values of R and a under the condition that the form of the ring is kept invariant) are satisfied.

We can obtain the considerations regarding the stability of the ring form from the calculations of the stability of a straight cylinder (see Appendix). In the case where the whole of the current flows along the surface of the cylinder, an internal magnetic field, directed along the axis, will help the stabilizing action on the perturbed form of the cylinder. The most "dangerous" perturbations are of the form $\cos(2\pi z/\lambda)$ and $\cos(2\pi z/\lambda - \varphi)$, where z is the coordinate along the axis of the cylinder, φ the azimuthal angle, and λ the wave length of the perturbation. The instability of the first kind, "wriggling," is produced, on the whole, by the tendency of a gravitating mass to take on spherical form; the instability of the second kind (a winding cylinder) is connected with the instability of a straight current. If $2\pi a/\lambda \ll 1$ the stability criterion has the form (Eq. (14) of the Appendix)

$$1)\ H_{10}^2 > \frac{H_{20}^2}{2} + 2H_G^2\left(\ln\frac{\lambda}{\pi a} - C - \frac{1}{2}\right), \quad 2)\ H_{10}^2 > H_{20}^2\left(\ln\frac{\lambda}{\pi a} - C\right) - H_G^2\left(\ln\frac{\lambda}{\pi a} - C + \frac{1}{4}\right), \quad (26)$$

with C = 0.577.

From Eq. (24) it is clear that $H_{10}^2 \lesssim 2H_G^2$, $H_{10}^2 \lesssim 2H_{20}^2$. Consequently the first inequality can not be satisfied for wavelengths $\lambda \gg 2\pi a$. A gravitating equilibrium ring is thus unstable against a perturbation of the form $\cos(2\pi z/\lambda)$ (wriggling).

2. RING WITH CURRENT IN A GAS

Let us now consider a ring with current embedded in a gaseous atmosphere, the pressure of which exceeds the gas pressure inside the ring by an amount p_e. This excess pressure together with the pressure due to the external magnetic field H_2 balances the pressure of the internal axial field H_1. As in the previous section we consider an idealized situation where the current flows in an infinitely thin surface layer of the ring. In this case the magnetic field exerts a normal pressure on the surface and the condition that the forces on the surface (s) of the ring are in equilibrium is

$$H_1^2|_s = H_2^2|_s + 8\pi p_e|_s, \quad (27)$$

where according to (8) $H_1 = 2I_1/cr$.

Condition (27) must be used to determine the form of the cross section of the equilibrium configuration. For a thin ring this cross section is a circle with $r|_s = R + a\cos\omega$ and

$$H_1^2|_s = H_{10}^2\left(1 - 2\frac{a}{R}\cos\omega\right). \quad (28)$$

The distribution of the magnetic field due to an axial current can be obtained from the paper by Fock[14] in which he studies the skin-effect in a torus. For $R \gg a$ the surface current density is distributed as follows

$$j_\varphi = \text{const}\left\{1 - \frac{a}{R}\left(\ln\frac{8R}{a} - \frac{1}{2}\right)\cos\omega\right\};$$

Consequently,

$$H_2^2|_s = H_{20}^2\left\{1 - 2\frac{a}{R}\left(\ln\frac{8R}{a} - \frac{1}{2}\right)\cos\omega\right\}. \quad (29)$$

If we now substitute (28) and (29) into (27) and compare terms independent of ω, and terms in $\cos\omega$ we obtain

$$H_{10}^2 = H_{20}^2 + 8\pi p_e, \quad H_{20}^2 = 8\pi p_e \Big/ \left(\ln\frac{8R}{a} - \frac{3}{2}\right)$$

or

$$H_{10}^2 = H_{20}^2\left(\ln\frac{8R}{a} - \frac{1}{2}\right), \quad H_{10}^2 = 8\pi p_e\left(\ln\frac{8R}{a} - \frac{1}{2}\right) \Big/ \left(\ln\frac{8R}{a} - \frac{3}{2}\right). \tag{30}$$

It is true that this equilibrium condition can also be obtained from (24) if put there $G = 0$ and take it into account that for p_i we must take the difference between the internal and external pressure, which now is negative, $p_i = -p_e < 0$.

