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139

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DE HAAS–VAN ALPHEN EFFECT IN A VARIABLE MAGNETIC FIELD

A. M. KOSEVICH

Chernovtsy State University

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A theoretical study is made of the characteristic features of the oscillations of the magnetic moment of a metallic specimen at low temperatures in a pulsed magnetic field.

IN connection with the use of the pulse method for the study of the De Haas-Van Alphen effect in strong magnetic fields,[1] it is of interest to clarify the peculiarities of the oscillations of the magnetic moment of a metal at low temperatures in a variable magnetic field of a pulsed type.

In Sec. 1 of the present paper, we give qualitative consideration to this problem, and show that the oscillation properties of the magnetic moment of a metallic specimen in a pulsed field depend essentially on the ratio of the characteristic dimension in the problem of the penetration of the variable field into the metal to the dimension of the sample itself. In Sec. 2 and 3, formulas are obtained for the oscillating part of the magnetic moment for different values of this ratio, and their analysis is given.

1. OSCILLATIONS OF THE MAGNETIC MOMENT OF A METALLIC SAMPLE IN A PULSED FIELD

(Qualitative Considerations)

The oscillations of the magnetic moment of a metal are determined by the quantized motion of the conduction electrons (current carriers) in the magnetic field; their periods and amplitudes are connected with the form of the electron dispersion law near the Fermi surface.[2] There are usually some groups of electrons in the metal with different dispersion laws, and each of these makes its own contribution to the oscillations of the magnetic moment. In what follows, we shall consider the contribution to the oscillating part of the magnetic moment of only one such group of electrons with the particular dispersion law $\epsilon = \epsilon(\mathbf{p})$ (ϵ = energy, $\mathbf{p}$ = quasi-momentum of the electron).

If the homogeneous field H is constant, then the oscillating part of the magnetic moment* of the electron gas is given by the following formula:[2]

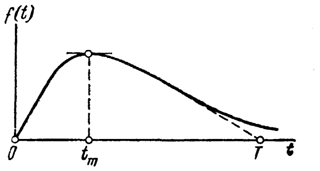
$$M_{\text{osc}} = V \sum_{n=1}^{\infty} \Psi_n(H) \cos\left(\frac{\alpha n}{H} + \varphi_n\right); \quad (1)$$

V = volume occupied by the electron gas (the volume of the metal); $\Psi_n(H) \equiv \Psi_n(H, T)$ = some slowly changing function of H and the temperature T; $\alpha = cS_m(\zeta)/e\hbar$, where $S_m(\zeta)$ = the ex-

*We are considering the component of the magnetic moment in the direction of the magnetic field.

tremal area of the surface formed by the intersection of the Fermi surface $\epsilon(p) = \zeta$ with the plane $p \cdot H = \text{const}$, and the notation of the constants is standard; φ_n = some "initial" phases of the oscillations, independent of H. We recall that Eq. (1) is valid for $kT/\zeta \ll 1$ and $H/\alpha \ll 1$; we shall consider these inequalities to be satisfied.

We consider that the metallic specimen is placed in a variable homogeneous magnetic field $H_0 = f(t)$, parallel to its surface, and that $f(t)$ have the form of a pulse (see figure) of duration T [$f(t)$ differs essentially from zero only in the interval $0 < t < T$ and has a single maximum for $t = t_m$; $f(0) > 0$].



Inasmuch as the external magnetic field is variable in time, it induces eddy currents in the conductor which distort the imposed field in the body of the metal; therefore, the magnetic field inside the specimen is not homogeneous throughout its thickness.

By virtue of the inhomogeneity of the magnetic field in the metal and the large value of the ratio α/H_0, the oscillations of the magnetic moments of neighboring layers of the metal differ significantly in phase; in averaging them over the thickness, a smoothing out takes place in the oscillations of the total magnetic momentum. In this regard, the following are exceptional: (1) the layers of metal adjoining the surface of the sample, where the averaging takes place only on one side (in the depth of the metal), and (2) the layer of metal in which the magnetic field is stationary (i.e., the layer in which $\partial H/\partial x = 0$, where x is the coordinate measured along the normal to the surface, into the depth of the specimen), and there is no shift in the phase of oscillation. Only these layers of the metal make an appreciable contribution to the oscillating part of the total magnetic moment of the metal.

