PROPAGATION OF SOUND ACROSS A BOUNDARY BETWEEN TWO SUPERFLUID PHASES

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The transmission of first and second sound across a boundary between two superfluid liquids is considered. The phenomenon of conversion of sound waves of one type into the other is revealed. Formulas are given for the energy fluxes in the reflected, refracted and converted waves. Separate consideration is given to the case in which one of the two liquids is not a superfluid.

THE hydrodynamics of concentrated solutions of He^3 in He^4 were developed in Ref. 1. In the system of equations derived there, it was assumed that all the He^3 takes part only in normal motion. This latter circumstance was proved rigorously only for the case of low concentration, and for all concentrations at absolute zero. Recently, it has been possible to carry out experiments with concentrated solutions of He^3 in He^4, which permit us to verify the validity of the hydrodynamical equations obtained in Ref. 1 by studying the propagation of sound vibrations in solutions.

As is well known, in the propagation of first sound in pure He^4, the pressure and density oscillations take place at constant entropy and temperature, while in the propagation of second sound, the temperature and entropy oscillations take place at constant pressure and density. In solutions, this is no longer the case. Propagation of first sound in solutions is accompanied by oscillations not only of pressure and density, but also of temperature and concentration. A similar picture is observed also in the propagation of second sound. The connection of the amplitudes of the vibrations of the various thermodynamical quantities can be found in the usual way[2] from the equations of hydrodynamics, taking into account the expressions for the velocities of first and second sounds in solutions (we denote by a prime the variable parts of quantities; the indices I and II refer to first and second sound, respectively: $p' = p'_{I} + p'_{II}$):

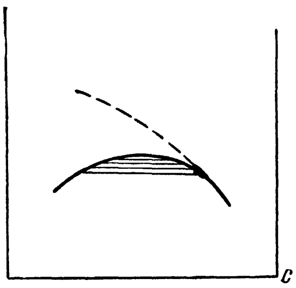

FIG. 1. Phase diagram and curve of the λ-transitions (dotted line) of the mixture of He^3 in He^4. The hypothetical region of two superfluid liquids is shaded.

$$c' = -\frac{\rho_s}{\rho_n}\frac{c^2}{\rho^2}\frac{\partial\rho}{\partial c}\frac{1}{s^2}p' + \frac{\partial c}{\partial\rho}\frac{1}{u^2}p'_{II},$$

$$T' = \left(\sigma - c\frac{\partial\sigma}{\partial c}\right)\frac{\partial T}{\partial\sigma}\frac{c'}{c}, \quad \left(\mu - \frac{Z}{\rho}c\right)' = \frac{1}{\rho}\left(1 + \frac{c}{\rho}\frac{\partial\rho}{\partial c}\right)p'_{I} + \frac{1}{\rho}\left(1 - \frac{\rho_n}{\rho_s}\frac{\rho}{c}\frac{\partial c}{\partial\rho}\right)p'_{II}, \tag{1}$$

$$v_n = \frac{1}{s\rho}\left(1 - \frac{\rho_s}{\rho_n}\frac{c}{\rho}\frac{\partial\rho}{\partial c}\right)p'_{I} + \frac{1}{u\rho}\left(1 + \frac{\rho}{c}\frac{\partial c}{\partial\rho}\right)p'_{II}, \quad j - \rho v_n = \frac{\rho_s}{\rho_n}\frac{c}{s\rho}\frac{\partial\rho}{\partial c}p'_{I} + \frac{\rho}{uc}\frac{\partial c}{\partial\rho}\rho'_{II}.$$

Here ρ is the density of the mixture, c the weight concentration of He^3, $\mu - Zc/\rho = \mu_4$ the chemical potential of He^4 per gram, σ the entropy (per gram) of the mixture, p the pressure, T the temperature, s the velocity of first sound, u the velocity of second sound, ρ_s, ρ_n the superfluid and normal densities, v_s, v_n the superfluid and normal (particle) velocities, and $j = \rho_s v_s + \rho_n v_n$ the current density. The oscillations of the vector quantities $\mathbf{v}_n$ and $\mathbf{j} - \rho\mathbf{v}_n$ are directed along the wave vector **k**. The latter two equations represent the amplitude of these vibrations.