For p_e = constant we find from (23) and (17)

$$\frac{\partial F_a}{\partial a} = -\frac{2\pi I_2^2}{c^2 a^2} R\left[2\ln\frac{8R}{a} - 2 + \left(\ln\frac{8R}{a} - 2\right)^{-1}\right] \quad \frac{\partial F_R}{\partial R} = -\frac{2\pi I_1^2}{c^2 R}\left[2\ln\frac{8R}{a} - 1 + 2\left(\ln\frac{8R}{a} - 2\right)^{-1}\right]. \tag{31}$$

Since $\partial F_a/\partial a < 0$ and $\partial F_R/\partial R < 0$ the equilibrium is stable with respect to changes in R and a. The criterion for stability with respect to a change in form, obtained for a homogeneous cylinder, has the form (see Eq. (14) of the Appendix)

$$H_{10}^2 > H_{20}^2\left(\ln\frac{\lambda}{\pi a} - C\right).$$

According to the equilibrium condition (30)

$$H_{10}^2 = H_{20}^2\left(\ln\frac{8R}{a} - \frac{1}{2}\right).$$

Assuming that the wavelength of the perturbation has its maximum value possible in the ring, $\lambda = \pi R$, we see that the criterion for stability is satisfied.

We see thus that the configurations of magnetic fields which produce a helical current flowing along the surface of the ring and in which the external gas pressure is maintained are very probably stable configurations. This conclusion should be verified by a direct calculation of the stability of the ring. Whether such a ring could be observed in an atmosphere of a non-conducting gas under laboratory conditions is an interesting problem. The external gas can be prevented from penetrating inside the ring by the ionized envelope along which the current flows. We write down the connection between the current necessary for equilibrium and the excess of the external pressure. Expressing H_{10} in terms of I_1 and H_{20} in terms of I_2 we get from Eq. (30)

$$\frac{I_2^2}{a^2} = I_1^2 \Big/ R^2\left(\ln\frac{8R}{a} - \frac{1}{2}\right) = 2\pi p_e c^2 \Big/ \left(\ln\frac{8R}{a} - \frac{3}{2}\right). \tag{32}$$

If the currents are expressed in amperes and the pressure in atmospheres, and if to be specific we put under the logarithm sign $R/a = 10$, we have

$$I_2/a = I_1/2R = 15\cdot 10^3 p_e. \tag{33}$$

For instance, if $a \sim 1$ cm, and $p_e \sim 1$ atmos the current must be of the order of tens of thousands of amperes.

The period during which such a ring will exist unsupported from without will be determined by the time in which the originally produced current will die away due to the collisions of the electrons with ions and atoms. If collisions with ions are the determining factor, this relaxation time does not depend on the gas or its dimensions, but is determined by the magnitude of the conductivity $\sigma = e^2 n\tau/m$ $[n\tau \approx 0.1(mv^2/e^2)^2]$ and is given by the equation

$$t_\sigma = 4\pi\sigma a^2/c^2 \approx 10\, a^2 (v/c)^3.$$

Let, for instance $a \sim 1$ cm and the electron velocity be $v \sim 10^{-2}$ c (such a velocity is normal for discharges with a current strength $I \sim 10^4$ amp); in that case $t_\sigma \sim 10\ \mu$ sec. The time of diffusion of the external gas into the current channel $t_D \sim a^2/D$ (D is the diffusion coefficient; for air under normal conditions $D \sim 0.1$) is larger than the relaxation time of the current right up to pressures $p \approx 0.001$ atmos and consequently can be neglected.

3. RING CURRENT IN AN EXTERNAL MAGNETIC FIELD

We shall assume that the ring with an axial current I_2 is located in a homogeneous magnetic field H_0 which is perpendicular to the plane of the ring. We can obtain expressions for the forces F_a and F_R from (23), if we assume there that $I_1 = 0$ and $G = 0$, and add in F_R the term corresponding to the Lorentz force $c^{-1} I_2 H_0 2\pi R$,

$$F_a = -\frac{2\pi I_2^2}{c^2}\frac{R}{a} + \frac{2NT}{a}, \qquad F_R = \frac{I_2^2}{2c^2}\frac{\partial L_2}{\partial R} + \frac{NT}{R} + \frac{1}{c} I_2 H_0 2\pi R. \tag{34}$$