It is easy to estimate the thicknesses of the above-mentioned layers, and to clarify the character of the dependence of the corresponding periods of oscillation on the external field. As is seen from (1), the oscillations of the magnetic moment are produced by the change in the values of α/H in the argument of the cosine, where $H = H(x, t)$ is different in the different layers of the metal. The magnetic field at the surface of the metal coincides with the external field H_0; consequently, the period of oscillation of the magnetic moment of the given layer with change in the external field is the same as for the constant field.[2] The thickness of this layer Δx is of the order of magnitude of the distance into the body of the metal at which the phase shift is equal to the period of oscillation.

$$\Delta\left(\frac{\alpha}{H}\right) = \alpha\left|\frac{\partial}{\partial x}\frac{1}{H}\right|\Delta x = \frac{\alpha}{H^2}\left|\frac{\partial H}{\partial x}\right|\Delta x \sim 2\pi. \tag{2}$$

It is appropriate to estimate the order of magnitude of $\partial H/\partial x$ by starting out from the form of the differential equation which defines the penetration of the magnetic field into the metal. We can consider the equation for the magnetic field in a specimen of cylindrical form without restricting the generality of the treatment:

$$\frac{1}{\rho}\frac{\partial H}{\partial \rho} + \frac{\partial^2 H}{\partial \rho^2} = \frac{1}{q}\frac{\partial H}{\partial t}. \tag{3}$$

Here ρ = distance from the axis of the cylinder, and $q = c^2/4\pi\mu\sigma$ (μ = magnetic susceptibility, σ = specific electrical conductivity of the metal). If the pulse duration of the magnetic field is T, then the problem has as its characteristic dimension the length $l \sim \sqrt{qT}$.

In the case in which the radius of the cylinder R is much greater than the length l ($R \gg l$), the problem reduces to consideration of the penetration of the magnetic field into the semi-infinite metal, and the only characteristic length is l, so that

$$|\partial H/\partial x| = |\partial H/\partial \rho| \sim H/l. \tag{4}$$

It follows from (2) and (4) that for $R \gg l$, the thickness of the surface layer $\Delta x \sim l(H/\alpha)$, and its volume is $\Delta V \sim Gl(H/\alpha)$, where G is the surface area of the specimen. It is clear that these estimates hold also for $R \sim l$.

For the case $R \ll l$, when it is necessary to take into consideration the presence of two different characteristic dimensions (R and l), it is easiest to estimate the value of $\partial H/\partial \rho$ from Eq. (3):

$$|\partial H/\partial \rho| \sim (\rho/q)|\partial H/\partial t| \sim RH/qT \sim (R/l)H/l. \tag{5}$$

Substituting the estimate (5) in (2), we find that for $R \ll l$, the thickness of the surface layer $\Delta x \sim l(l/R)H/\alpha$ and its volume is $\Delta V \sim Gl(l/R)H/\alpha$.

The thickness of the layer in which the magnetic field is stationary in depth is determined by a relation similar to (2), namely,

$$\Delta\left(\frac{\alpha}{H}\right) = \frac{\alpha}{2}\left|\frac{\partial^2}{\partial x^2}\frac{1}{H}\right|\Delta x^2 = \frac{\alpha}{2H^2}\left|\frac{\partial^2 H}{\partial x^2}\right|\Delta x^2 \sim 2\pi. \tag{6}$$

The order of magnitude of $\partial^2H/\partial x^2$ does not depend on the dimensions of the specimen to such a degree as the value of $\partial H/\partial x$, and Eq. (3) always gives the expression for it:

$$|\partial^2 H/\partial x^2| = |\partial^2 H/\partial \rho^2| \sim H/l^2. \tag{7}$$

It follows from (6) and (7) that the thickness of the layer with stationary field is $\Delta x \sim l(H/\alpha)^{1/2}$ and consequently it exceeds the thickness of the surface layer (in the case $R \gtrsim l$) by the ratio $(\alpha/H)^{1/2}$.

It should be observed that in a cylindrical specimen one such layer corresponds to the neighborhood of the axis of the cylinder. The volume of this "layer" is $\Delta V \sim Ll^2(H/\alpha)$, where L is the length of the cylinder. If $l \ll R$, then the contribution of the region about the axis of the cylinder to the oscillating part of the total magnetic moment of the specimen gives the same order of contribution to the oscillations of the total magnetic moment as does the surface layer; however, this part has a different period (corresponding to the magnetic field on the axis of the cylinder).

So far as other layers with magnetic field stationary with respect to depth are concerned, they can exist in the metal only beyond the passage of the external field $H_0(t)$ through a maximum, i.e., for $t > t_m$. Actually, when $H_0(t)$ increases, by the law of electromagnetic induction, the rate of increase of the magnetic field inside the metal falls off with the depth; therefore, at each particular moment of time, the magnetic field decreases with depth: $\partial H/\partial x < 0$. When $H_0(t)$ passes through the maximum and begins to decrease, then the opposite takes place: the drop of the magnetic field in the body of the metal does not take place at the expense of $H_0(t)$, and at a certain instant t_0 the field in the surface layer is comparable with $H_0(t)$. At this moment, $(\partial H/\partial x)_{x=0} = 0$, and at the surface of the specimen, a layer arises with stationary magnetic field. Upon further lag of the drop of the field in the body of the metal behind the decrease in $H_0(t)$, the field at the surface of the specimen will increase with depth and therefore, for $t > t_0$, $(\partial H/\partial x)_{x=0} > 0$. However, at a sufficient depth x, the situation undergoes no change, and, as before, $\partial H/\partial x < 0$. Consequently, at the same depth $x_0 > 0$, the derivative $\partial H/\partial x$ passes through zero, i.e., there exists a layer with a magnetic field that is stationary with respect to depth. The function $x_0 = x_0(t)$ gives the depth of this field at each moment of time $t > t_0$ [evidently, $x_0(t_0) = 0$]. The period of oscillation of the magnetic moment of such a layer is determined by the magnetic field at the depth $x = x_0(t)$ and has a complicated dependence on $H_0(t)$.