It is evident from the formulas that the "intermeshing" of the temperature and pressure comes about through the derivative $\partial\rho/\partial c$ (and not through the very small quantity $\partial\rho/\partial T$, as in pure He^4).

New experimental possibilities for the study of sound propagation were opened up recently by Walters and Fairbank[3] with the discovery of the phenomenon of the separation of a solution of He^3 in He^4 into two phases of different concentration. The authors carried out their measurements by the method of magnetic resonance. Their results were confirmed by direct optical observations by Zinov'ev and Peshkov.[4] It was shown that the critical point of the phase diagram lies at a temperature of about 0.85°K and a He^3 concentration of about 60% (Fig. 1). The concentration dependence of the temperature of the λ-transition is known with a very small degree of accuracy. If this curve (see Ref. 5) intersects the phase diagram to the right of the critical point, then there ought to exist two superfluid phases, which are in equilibrium with one another. Investigation of sound propagation across a boundary separating two phases should give additional information on the hydrodynamics of liquid helium.

Let us formulate the boundary conditions of the problem of the propagation of sound across a boundary separating two superfluid phases (the case of the boundary of a superfluid phase with a nonsuperfluid is obtained from the corresponding equations as the limit of $\rho_s \to 0$). We note that here, in addition to refracted and reflected waves in both phases, there arise waves which correspond to the conversion of first sound into second, and vice versa. Obviously, the excited sound will have the same values ω, k_y, k_z as the incident.[6] The case of practical interest is that in which only first or only second sound is incident on the boundary; there are then incident, reflected, refracted, and two converted waves. Because of the linearity of the problem, the general case is obtained as a superposition. As usual, we can consider that the plane of incidence of the wave is x, y; then $k_z = 0$ and the directions of the incident, reflected and the two converted waves lie in a single plane (Fig. 2). We shall consider all quantities proportional to $\exp\{i(k_x x + k_y y - \omega t)\}$. The condition $\mathrm{curl}\,\mathbf{v}_s = 0$ on the boundary gives $v_{sy1} = v_{sy2}$ (we shall omit the arabic numerals for the different phases). From the equation

$$\frac{\partial v_{sy}}{\partial t} + \frac{\partial}{\partial y}\left(\mu - \frac{Z}{\rho} c\right) = 0$$

we obtain

$$v_{sy} = (k_y/\omega)(\mu - Zc/\rho)'.$$

Consequently, the quantity $\mu - Zc/\rho$ must be continuous. (Here and below, we shall denote the variable part of quantities by means of a prime.) It is also evident that the pressure should be continuous at the boundary: $p_1 = p_2$.

For the derivation of two more (vector) conditions, we write down the condition of continuity of energy flow:

$$\mathbf{q} = \mathbf{j}\left(\mu - \frac{Z}{\rho} c\right) + \rho c \mathbf{v}_n \left(\frac{Z}{\rho} + \frac{\sigma}{c} T\right)$$

across the boundary ($q_{x1} = q_{x2}$) and consider that, because of the equations

$$\frac{\partial}{\partial t}(\rho\sigma) + \mathrm{div}(\rho\sigma\mathbf{v}_n) = 0, \quad \frac{\partial(\rho c)}{\partial t} + \mathrm{div}(\rho c \mathbf{v}_n) = 0$$

the quantity σ/c remains constant. Separating in each of the cofactors the part that is linear in the oscillations, we easily obtain

$$\mathbf{q} = (\mathbf{j} - \rho\mathbf{v}_n)\left(\mu - \frac{Z}{\rho} c\right)' + \mathbf{v}_n p', \tag{2}$$

from which we can conclude that $j_x - \rho v_{nx}$ and v_{nx} are also continuous; the latter is of course obvious on other grounds, since the gas excitation must be at rest relative to the boundary, while the continuity of $j_x - \rho v_{nx} = -\rho_s(v_{nx} - v_{sx})$ signifies that the superfluid part passes freely through the boundary.