From the condition $F_a = 0$ we get the well-known formula for the pinch effect,

$$I_2^2 = 2c^2 NT / 2\pi R. \tag{35}$$

Expressing in NT the equation $F_R = 0$ in terms of I_2^2 we find the magnitude of the magnetic field supporting the equilibrium current I_2 flowing in a ring of radius R,

$$H_0 = -\frac{I_2}{cR}\left(\frac{1}{4\pi}\frac{\partial L_2}{\partial R} + \frac{1}{2}\right). \tag{36}$$

If there is a strong skin effect, $L_2 = 4\pi R\,[\ln(8R/a) - 2]$ and consequently

$$H_0 = -\frac{I_2}{cR}\left(\ln\frac{8R}{a} - \frac{1}{2}\right). \tag{37}$$

If the current density is constant over a cross section, $L_2 = 4\pi R\,[\ln(8R/a) - 7/4]$ and

$$H_0 = -\frac{I_2}{cR}\left(\ln\frac{8R}{a} - \frac{1}{4}\right). \tag{38}$$

One can easily obtain Eq. (37) from the requirement that in equilibrium the value of the magnetic field on the surface of a perfectly conducting ring must be constant over a cross section. According to (29) the field due to the current is on the surface equal to

$$H_2 = H_{20}\left\{1 - \frac{a}{R}\left(\ln\frac{8R}{a} - \frac{1}{2}\right)\cos\omega\right\}.$$

The distribution of the external field for the case $R \gg a$ is determined in the same way as in the case of a perfectly conducting cylinder located in a homogeneous field

$$H = \nabla\varphi, \quad \varphi = aH_0\left(\frac{\rho}{a} + \frac{a}{\rho}\right)\sin\omega$$

where ρ is the polar radius and ω the azimuthal angle. Hence we have

$$H_\omega\Big|_{\rho=a} = \frac{1}{\rho}\frac{\partial\varphi}{\partial\omega}\Big|_{\rho=a} = 2H_0\cos\omega.$$

Consequently the total field on the surface of the ring is equal to

$$H_{20} + 2\left\{H_0 - \frac{H_{20}}{2}\frac{a}{R}\left(\ln\frac{8R}{a} - \frac{1}{2}\right)\right\}\cos\omega.$$

Putting the factor in front of $\cos\omega$ equal to zero we obtain condition (37).

A simple calculation of the derivatives of F_a and F_R leads to

$$\frac{\partial F_a}{\partial a} = -\frac{4\pi I_2^2}{c^2}\frac{R}{a}\left(\gamma - 1 + \frac{4\pi R}{L_2}\right), \quad \frac{\partial F_R}{\partial R} = \frac{2\pi I_2 H}{c}\left\{\frac{3}{2} - \left[(5 - 2\gamma)\frac{2\pi R}{L_2} - 1\right] \Big/ \frac{4\pi R}{L_2}\left(3 + \frac{2\pi R}{L_2}\right)\right\}. \tag{39}$$

Since both derivatives are negative the equilibrium is stable with respect to a change in R or a.

As we shall show in the next section, a ring current in a magnetic field has an interesting counterpart in hydrodynamics, namely the well known smoke rings.

4. HYDRODYNAMICAL ANALOGY OF EQUILIBRIUM CONFIGURATIONS

The theory of the equilibrium of magnetohydrodynamical systems shows an interesting analogy with the theory of hydrodynamical vortices in an incompressible fluid. This analogy can be formulated in the form of the following theorem: the magnetic field and current density in an equilibrium magnetohydrodynamical configuration are respectively (expressed in suitably defined units) identical with the velocity field of a hydrodynamical vortex which is at rest and the vorticity in the system itself. If there are gravitational forces present the applicability of the analogy is limited by the requirement that the density in the configuration be uniform. The proof of this theorem follows from comparison of the equations describing the kinematics of a vortex with the magnetohydrostatical equations. We have written these last equations below in rationalized electro-magnetic units in order to let the analogy be more complete; one may assume that the gravitational terms are included in p:

Equations for the kinematics of a vortex	Magnetohydrostatic equations
$\operatorname{div} \mathbf{v} = 0$	$\operatorname{div} \mathbf{H} = 0$
$\operatorname{curl} \mathbf{v} = \mathbf{\Omega}$	$\operatorname{curl} \mathbf{H} = \mathbf{j}$
$[\mathbf{\Omega} \times \mathbf{v}] = -\nabla (p/\rho + v^2/2)$	$[\mathbf{j} \times \mathbf{H}] = \nabla p$

It is clear that the first two equations for $\mathbf{v}$ and $\mathbf{\Omega}$ on the one side, and for $\mathbf{H}$ and $\mathbf{j}$ on the other side are identical. The last equations impose on these quantities the identical conditions $\operatorname{curl} [\mathbf{\Omega} \times \mathbf{v}] = 0$ and $\operatorname{curl} [\mathbf{j} \times \mathbf{H}] = 0$. The boundary conditions for $\mathbf{v}$ and $\mathbf{H}$ are also identical. Consequently the distributions of $\mathbf{v}$ and $\mathbf{H}$ will be identical.

The functions $p/\rho + v^2/2$ on the one side and p on the other side, the gradients of which are determined from the last equations, may be different from zero. The analogy established a moment ago is, of course, not extended to the dynamics of a system described by equations which are no longer identical. In particular, conclusions about the stability of the one system can not be transferred to its analogue. For instance, it is well known that the plasma string with a current ("pinch effect") is unstable. The corresponding analogue — a single vortex filament in a fluid — is a stable system.

The fact that there exists this identity between the distributions of the velocity field of a vortex and of the magnetic field of a magnetohydrodynamical configuration makes it possible to transfer methods and results of investigations in the one field to the other field. For instance, from the point of view of looking for the analogue, a ring current in a magnetic field corresponds to a circular vorticity differing from zero in a ring of large radius R and small radius a. It is well known that the kinematical condition for the existence of such a vortex is the motion of the vortex ring along its axis of symmetry. According to our theorem the velocity of the current flowing along a ring, in the frame of reference in which this is at rest, corresponds to an external magnetic field maintained in equilibrium by the ring current. Knowing the magnitude of this field we can thus immediately write down also an expression for the velocity of the vortex ring motion in a non-conducting gas which will be equal to the velocity of the gas current in the ring, but with opposite sign. Introducing the intensity of vorticity $\kappa = \int \mathbf{\Omega} d\mathbf{s}$ corresponding to the strength of the current, and going over to absolute rationalized electromagnetic units we obtain from (38) the velocity of the vortex ring for the case where the vorticity is constant along a cross section (see Ref. 15 p. 241)

$$v = \frac{\varkappa}{4\pi R}\left(\ln\frac{8R}{a} - \frac{1}{4}\right). \tag{41}$$

We investigated in Secs. 1 to 3 systems with a surface current; these correspond in hydrodynamics to stationary currents of an incompressible fluid with a surface where the tangential velocoty is discontinous.

Using the hydrodynamical analogy we can find a number of examples of magnetohydrodynamical equilibrium configurations with a distributed current. Thus, for instance, corresponding to the well known spherical vortex of Hill's (Ref. 15, p. 245) in hydrodynamics, we have an equilibrium configuration, which might be called a spherical plasmoid, consisting of a conducting gas inside a sphere $r \leq a$ in an external field H_0 directed along the z-axis. We give here this solution in spherical coordinates (r, ϑ, φ):

$$\begin{array}{ll} r \leqslant a & r > a \\ p = \dfrac{45}{32\,\pi} H_0^2 \dfrac{r^2}{a^2}\left(1 - \dfrac{r^2}{a^2}\right)\sin^2\vartheta & p = 0 \\ j_\varphi = \dfrac{15}{8}\dfrac{cH_0}{\pi a^2} r \sin\vartheta & j_\varphi = 0 \\ H_r = -\dfrac{3}{2} H_0\left(1 - \dfrac{r^2}{a^2}\right)\cos\vartheta & H_r = H_0\left(1 - \dfrac{a^3}{r^3}\right)\cos\vartheta \\ H_\vartheta = -3\,H_0\left(\dfrac{r^2}{a^2} - \dfrac{1}{2}\right)\sin\vartheta & H_\vartheta = -H_0\left(1 + \dfrac{a^3}{2\,r^3}\right)\sin\vartheta. \end{array} \tag{42}$$

One can also easily obtain the analogous spheroidal configuration. We note that Lundquist[1] and Chandrasekhar and Prendergast[7] also mentioned the analogy between a stationary flow of a fluid with a vorticity different from zero and a stationary magnetic field. However, in those papers the details of this correspondence were not established.