Thus the oscillations of the magnetic moment of a cylindrical specimen in a pulsed field are characterized by the following peculiarities:

1. Case $l \ll R$. In the time interval $0 < t < t_0$ the oscillations are completely determined by the surface layer of the specimen. The amplitude of oscillation is less, in the ratio $(l/R)(H_0/\alpha)$, than the amplitude of oscillation in a constant field, and the period of oscillation with change of the external magnetic field $\Delta(1/H_0)$ is the same as for a constant field.

At the time t_0 the amplitude of oscillation increases by a factor $(\alpha/H_0)^{1/2}$, and the oscillations of the magnetic moment begin to be determined by the layer of the metal with stationary field. For $t > t_0$, the surface layer of the specimen gives an insignificant contribution to the oscillations, and the period of the oscillation is determined by the magnetic field at the depth $x = x_0(t)$.

2. Case $l \gtrsim R$. If $1 \gtrsim (R/l)^2 \gg H_0/\alpha$, then in the time interval $0 < t < t_0$ the oscillating part of the magnetic moment consists of two components of the same order of magnitude: the component determined by the surface layer of the metal and the component which corresponds to the region around the axis of the cylinder. The amplitudes of these components are less by a factor $(l/R)^2(H_0/\alpha)$ than the amplitudes of the oscillations in constant field. The periods of the oscillations of these two components are different, since for $l \sim R$, the magnetic field on the axis of the cylinder differs from the imposed field H_0.

At the time t_0, as in case 1, the amplitude of the oscillation increases by the factor $(R/l)(\alpha/H_0)^{1/2}$, and in what follows $(t > t_0)$ the oscillations are determined by the layer of metal with stationary field, and the period of the oscillations corresponds to the magnetic field in this layer.

It can be pointed out that at a certain moment of time t_1 $(t_0 < t_1 < T)$, the layer with stationary magnetic field, which "splits off" from the surface of the specimen, coincides with the region near the axis of the cylinder. Then the amplitude of the oscillation decreases and the oscillation will again be determined by the same two components as for $t < t_0$.

If $R \sim \ell(H_0/\alpha)^{1/2}$, then the oscillating part of the magnetic moment of the specimen is determined by its thickness, and the period of oscillation corresponds to a certain mean magnetic field over the thickness of the specimen.

If $R \ll \ell(H_0/\alpha)^{1/2}$, then the magnetic field inside the metal can be considered homogeneous throughout the thickness of the specimen and Eq. (1) is valid for oscillations of the magnetic moment.

Finally, one should point out the possibility of experimental observation of these oscillations, the amplitude of which is, in the most interesting cases ($\ell \lesssim R$), significantly less than the for constant field. The fact is that in the use of the pulse method for the investigation of the De Haas—Van Alphen effect, we measure dM_{osc}/dH, not M_{osc}.[1] But, as is evident from (1), the following relation holds:

$$dM_{osc}/dH \sim (\alpha/H)\, M_{osc}/H,$$

therefore, the small factor H_0/α which appears in the amplitude of the oscillation can be compensated.

Taking into consideration the qualitative difference of the effect in for different ratios of ℓ/R, we proceed to the detailed quantitative analysis of the cases enumerated above.

2. CASE $\ell \ll R$. THE OSCILLATING PART OF THE MAGNETIC MOMENT OF A FLAT METALLIC SPECIMEN

Let us consider first the character of the inhomogeneity of the magnetic field inside the specimen. For $\ell \ll R$, the surface of the metal can be considered flat and therefore the magnetic field inside the sample, $H(x, t)$, is a function of t and the cartesian coordinate x, whose axis is perpendicular to the surface of the metal. The problem of the distortion of the given variable magnetic field $H_0 = f(t)$ inside a plane, half-bounded metal reduces to the solution of a one-dimensional thermal conductivity equation with corresponding boundary conditions:

$$H(x, t) = \frac{2}{\sqrt{\pi}} \int\limits_{x/2\sqrt{qt}}^{\infty} f\left(t - \frac{x^2}{4q\xi^2}\right) e^{-\xi^2} d\xi. \tag{8}$$

In this case, naturally, it is assumed that q in Eq. (8) does not depend on the magnetic field.