First we shall write down the set of boundary conditions for normal incidence of the sound wave, which we shall consider to be a superposition of waves of first and second sound of the same frequency

$$p' = p'_{\mathrm{I}} + p'_{\mathrm{II}}, \quad p'_{\mathrm{I}} = a_0 e^{i(k_{\mathrm{I}} x - \omega t)}, \quad p'_{\mathrm{II}} = b_0 e^{i(k_{\mathrm{II}} x - \omega t)}.$$

We can easily obtain a set of equations for the reflected (a_1, b_1) and refracted (a_2, b_2) waves from the set of boundary conditions:

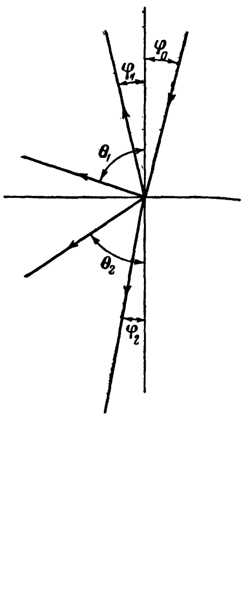

FIG. 2. The incident ray of second sound forms an angle φ_0 with the normal. The reflected ray of second sound makes an angle $\varphi_1 = \varphi_0$ with the normal. Moreover, the refracted ray of second sound and the converted ray of first sound in the two media make the respective angles φ_2, θ_1, θ_2 with the normal.

$$a_0 + a_1 + b_0 + b_1 = a_2 + b_2, \quad \frac{1-\alpha_1}{\rho_1}(a_0+a_1) + \frac{1-\beta_1}{\rho_1}(b_0+b_1) = \frac{1-\alpha_2}{\rho_2}a_2 + \frac{1-\beta_2}{\rho_2}b_2,$$
$$\frac{\beta_1-1}{s_1\rho_1\beta_1}(a_0-a_1) + \frac{\alpha_1-1}{u_1\rho_1\alpha_1}(b_0-b_1) = \frac{\beta_2-1}{\rho_2 s_2\beta_2}a_2 + \frac{\alpha_2-1}{\rho_2 u_2\alpha_2}b_2, \quad \frac{1}{s_1\beta_1}(a_0-a_1) + \frac{1}{u_1\alpha_1}(b_0-b_1) = \frac{1}{s_2\beta_2}a_2 + \frac{1}{u_2\alpha_2}b_2. \tag{3}$$

For brevity, we introduce the notation

$$\alpha = \rho c\frac{\partial}{\partial c}\frac{1}{\rho}; \quad \beta = -\frac{\rho_n}{\rho_s\alpha}.$$

The general solution of the system (3) can be obtained by making use of the smallness of the velocity of second sound in comparison with the velocity of first sound.

We shall first assume that $a_0 = 1$, $b_0 = 0$ (only first sound is incident). As will be evident from what follows, in this case, $a_0 + a_1 \sim a_2 \sim b_1 \sim b_2 \sim u/s$, i.e., in particular, we can consider $a_1 \approx -1$ (the first sound is totally reflected from the superfluid liquid); in the last two equations of (3) we can neglect the term with a_2 on the right hand side, and also set $a_0 - a_1 \approx 2$. Solving the resultant system, we find for b_1 and b_2:

$$b_1 = -2\frac{u_1\alpha_1}{s_1\beta_1}\left(\frac{1-\beta_1}{\rho_1} - \frac{1-\alpha_2}{\rho_2}\right)\Big/\Delta\left(\frac{1-\alpha}{\rho}\right), \qquad b_2 = -2\frac{u_2\alpha_2}{s_2\beta_2}\frac{\beta_1-\alpha_1}{\rho_1}\Big/\Delta\left(\frac{1-\alpha}{\rho}\right), \tag{4}$$

where the symbol ΔA denotes the difference $A_2 - A_1$.