5. GENERAL EQUILIBRIUM CONDITIONS FOR AN AXIALLY SYMMETRIC SYSTEM

We must note that in the considerations of Secs. 2 to 4 of the simplest bounded equilibrium configurations which had the form of thin rings we assumed that they did not depend strongly on the distribution of the current over the cross section. However, they correspond to only one of the limiting cases of a more general kind of configuration which has the form of a toroid with a non-circular cross section and with arbitrary relations between the linear dimensions. Other limiting cases of such configurations are the configurations of the kind of spherical (or spheroidal) plasmoids. It is of interest to discuss general conditions for the existence of such equilibrium configurations.

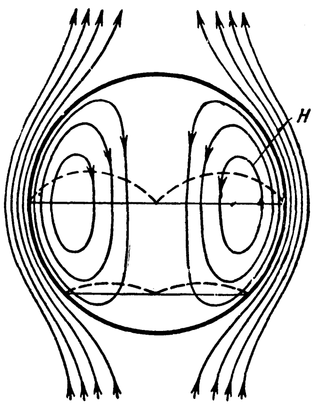

FIG. 2

We suppose that gravitational effects can be neglected (or, if it is necessary to take them into account that the gas can be considered to be incompressible). In that case we need obviously not write down the term $\rho\Phi$ and may assume it to be included in p. Using the analogy developed in the previous section we take over into the theory of equilibrium of magnetohydrodynamical configurations (magnetohydrostatics) the hydrodynamical theory of the stationary motion of an incompressible fluid. We shall consider the equilibrium conditions for axially symmetric magnetohydrodynamical configurations. In that case the set of equations describing the equilibrium conditions can be condensed into one equation according to the following scheme. We introduce the functions

$$\psi = \int_0^r H_z \cdot 2\pi r dr, \quad I = \int_0^r j_z \cdot 2\pi r dr = \frac{cr}{2} H_\varphi; \tag{43}$$

in terms of which the r- and z-components of the field and the current are expressed. The surfaces of constant ψ and of constant I are the surfaces corresponding to the magnetic lines of force and the current lines. From the equilibrium condition $-\nabla p + [\mathbf{j} \times \mathbf{H}]/c = 0$ it follows that each of these surfaces coincides with a surface of constant pressure, that is, $p = p(\psi)$, $I = I(\psi)$. From the equilibrium condition which is not used we can determine the current density component j_φ as a function of ψ and r. Finally, the equation $\partial H_r/\partial z - \partial H_z/\partial r = 4\pi j_\varphi/c$ gives us the relation we are looking for. All this can conveniently be collected in the following table,

$$\operatorname{div}\mathbf{H} = 0 \qquad \frac{1}{r}\frac{\partial (rH_r)}{\partial r} + \frac{\partial H_z}{\partial z} = 0 \qquad H_z = \frac{1}{2\pi r}\frac{\partial \psi}{\partial r}, \quad H_r = -\frac{1}{2\pi r}\frac{\partial \psi}{\partial z} \tag{44}$$

$$(\operatorname{curl}\mathbf{H})_{r,z} = \frac{4\pi}{c}\mathbf{j}_{r,z} \qquad \frac{1}{r}\frac{\partial (rH_\varphi)}{\partial r} = \frac{4\pi}{c} j_z, \quad -\frac{\partial H_\varphi}{\partial z} = \frac{4\pi}{c} j_r \qquad j_z = \frac{1}{2\pi r}\frac{\partial I}{\partial r}, \quad j_r = -\frac{1}{2\pi r}\frac{\partial I}{\partial z} \tag{45}$$

$$-\nabla p + \frac{1}{c}[\mathbf{j}\times\mathbf{H}] = 0 \qquad (\mathbf{H}\nabla p) = 0, \quad (\mathbf{j}\nabla p) = 0 \qquad p = p(\psi), \ I = I(\psi) \tag{46}$$