If T is not very small (so that ℓ is significantly larger than the mean radius of the electron orbit in the magnetic field), then the magnetic field in the metal changes so smoothly with increase in x that for the determination of the oscillating part of the magnetic moment in the variable field we can use the formula

$$M_{osc}(t) = G \sum_{n=1}^{\infty} \int\limits_0^{\infty} \Psi_n [H(x, t)] \cos\left(\frac{\alpha n}{H(x, t)} + \varphi_n\right) dx; \tag{9}$$

G is the surface area of the metal and $H(x, t)$ is determined from (8).

Making use of the fact that $\alpha/H_0 \gg 1$, and that $\Psi_n(H)$ is a smooth function of H, we easily obtain the asymptotic expansion of (9) in powers of H/α. A similar expression for the asymptotic estimates, which is computed by the method of "critical points,"[4] is different for the interval $0 < t < t_0$ [when there is no point of stationary phase of the oscillating factor in Eq. (9)] and the interval $t > t_0$ [when there is a point of stationary phase at the depth $x = x_0(t)$]. The instant of time t_0 at which $\partial H/\partial x = 0$ for $x = +0$ can be estimated by noting that it is determined from the condition

$$\int\limits_1^{\infty} f'\left[t_0\left(1 - \frac{1}{u^2}\right)\right] \frac{du}{u^2} = 0. \tag{10}$$

For the estimate it suffices to approximate the function $f(t)$ by means of two quadratic parabolas which have a common apex coinciding with f_{max} and which pass through zero at the points $t = 0$ and $t = T$. Then $f(t)$ is represented in the form

$$f(t) = \begin{cases} 2f_{max} t_m^{-2} (t_m - t), & 0 < t < t_m, \\ 2f_{max} (T - t_m)^{-2} (t_m - t), & t_m < t < T, \end{cases} \tag{11}$$

and for t_0 we get [from (10)] a third-degree algebraic equation. The analysis of this equation shows that the quantity t_0 depends on the ratio of the time t_m in which the external field increases in the pulse, to the total duration of the pulse T, and increases almost linearly with T:*

$$t_0 = {}^3/_2\, t_m = {}^3/_4\, T \quad \text{for} \quad T = 2t_m,$$
$$t_0 \approx 0.6\,(1 + 0.4\, t_m/T)\, T \quad \text{for} \quad T \gg t_m. \tag{12}$$

Inasmuch as the magnetic field in real pulses does not disappear for $t > T$, but only falls off to zero asymptotically (as is shown in the drawing above), then the real value of $|f(t)|$ (for $t \approx T$) is less than assumed in (11); therefore, Eqs. (12) can give a somewhat smaller value of t_0.

As an example of a magnetic field of the pulse type $H_0 = f(t)$, we consider the pulse

$$f(t) = Ae^{-\beta_1 t}\sinh\beta_2 t, \quad \beta_1 > \beta_2 > 0. \tag{13}$$

Such a field arises in a solenoid upon passage through it of the discharge current of a condenser.

If we assume that $\beta_1 \gg \beta_2$, then $t_m \approx 1/\beta_1$, and $t_0 \approx 2.7\, t_0$. So far as the duration of the pulse (13) is concerned, it appears natural to determine it in the following way. At the point of inflection of the curve $f = f(t)$ (for the case $t = 2t_m$), we draw a tangent to it, and continue the tangent to its intersection with the abscissa. The intercept gives the duration of the pulse T. It is seen that $T \approx 4t_m$, and the value of the function $f(t)$ at this point is $f(T) \approx 0.2 f_{max}$. For comparison with (12), we note that $t_0 \approx 0.68\,T$. Although there is some arbitrariness in the determination of the length of the actual pulse, we can still assume that the estimate (11) is in excellent agreement with the value of t_0 for the pulse of form (13).

For $t > t_0$, as was noted above, the interval of integration in (9) contains a point of stationary phase of the oscillating factor, the position of which changes with time. The dependence $x_0 = x_0(t)$ is completely determined by the form of the function $f(t)$, and is found from the equation

$$\int_1^\infty f'\left[t\left(1 - \frac{1}{u^2}\right)\right] \exp\left(-\frac{x^2}{4qt}u^2\right)\frac{du}{u^2} = 0. \tag{14}$$

To solve the transcendental equation (14) in the general case is not possible; however, we can establish the dependence of $x_0 = x_0(t)$ for small $t - t_0$ and x. This dependence is linear:

$$x_0 = \frac{qd}{|df/dt_0|}(t - t_0), \quad d = (\partial^2 H/\partial x\, \partial t)_{\substack{x=+0\\ t=t_0}} > 0.$$

Taking into account the different character of the inhomogeneity $H(x, t)$ in the time intervals $0 < t < t_0$ and $t > t_0$, we can easily obtain the principal term in the asymptotic expansion of $M_{osc}(t)$ in increasing power of H/α:

1. $0 < t < t_0$

$$M_{osc}(t) = G\sum_{n=1}^\infty \frac{\sqrt{\pi}}{\sqrt{2\alpha n b(0)}} \Psi_n[H_0(t)] \left\{\left[\frac{1}{2} - C\left(\frac{a(0)}{2}\sqrt{\frac{\alpha n}{b(0)}}\right)\right]\cos\left(\frac{\alpha n}{H_0(t)} - \frac{\alpha n}{4}\frac{a^2(0)}{b(0)} + \varphi_n\right)\right.$$
$$\left. - \left[\frac{1}{2} - S\left(\frac{a(0)}{2}\sqrt{\frac{\alpha n}{b(0)}}\right)\right]\sin\left(\frac{\alpha n}{H_0(t)} - \frac{\alpha n}{4}\frac{a^2(0)}{b(0)} + \varphi_n\right)\right\}. \tag{15}$$

2. $t > t_0$

$$M_{osc}(t) = G\sum_{n=1}^\infty \frac{\sqrt{\pi}}{\sqrt{2\alpha n b(x_0)}} \Psi_n[H(x_0, t)] \left\{\left[\frac{1}{2} + C\left(\frac{qd(t - t_0)\sqrt{\alpha n b(x_0)}}{|dH_0/dt|}\right)\right]\cos\left(\frac{\alpha n}{H(x_0, t)} + \varphi_n\right)\right.$$
$$\left. - \left[\frac{1}{2} + S\left(\frac{qd(t - t_0)\sqrt{\alpha n b(x_0)}}{|dH_0/dt|}\right)\right]\sin\left(\frac{\alpha n}{H(x_0, t)} + \varphi_n\right)\right\}. \tag{16}$$

The following notation is introduced and emplyed in Eqs. (15) and (16): $C(z)$ and $S(z)$ are the Fresnel cosine and sine integrals, respectively;

$$a(x) \equiv a(x, t) = \frac{\partial}{\partial x}\left(\frac{1}{H(x, t)}\right); \quad b(x) \equiv b(x, t) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\frac{1}{H(x, t)}\right).$$

*For the pulses usually obtained in experiment, $T \geq 2t_m$; therefore in what follows we shall be interested only in this case.

For all time intervals which do not include the small region in the neighborhood of t_0, Eqs. (15) and (16) are considerably simplified:

1. $t > 0$, но $t_0 - t \gg H_0^2(t_0)\sqrt{b(0, t_0)}/d\sqrt{\alpha}$ $\quad M_{osc}(t) = G\sum_{n=1}^{\infty}\frac{1}{\alpha n a(0)}\Psi_n(H_0)\cos\left(\frac{\alpha n}{H_0} + \varphi_n + \frac{\pi}{2}\right)$.

2. $t - t_0 \gg |dH_0/dt_0|/qd\sqrt{\alpha b(0, t_0)}$ $\quad M_{osc}(t) = G\sum_{n=1}^{\infty}\frac{\sqrt{\pi}}{\sqrt{\alpha n b(x_0)}}\Psi_n[H(x_0, t)]\cos\left(\frac{\alpha n}{H(x_0, t)} + \varphi_n + \frac{\pi}{4}\right)$.

These formulas confirm the fact that: (1) In the interval of time $0 < t < t_0$, not including the small interval about t_0, the period of the oscillation of the magnetic moment with change in the external magnetic field $\Delta(1/H_0)$ is the same as for constant field. For $t > t_0$, the period of the oscillation is determined by the value of the magnetic field intensity at the depth $x = x_0(t)$, and consequently depends in a complicated fashion on H_0; (2) The dependence of the amplitude of the oscillation on H_0 is different than in (1); the value of the amplitude of the oscillation is considerably smaller than in a constant field, since it is proportional not to the volume of the entire metal, but to the volume of that layer which gives the fundamental contribution to the oscillation of the magnetic moment. For $t < t_0$, this layer is the surface layer, whose thickness $\sim 1/\alpha a(0) \sim \ell(H_0/\alpha)$, while for $t > t_0$, the layer is found at the depth $x = x_0(t)$ and has the thickness

$$\sim (\alpha b(x_0))^{-1/2} \sim l(H(x_0)/\alpha)^{1/2}.$$

3. CASE $\ell \gtrless R$. CYLINDRICAL SPECIMEN IN A PULSED FIELD

For the case of definite dimensions of the metal, we consider a specimen in the form of a circular cylinder of radius R, whose axis is parallel to the applied field. The variable magnetic field in such a specimen, $H(\rho, t)$, is a function of the distance ρ from the axis of the cylinder, and the solution of the differential equation (3) with boundary conditions $H(R, t) = f(t)$ leads to the following expression:*

$$H(\rho, t) = f(t) - 2\sum_{k=1}^{\infty}\frac{J_0(y_k\rho/R)}{y_k J_1(y_k)}\exp\left\{-\left(y_k\frac{l}{R}\right)^2\frac{t}{T}\right\}\int_0^{t/T}\frac{df}{d\tau}\exp\left\{\left(y_k\frac{l}{R}\right)^2\tau\right\}d\tau, \tag{17}$$

where $J_n(z)$ is the Bessel function of order n, and y_k is the k-th root of the equation $J_0(z) = 0$.