To find a_2, we eliminate $a_0 + a_1$ from the first pair of equations of (3) and substitute the above values of b_1 and b_2:

$$a_2 = 2\frac{\beta_1-\alpha_1}{\rho_1}\left[\frac{u_1\alpha_1}{s_1\beta_1}\left(\frac{1-\beta_1}{\rho_1} - \frac{1-\alpha_2}{\rho_2}\right) + \frac{u_2\alpha_2}{s_1\beta_1}\left(\frac{1-\beta_2}{\rho_2} + \frac{1-\alpha_1}{\rho_1}\right)\right]\left[\Delta\left(\frac{1-\alpha}{\rho}\right)\right]^{-2}. \tag{4a}$$

Now let us consider the case in which a pure second sound ($a_0 = 0$, $b_0 = 1$) is incident. Here it is seen that $b_0 - b_1 \sim b_2 \sim u/s$, i.e., the second sound is reflected almost totally. In this case we find from the first pair of equations in (3):

$$a_1 = 2\left(\frac{1-\beta_1}{\rho_1} - \frac{1-\alpha_2}{\beta_2}\right)\Big/\Delta\left(\frac{1-\alpha}{\rho}\right), \quad a_2 = -2\frac{\beta_1-\alpha_1}{\rho_1}\Big/\Delta\left(\frac{1-\alpha}{\rho}\right). \tag{5}$$

Eliminating $b_0 - b_1$ from the second pair of equations in (3), we find b_2:

$$b_2 = 2\frac{\beta_1-\alpha_1}{\rho_1}\left[\frac{u_2\alpha_2}{s_1\beta_1}\left(\frac{1-\beta_1}{\rho_1} - \frac{1-\alpha_2}{\rho_2}\right) + \frac{u_2\alpha_2}{s_2\beta_2}\left(\frac{1-\beta_2}{\rho_2} - \frac{1-\alpha_1}{\rho_1}\right)\right]\left[\Delta\left(\frac{1-\alpha}{\rho}\right)\right]^{-2}. \tag{5a}$$

The resultant formulas lose their applicability whenever the concentrations (and consequently the thermodynamic functions found in the equilibrium of phases) are close to one another. By virtue of the small difference between the properties of the phases, sound is almost completely transferred from the first phase to the second ($a_2 \sim a_0$; $b_2 \sim b_0$). To find the amplitudes of the reflected and converted waves, we must expand (3) in powers of Δc. As a result of the calculation, we obtain for the case of incidence of first sound ($a_1 = 1$, $b_0 = 0$):

$$a_1 = \frac{1}{2s\beta}\Delta s\beta + \frac{\rho}{2(\beta-\alpha)}\Delta\frac{2-\alpha-\beta}{\rho}; \quad a_2 = 1 - \frac{\rho s\beta}{2(\beta-\alpha)}\Delta\frac{\beta-\alpha}{\rho};$$
$$b_1 = \frac{\rho}{2(\beta-\alpha)}\left[-\Delta\frac{1-\alpha}{\rho} - \frac{u\alpha}{s\beta}\Delta\frac{1-\beta}{\rho}\right]; \quad b_2 = \frac{\rho}{2(\beta-\alpha)}\left[\Delta\frac{1-\alpha}{\rho} - \frac{u\alpha}{s\beta}\Delta\frac{1-\beta}{\rho}\right], \tag{6}$$

and for the case of incidence of second sound ($a_0 = 0$; $b_0 = 1$):

$$b_1 = \frac{1}{2u\alpha}\Delta u\alpha - \frac{\rho}{2(\beta-\alpha)}\Delta\frac{2-\alpha-\beta}{\rho}; \quad b_2 = 1 - \frac{\rho u\alpha}{2(\beta-\alpha)}\Delta\frac{\beta-\alpha}{\rho u\alpha};$$
$$a_2 = \frac{\rho}{2(\beta-\alpha)}\left[\frac{s\beta}{u\alpha}\Delta\frac{1-\alpha}{\rho} + \Delta\frac{1-\beta}{\rho}\right]; \quad a_2 = \frac{\rho}{2(\beta-\alpha)}\left[\frac{s\beta}{u\alpha}\Delta\frac{1-\alpha}{\rho} - \Delta\frac{1-\beta}{\rho}\right]. \tag{7}$$

There still remains the case in which one of the liquids is not a superfluid. The corresponding system of equations can be obtained from (3) by passage to the limit for $\rho_s \to 0$, i.e., $\beta \to \infty$. Then the second equation of (3) yields $b = 0$, while in the remaining equations, we can set $b = 0$, $\beta = \infty$.