$$-\frac{\partial p}{\partial r} + \frac{1}{c}(j_\varphi H_z - j_z H_\varphi) = 0 \qquad r j_\varphi d\psi = \frac{1}{c} dI^2 + 2\pi c r^2 dp \tag{47}$$

$$(\operatorname{curl}\mathbf{H})_\varphi = \frac{4\pi}{c}\mathbf{j}_\varphi \qquad \frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = \frac{4\pi}{c} j_\varphi \qquad \frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} = -\frac{8\pi^2}{c^2} r j_\varphi = -\frac{16\pi^2}{c^2} I(\psi)\frac{dI(\psi)}{d\psi} - 16\pi^3 r^2 \frac{dp(\psi)}{d\psi} \tag{48}$$

Equation (48), which contains two arbitrary functions, gives us the equilibrium condition. This equation is essentially an extension into magnetohydrostatics of the condition for a stationary flow of an incompressible fluid which has been investigated before (Lamb[15]). It was obtained, in slightly different form, by Chandrasekhar and Prendergast who instead of the functions ψ and I (the meaning of which is respectively the induction flux and the current through a circular cross section of radius r at z = constant) introduced the functions p and T which are connected with ψ and I by relations $p = \psi/2\pi r^2$, $T = 2I/cr^2$. If these functions are introduced the meanings of the different terms of the equation are no longer clear.

Let us note a few consequences of the equations we just obtained.

From Eq. (47) for j_φ it follows that, (1) if $j_\varphi = 0$ the functions I and p do not depend on z. Consequently if there is no gravitation a bounded equilibrium configuration can exist only if there is an azimuthal current present; it also follows that, (2) if $j_\varphi = \text{const}/r$ a bounded configuration can exist only if $p = \text{const}$ (weak fields), and that, (3) if $j_\varphi = \text{const.}\, r$ the solution depending on z exists only if $I = \text{const}$, i.e., if $H_\varphi = \text{const}/r$. In particular, solution (42) corresponding to Hill's vortex is obtained in the case where $I = 0$.

In conclusion I should like to express my gratitude to Academician M. A. Leontovich for discussing this paper with me and for a number of helpful suggestions.

APPENDIX

Criteria for the Stability of an Perfectly Conducting Cylinder with a Surface Current

We investigate the stability of a homogeneous perfectly-conducting cylinder (a) taking gravitation into account, and (b) in the presence of external pressure. The program of investigation is the same as in Ref. 9 where these effects were not taken into account. Let there be imposed on the cylinder a perturbation of the form $\exp\{i(kz + m\omega) + \Omega t)\}$. An arbitrary perturbation can be represented in the form of a superposition a such perturbations with different values of k and m ($m = 0, \pm 1, \pm 2, \ldots$). For given values of k and m the equations of motion together with the boundary conditions will determine the corresponding eigenvalue Ω^2.

If we are only interested in the stability criterion we can simplify our considerations by assuming the gas to be incompressible. Indeed, the boundary of the unstable region lies at the value $\Omega^2 = 0$. This value of Ω^2 corresponds to a stationary perturbation when the compressibility of the gas does not play a role.

(a) Gravitating cylinder. The original magnetohydrodynamical equations have the form

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla\left(p - \rho\Phi + \frac{H^2}{8\pi}\right) + \frac{1}{4\pi}(\mathbf{H}\nabla)\mathbf{H},\quad \partial\mathbf{H}/\partial t = \operatorname{curl}[\mathbf{v}\times\mathbf{H}],\quad \operatorname{div}\mathbf{v} = 0,\quad \nabla^2\Phi = -4\pi G\rho. \tag{1}$$

We give the values of the different variables in the unperturbed state:

$$\begin{array}{cc} \text{Inside the cylinder} & \text{Outside the cylinder} \\ \rho = \text{const} & \rho = 0 \\ H_\omega = 0 & H_\omega = H_{20}a/r \\ H_z = H_{10} = \text{const} & H_z = 0 \\ \Phi_i = -\pi G\rho r^2 & \Phi_e = -2\pi G\rho a^2 \ln r + \text{const} \\ p = p_0 - \pi G\rho^2 r^2 & p = 0 \end{array} \tag{2}$$