The oscillating part of the magnetic moment of a cylindrical specimen is determined by a formula similar to (9), namely,

$$M_{osc}(t) = 2\pi L\sum_{n=1}^{\infty}\int_0^R\Psi_n[H(\rho, t)]\cos\left(\frac{\alpha n}{H(\rho, t)} + \varphi_n\right)\rho d\rho; \tag{18}$$

L is the length of the cylinder and $H(\rho, t)$ was defined in (17).

As we saw in Sec. 2, the result of integration in (18) depends on the position of the instant $t = t_0$ on the time axis; therefore, we shall first find (or estimate) t_0.

The instant t_0 is most simply determined in the case $\ell \gg R$, with the analysis of which we begin. We turn our attention to the fact that the right hand side of (17) contains (under the summation sign) integrals of the type

$$I = \int_0^z\psi(\tau)e^{w\tau}d\tau,$$

which depend on the large parameter $w = (y_k\ell/R)^2 \gg 1$. We can easily convince ourselves that the asymptotic value of I for $w \gg 1$ has the form:

$$I \approx \frac{1}{w}e^{wz}\left\{\psi(z) - \frac{1}{w}\psi'(z) + \ldots\right\}, \quad w \gg 1. \tag{19}$$

*Here and in what follows, we shall consider that $\ell = \sqrt{qT}$.

If we make use of the asymptotic value of (19), we then obtain the expansion of $H(\rho, t)$ in increasing powers of $(R/\ell)^2$ from (17):

$$H(\rho, t) = f(t) + \frac{1}{4} T\dot{f}(t)\frac{\rho^2 - R^2}{l^2} + \frac{1}{64} T^2\ddot{f}(t)\frac{\rho^4 - 4\rho^2 R^2 + 3R^4}{l^4} + \ldots \qquad (20)$$

In order to put the expression for $H(\rho, t)$ in the form (20), we have introduced an obvious expansion of the functions z^{2p} in Fourier-Bessel series.

From (20) there follows the expansion of the derivative

$$\frac{\partial H}{\partial \rho} \approx \frac{\rho T}{2l^2}\left\{\dot{f}(t) + \frac{1}{8} T\ddot{f}(t)\frac{\rho^2 - 2R^2}{l^2}\right\}.$$

The instant of time t_0 is determined by the condition $(\partial H/\partial \rho)_{\rho = R} = 0$, from which it follows that

$$\dot{f}(t_0) = {}^1/_8 (R/l)^2 T\ddot{f}(t_0); \qquad (21)$$

$$|\dot{f}(t_0)| \ll T|\ddot{f}(t_0)| \sim f_{\max}/T. \qquad (22)$$

The inequality (22) shows that the absolute value of the derivative $\dot{f}(t_0)$ is very small, while the latter signifies that t_0 is close to the point of the maximum of the external magnetic field t_m.

In this connection we can set

$$\dot{f}(t_0) = (t_0 - t_m)\ddot{f}(t_m), \quad \ddot{f}(t_0) \approx \ddot{f}(t_m),$$

and we then obtain the value of t_0 from (21):

$$t_0 = t_m + {}^1/_8 (R/l)^2 T. \qquad (23)$$

For $t > t_0$, the range of integration in (18) contains some value $\rho = \rho_0$, which corresponds to the stationary phase of the cosine. The position of the point $\rho = \rho_0(t)$ is determined by the formula

$$\rho_0 = R\sqrt{1 - 8(l/R)^2(t - t_0)}.$$

which comes from the condition $\partial H/\partial \rho = 0$.

But, inasmuch as in the given case, $\ell^2 \gg R^2$, then $\rho_0(t)$ falls off very quickly with increase in $t - t_0$, and after a short interval of time Δt, the point of stationary phase "falls back" on the axis of the cylinder. This time interval is given by

$$\Delta t = {}^1/_8 (R/l)^2 T. \qquad (24)$$

We note that we can make use of Eqs. (23) and (24) over a rather wide range of values of ℓ/R. Actually, both of these are derived under the assumption $w \gg 1$, but inasmuch as $y_1 \approx 2.4$, $y_2 \approx 5.5$, etc.; then, even for $\ell = 2R$ the parameter $w \gg 1$, and Eqs. (23), (24) remain in force. This allows us to think that Eqs. (23) and (24) can give the correct estimate of the order of magnitude of t_0 and Δt for all $\ell \gtrsim R$.

We now proceed to calculate $M_{osc}(t)$.

(a) <u>Pulses of long duration</u>: $\ell \gtrsim R(\alpha/H)^{1/2}$.