Thus, if the sound is propagated from the first, superfluid medium into the nonsuperfluid ($\beta_2 = \infty$), we then get the set of equations

$$a_0 + a_1 + b_0 + b_1 = a_2, \quad \frac{1-\beta_1}{\rho_1 s_1 \beta_1}(a_0 - a_1) + \frac{1-\alpha_1}{\rho_1 u_1 \alpha_1}(b_0 - b_1) = -\frac{1}{\rho_2 s_2} a_2, \quad (a_0 - a_1)/s_1\beta_1 + (b_0 - b_1)/u_1\alpha_1 = 0.$$

For incidence of pure first sound ($a_0 = 1$, $b_0 = 0$), we easily get, assuming that $u \ll s$,

$$a_1 = \left(1 - \frac{\alpha_1}{\beta_1} - \frac{\rho_1 s_1}{\rho_2 s_2}\right) \Big/ \left(1 - \frac{\alpha_1}{\beta_1} + \frac{\rho_1 s_1}{\rho_2 s_2}\right); \quad a_2 = 2\left(1 - \frac{\alpha_1}{\beta_1}\right) \Big/ \left(1 - \frac{\alpha_1}{\beta_1} + \frac{\rho_1 s_1}{\rho_2 s_2}\right); \quad b_1 = 2\frac{u_1\alpha_1}{s_1\beta_1}\frac{\rho_1 s_1}{\rho_2 s_2} \Big/ \left(1 - \frac{\alpha_1}{\beta_1} + \frac{\rho_1 s_1}{\rho_2 s_2}\right). \tag{8}$$

If only second sound is incident ($a_0 = 0$, $b_0 = 1$)

$$a_1 = -2 \Big/ \left(1 + \frac{\rho_2 s_2}{\rho_1 s_1}\frac{\beta_1 - \alpha_1}{\beta_1}\right); \quad a_2 = 2 \Big/ \left(1 + \frac{\rho_1 s_1}{\rho_2 s_2}\frac{\beta_1}{\beta_1 - \alpha_1}\right); \quad b_1 = 1 + 2\frac{u_1\alpha_1}{s_1\beta_1} \Big/ \left(1 + \frac{\rho_2 s_2}{\rho_1 s_1}\frac{\beta_1 - \alpha_1}{\beta_1}\right). \tag{9}$$

If the nonsuperfluid is the first phase ($\beta_1 = \infty$) and the sound is propagated from it, then we get a similar set of equations:

$$a_1 = -\left(1 - \frac{\rho_2 s_2}{\rho_1 s_1}\frac{\beta_2}{\beta_2 - \alpha_2}\right) \Big/ \left(1 + \frac{\rho_2 s_2}{\rho_1 s_2}\frac{\beta_2}{\beta_2 - \alpha_2}\right); \quad a_2 = 2\frac{\rho_2 s_2}{\rho_1 s_1}\frac{\beta_2}{\beta_2 - \alpha_2} \Big/ \left(1 + \frac{\rho_2 s_2}{\rho_1 s_1}\frac{\beta_2}{\beta_2 - \alpha_2}\right); \quad b_2 = -\frac{u_2\alpha_2}{s_2\beta_2} a_2. \tag{10}$$

With the help of Eqs. (4) — (10) and the formulas for the energy flow (2) [which it is expedient to transform in the following fashion

$$q = (j - \rho v_n)\left(\mu - \frac{Z}{\rho} c\right)' + v_n p' = \frac{\beta - \alpha}{\rho s \beta} p_I'^2 + \frac{\alpha - \beta}{\rho u \alpha} p_{II}'^2 \equiv q_a + q_b\ ],$$

we can without difficulty calculate the reflection coefficients $R_I = |a_1|^2$, $R_{II} = |b_1|^2$ of the first and second sound, and also the ratio of the transmitted and converted energy fluxes to the incident. The results of the calculation are given in the Appendix.