After linearization equations (1) have the form

$$\rho\Omega^2\xi = -\nabla\left(p^{(1)} + \frac{H_z^{(1)}H_{10}}{4\pi} - \rho\Phi^{(1)}\right) + \frac{1}{4\pi}(\mathbf{H}_0\nabla)\mathbf{H}^{(1)},\quad \mathbf{H}^{(1)} = \operatorname{curl}[\xi\times\mathbf{H}_0] = (\mathbf{H}_0\nabla)\xi = ikH_{10}\xi,\quad \operatorname{div}\xi = 0,\quad \nabla^2\Phi^{(1)} = 0. \tag{3}$$

where $\xi = \mathbf{v}/\Omega = \xi(r)\exp\{i(kz + m\omega) + \Omega t\}$ is the displacement vector of the gas particles, and $p^{(1)}$ and $\mathbf{H}^{(1)}$ are the perturbation-induced deviations of the pressure and magnetic field from their equilibrium values. Eliminating $\mathbf{H}^{(1)}$ and introducing the notation $\tilde{p} = p^{(1)} + H_{10}H_z^{(1)}/4\pi$ we can write this set of equations in the following form,

$$\rho(\Omega^2 + k^2H_{10}^2/4\pi)\xi = -\nabla(\tilde{p} - \rho\Phi^{(1)}), \tag{4}$$

$$\nabla^2\Phi^{(1)} = 0;\quad \nabla^2\tilde{p} = 0. \tag{5}$$

The solution for $\tilde{p}$ which is bounded for $r = 0$ is of the form

$$\tilde{p} = \tilde{p}(a)\frac{I_m(kr)}{I_m(ka)}e^{i(kz+m\omega)+\Omega t}. \tag{6}$$

The boundary condition for $\tilde{p}$

$$\tilde{p}(a) + \xi_r(a)\frac{\partial p_0}{\partial r} = \frac{H_\omega^0 H_\omega^{(1)}}{4\pi} + \xi_r\frac{\partial}{\partial r}\left(\frac{H_{\omega 0}^2}{8\pi}\right) \tag{7}$$

differs from the boundary condition in Ref. 9 by the term $\xi_r \partial p_0/\partial r$. Using the results of Ref. 9 we get

$$\tilde{p}(a) = -\frac{\xi_r(a)}{4\pi}\left\{H_{20}^2 - \frac{m^2 H_{20}^2}{ka\{K_{m-1}(ka)/K_m(ka)\}+m} - 2H_G^2\right\}, \tag{8}$$

where

$$2H_G^2 = -4\pi\partial p_0(a)/\partial r = 8\pi^2 G\rho^2 a^2. \tag{9}$$

The solution of the equation $\nabla^2\Phi^{(1)} = 0$ inside and outside the cylinder can therefore be expressed by

$$\Phi_i^{(1)} = C_1 I_m(kr)\, e^{i(kz+m\omega)+\Omega t}; \quad \Phi_e^{(1)} = C_2 K_m(kr)\, e^{i(kz+m\omega)+\Omega t}. \tag{10}$$

The constants C_1 and C_2 are determined by the requirement that the potential and its derivative with respect to the normal must be continuous for $r = a + \xi_r$,

$$p_i^{(1)} + \xi_r\frac{\partial\Phi_{i0}}{\partial r} = \Phi_e^{(1)} + \xi_r\frac{\partial\Phi_{e0}}{\partial r}; \quad \frac{\partial\Phi_i^{(1)}}{\partial r} + \xi_r\frac{\partial^2\Phi_{i0}}{\partial r^2} = \frac{\partial\Phi_e^{(1)}}{\partial r} + \xi_r\frac{\partial^2\Phi_{e0}}{dr^2}. \tag{11}$$

We find thus

$$C_1 = 4\pi G\rho a K_m(ka)\xi_r(a); \quad C_2 = 4\pi G\rho a I_m(ka)\xi_r(a). \tag{12}$$