The magnetic field inside the specimen for $\ell \gg R$ changes little with depth, and if we leave only the first components in (20), we have the expression

$$H(\rho, t) = f(t) - {}^1/_4 (R/l)^2 T\dot{f}(t)[1 - (\rho/R)^2]. \qquad (25)$$

It is evident from (25) that the total change of the magnetic field throughout the depth of the specimen is equal to

$$H(R, t) - H(0, t) = {}^1/_4 (R/l)^2 T\dot{f}(t) \sim (R/l)^2 H. \qquad (26)$$

On the other hand, the change in the magnetic field ΔH which corresponds to the period of oscillation M_{osc} is determined by the relation

$$\Delta H = 2\pi (H/\alpha) H. \qquad (27)$$

Comparing (26) and (27), we can conclude that two different cases are possible:

1) $(R/\ell)^2 \ll H/\alpha \ll 1$. In this case, the magnetic field changes so slightly with change of ρ from R to 0 that the phase of the oscillating factor in the integrand of (18) remains practically unchanged. Con-

sequently, in the calculation of the oscillating part of the magnetic moment of the specimen, we can make use of Eq. (1), assuming $H = H_0(t) = f(t)$.

2) $R \sim l(H/\alpha)^{1/2}$. In this case the change in phase of the cosine in (18) over the interval of integration ($0 < \rho < R$) is of the same order of magnitude as its period. It therefore follows that in the interval it is not possible to neglect the dependence of the magnetic field on ρ, but asymptotic expansion of the integral is also not possible. We can make use only of the theorem of the mean:

$$M_{\rm osc}(t) = 2\pi L \sum_{n=1}^{\infty} \cos\left(\frac{\alpha n}{H^*(t)} + \varphi_n\right) \int_0^R \Psi_n[H(\rho, t)]\rho\, d\rho \approx V \sum_{n=1}^{\infty} \Psi_n[H_0(t)] \cos\left(\frac{\alpha n}{H^*(t)} + \varphi_n\right), \quad (28)$$

where $H^* = H(\rho^*, t)$ is the magnetic field at some internal point of the specimen $0 < \rho^*(t) < R$.

Inasmuch as the point of stationary phase of the cosine in (18) exists for a very short time $\Delta t = \frac{1}{8}(R/l)^2 T$, during which the magnetic field changes by the order of magnitude of a single period of oscillation, then consideration of the point of stationary phase does not change the equations (28) to any appreciable extent.

It follows from (28) that the oscillations of the magnetic moment in this case differ from the oscillations for a constant field only by some increased, complicated dependence of the argument of the oscillating factor on the applied field. As is evident from (25), the complication reduces to a shift in phase of the oscillation by an amount $\sim (R/l)^2(\alpha/H_0) \sim 1$, and to a small difference of the period from that in the constant field. If we denote the period of the oscillation of the magnetic moment with change in the external field by $\Delta(1/H_0)$, then it is determined by the expression:

$$\Delta(1/H_0) = \frac{2\pi}{\alpha}\left\{1 - \left(\frac{R}{l}\right)^2 \gamma(t)\left[1 - \left(\frac{\rho^*}{R}\right)^2\right]\right\}, \quad \gamma(t) = \frac{T}{4}\frac{d}{dt}\ln f(t); \quad (29)$$

$$2\pi/\alpha - \Delta(1/H_0) \sim (R/l)^2\, 2\pi/\alpha \sim (H_0/\alpha)\, 2\pi/\alpha. \quad (30)$$

The relations (30) indicate that the difference of $\Delta(1/H_0)$ from the period of oscillation in a constant field is very small.

(b) Short pulses: $l \gtrsim R \gg l(H/\alpha)^{1/2}$.

For $l \sim R$, a change in the magnetic field over the thickness of the specimen is of the order of the magnetic field itself; therefore, Eq. (17) is not simplified, and calculations [even with a concrete form of the the function $f(t)$] are more difficult. Significant simplification takes place only for $l^2 \gg R^2 \gg l^2(H/\alpha)$, when we can make use of the expansion (20). However, if we make asymptotic estimates of the integrals in (18) by the method of "critical points," it is not difficult to obtain general relations of types (15) and (16) for $M_{\rm osc}(t)$.

To simplify the description, we represent the oscillating part of the magnetic moment in the form of a sum

$$M_{\rm osc}(t) = M^1_{\rm osc}(t) + M^2_{\rm osc}(t), \quad (31)$$

where the first term is the contribution to the oscillations of the magnetic moment of the surface layer of the specimen and the layer with a magnetic field that is stationary in depth, and the second is the contribution of the "core" of the specimen, i.e., the vicinity of the axis of the cylinder.