From Eqs. (6), (8), (10), and (12) we determine $\rho\Phi^{(1)} - \tilde{p}$ and substituting this result into (4) we get

$$\Omega^2 = \left\{\frac{c^2}{a^2}\left(1 - \frac{m^2}{[kaK_{m-1}(ka)/K_m(ka)]+m}\right) + \frac{4c_G^2}{a^2}\left(K_m(ka)I_m(ka) - \frac{1}{2}\right)\right\}\left(ka\frac{I_{m-1}(ka)}{I_m(ka)} - m\right) - k^2c_H^2, \tag{13}$$

where

$$c^2 = H_{20}^2/4\pi\rho; \; c_G^2 = H_G^2/4\pi\rho; \; c_H^2 = H_{10}^2/4\pi\rho.$$

The stability condition $\Omega^2 < 0$ is for $c_G = 0$ identical with the criterion obtained in Ref. 9 for a compressible gas; for $c = 0$ and $m = 0$ it coincides with the result of the considerations of Chandrasekhar and Fermi,[2] obtained by a different method. For an infinite cylinder the criterion for stability can not be realized for $m = 0, 1$, $ka = 2\pi a/\lambda \to 0$, when it is of the form

$$H_{10}^2 > \frac{1}{2}H_{20}^2 + 2H_G^2\left(\ln\frac{\lambda}{\pi a} - C - \frac{1}{2}\right) \quad (m=0), \quad H_{10}^2 > H_{20}^2\left(\ln\frac{\lambda}{\pi a} - C\right) - H_G^2\left(\ln\frac{\lambda}{\pi a} - C + \frac{1}{4}\right) \quad (m=-1). \tag{14}$$

(b) Cylinder with external pressure. If the external gas is non-conducting the change in the previous arguments lies in the addition to the right hand side of Eq. (7) of a correction to the external pressure, $p_e^{(1)}$, determined from the equation $\Omega^2\xi = -\nabla p_e^{(1)}$, div $\xi = 0$. When $G = 0$ the dispersion relation is

$$\left(\rho_i\frac{\Omega^2}{k^2} + \frac{H_{10}^2}{4\pi}\right)\Big/\left(\frac{I_{m-1}(ka)}{kaI_m(ka)} - \frac{m}{(ka)^2}\right) + \rho_e\frac{\Omega^2}{k^2}\Big/\left(\frac{K_{m-1}(ka)}{kaK_m(ka)} + \frac{m}{(ka)^2}\right) = \frac{H_{20}^2}{4\pi}\left[1 - m^2\Big/\left(ka\frac{K_{m-1}(ka)}{K_m(ka)} + m\right)\right]. \tag{15}$$

The absolute value of Ω is reduced, but the stability criterion remains the same as for $\rho_e = 0$.

If the gas on the outside is perfectly conducting the calculation is slightly more complicated, but one can show that the stability criterion remains the same as before.

[1] S. Lundquist, Ark. f. Fys. **5**, 297 (1952).

[2] S. Chandrasekhar and E. Fermi, Astrophys. J. **118**, 116 (1953).

[3] M. Kruskal and M. Schwarzschild, Proc. Roy. Soc. **A223**, 348 (1954).

[4] V. C. A. Ferraro, Astrophys. J. **119**, 407 (1954).

[5] R. Lüst and A. Schlüter, Z. Astrophys. **34**, 263 (1954).

[6] S. Chandrasekhar, Proc. Nat. Acad. Sci. **42**, 1 (1956).

[7] S. Chandrasekhar and K. H. Prendergast, Proc. Nat. Acad. Sci. **42**, 5 (1956).

[8] K. H. Prendergast, Astophys. J. **123**, 498 (1956).

[9] V. D. Shafranov, Атомная энергия (Atomic Energy) **5**, 38 (1956).

[10] W. H. Bostick, Phys. Rev. **104**, 292 (1956); **104**, 1191 (1956).

[11] I. S. Stekol'nikov, Физика молнии и грозозащита (The Physics of Thunder and Lightning Protection) Moscow-Leningrad, 1943.

[12] P. L. Kapitza, Dokl. Akad. Nauk SSSR **101**, 245 (1955).

[13] H. Helmholtz, Vorlesungen über die Theorie der Wärme, Leipzig, 1922.

[14] V. A. Fock, Phys. Z. USSR **1**, 215 (1932).

[15] H. Lamb, Hydrodynamics, Cambridge University Press, 1932.

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