The component $M^1_{\rm osc}(t)$ is determined from (15) and (16), in which we replace x by $x = R - \rho$, and in which the area of the surface corresponding to the layer is defined by

$$1)\ 0 < t < t_0, \quad G = 2\pi LR, \qquad 2)\ t > t_0, \quad G = 2\pi L\rho_0(t) = 2\pi L[R - x_0(t)].$$

The principal part of the expansion of the second component in increasing powers of H/α is equal to

$$M^2_{\rm osc}(t) = 2\pi L \sum_{n=1}^{\infty} \frac{1}{2\alpha n g(t)} \Psi_n[H(t)] \cos\left(\frac{\alpha n}{H(t)} + \varphi_n + \frac{\pi}{2}\right), \quad (32)$$

where

$$H(t) = H(\rho, t)|_{\rho=0}, \quad g(t) = \frac{1}{2}\left(\frac{\partial^2}{\partial\rho^2}\frac{1}{H}\right)_{\rho=0} = b(R, t).$$

In the case $R^2 \ll \ell^2$, we can set

$$H(t) = H_0(t)[1 - (R/l)^2 \gamma(t)], \quad g(t) = -\gamma(t)/l^2 H_0(t). \tag{33}$$

For time intervals not including the small region about the time t_0, the equations stemming from (31) become simplified:

$$1)\ t > 0, \text{ но } t_0 - t \gg H_0^2(t_0)\sqrt{b(0, t_0)}/d\sqrt{\alpha}$$

$$M_{\rm osc}(t) = 2\pi L \sum_{n=1}^{\infty} \frac{1}{\alpha n}\left\{\frac{R}{a(0)}\Psi_n(H_0)\cos\left(\frac{\alpha n}{H_0(t)} + \varphi_n + \frac{\pi}{2}\right) + \frac{1}{2g(t)}\Psi_n[H(t)]\cos\left(\frac{\alpha n}{H(t)} + \varphi_n + \frac{\pi}{2}\right)\right\}. \tag{34}$$

$$2)\ t - t_0 \gg |dH_0/dt|/qd\sqrt{\alpha b(0, t)} \quad M_{\rm osc}(t) = 2\pi L \rho_0(t)\sum_{n=1}^{\infty}\frac{\sqrt{\pi}}{\sqrt{\alpha n b(x_0)}}\Psi_n[H(x_0)]\cos\left(\frac{\alpha n}{H(x_0)} + \varphi_n + \frac{\pi}{4}\right). \tag{35}$$

It can be shown that the time interval (0, T) contains the moment t_1, in which the point with stationary phases of the oscillating factor "falls back" on the axis: $\rho_0(t_1) = 0$. Then in a small region about t_1 [where Eq. (35) is comparable in value with (32) and thereafter completely disappears] and for $t > t_1$, $M_{osc}(t)$ is again determined by the expression (34).

In the case $\ell \sim R$, the oscillations of the magnetic moment are connected in very complicated fashion with the direction of the applied magnetic field. As is evident from (34), M_{osc} consists of two terms of the same order of magnitude in the time ragnes $0 < t < t_0$ and $t > t_1$. These terms have different periods, with different dependences on the external field. Over a sufficiently long time interval $\Delta t = t_1 - t_0 \sim T$, the dependence of $M_{osc}(t)$ on the external magnetic field becomes even more complicated.

Some singular characteristics of the oscillations appear in the case $1 \gg (R/\ell)^2 \gg H/\alpha$, which probably corresponds to experimental conditions.[1] Since for $R^2 \ll \ell^2$, we can limit ourselves in the calculations to the approximate formulas (33); then in the duration of the total pulse, after elimination of the short time interval $\Delta t = \frac{1}{8}(R/\ell)^2 T$ in the vicinity of the maximum point of the external field, $M_{osc}(t)$ is determined by the expression:

$$M_{\rm osc}(t) = 2\pi l^2 L \frac{H_0(t)}{\alpha\gamma(t)}\sum_{n=1}^{\infty}\frac{1}{n}\Psi_n[H_0(t)]\sin\left[\frac{\alpha n}{2H_0}\left(\frac{R}{l}\right)^2\gamma(t)\right]\sin\left\{\frac{\alpha n}{H_0(t)}\left[1 + \frac{1}{2}(R/l)^2\gamma(t)\right] + \varphi_n + \frac{\pi}{2}\right\}. \tag{36}$$

The oscillations of the magnetic moment in (36) have the characteristic binary form. The period of the fundamental vibration is determined by a formula of the type (29):

$$\Delta(1/H_0) = (2\pi/\alpha)\left[1 - \frac{1}{2}(R/l)^2\gamma(t)\right],$$

while the "binary period" has the following order of magnitude:

$$\Delta_6(1/H_0) \sim (2\pi/\alpha)(l/R)^2 \gg 2\pi/\alpha.$$

In the experiments of Shoenberg,[1] the oscillating binary form was actually observed.

Finally, it should be emphasized that in the analysis of experiments according to the measurement of the oscillating part of the magnetic moment of the metal, it is necessary, generally speaking, to take into account the presence of certain groups of electrons which create well known additional difficulties in the interpretation of experimental results.

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140