These formulas can easily be generalized to the case of the incidence of the sound at some arbitrary angle by making the substitution $s \to s/\cos\theta$; $u \to u/\cos\varphi$ and adding to the conditions for the determination of θ_1, θ_2, φ_1, φ_2 those following from k_y = const:

$$\frac{\sin\theta_0}{s_1} = \frac{\sin\theta_1}{s_1} = \frac{\sin\theta_2}{s_2} = \frac{\sin\varphi_0}{u_1} = \frac{\sin\varphi_1}{u_1} = \frac{\sin\varphi_2}{u_2}.$$

Since $u \ll s$ always, then for incidence of the first sound at an angle $\theta_0 \sim 1$, the generated ray of second sound will make an angle $\varphi \sim u/s \ll 1$ with the normal, i.e., it will go essentially along the normal to the surface. For incidence of second sound on the boundary of separation, its conversion into first sound will take place only if $\varphi_0 < \sin^{-1}(u/s)$; in the opposite case, satisfaction of the equality $\sin\theta/s = \sin\varphi/u$ will not be possible.

The values of $\rho(c)$ and the velocities of first (s) and second (u) sound determined by experiment enter easily into these formulas. The investigation of the passage of sound across the boundary between phases permits us to resolve the problem of the validity of the equations of hydrodynamics for concentrated solutions.

I express my gratitude to Academician L. D. Landau for discussion of the work.

APPENDIX

Relation of the reflected, refracted and converted energy fluxes to the incident.
[Notation: $(1-\alpha)/\rho \equiv A$, $(1-\beta)/\rho \equiv B$]

1. Both liquids superfluid, the values of the concentrations not too close to one another [$(\Delta c)^2 s\beta/\mu\alpha \gg 1$].

(a) Incidence of first sound:

$$\frac{q_{a1}}{q_{a0}} = 1 - \frac{4}{(A_1 - A_2)^2}\left[(B_1 - A_2)^2\frac{u_1\alpha_1}{s_1\beta_1} + (B_1 - A_1)(B_2 - A_2)\frac{u_2\alpha_2}{s_1\beta_1}\right],$$

$$\frac{q_{a2}}{q_{a0}} = 4\frac{s_1\beta_1}{s_2\beta_2}\frac{(B_1 - A_1)(B_2 - A_2)}{(A_1 - A_2)^4}\left[(B_1 - A_2)\frac{u_1\alpha_1}{s_1\beta_1} + (B_2 - A_1)\frac{u_2\alpha_2}{s_1\beta_1}\right]^2,$$

$$\frac{q_{b1}}{q_{a0}} = 4\frac{u_1\alpha_1}{s_1\beta_1}\frac{(B_1 - A_2)^2}{(A_1 - A_2)^2}, \quad \frac{q_{b2}}{q_{a0}} = 4\frac{u_2\alpha_2}{s_1\beta_1}\frac{(A_1 - B_1)(A_2 - B_2)}{(A_1 - A_2)^2}.$$

(b) Incidence of second sound:

$$\frac{q_{b1}}{q_{b0}} = 1 - \frac{4}{(A_1 - A_2)^2}\left[(B_1 - A_2)^2 \frac{u_1\alpha_1}{s_1\beta_1} + (A_1 - B_1)(A_2 - B_2)\frac{u_1\alpha_1}{s_2\beta_2}\right],$$

$$\frac{q_{b2}}{q_{b0}} = 4\,\frac{u_1\alpha_1}{u_2\alpha_2}\,\frac{(A_1 - B_1)(A_2 - B_2)}{(A_1 - A_2)^4}\left[(B_1 - A_2)\frac{u_2\alpha_2}{s_1\beta_1} + (B_2 - A_1)\frac{u_2\alpha_2}{s_2\beta_2}\right]^2,$$

$$\frac{q_{a1}}{q_{b0}} = 4\,\frac{u_1\alpha_1}{s_1\beta_1}\,\frac{(B_1 - A_2)^2}{(A_1 - A_2)^2};\quad \frac{q_{a2}}{q_{b0}} = 4\,\frac{u_1\alpha_1}{s_2\beta_2}\,\frac{(A_1 - B_1)(A_2 - B_2)}{(A_1 - A_2)^2}.$$

2. Both liquids superfluid, values of concentration close together $[(\Delta c)^2 s\beta/u\alpha \ll 1]$.

(a) Incidence of first sound:

$$\frac{q_{a1}}{q_{a0}} = \left[\frac{1}{2s\beta}\Delta s\beta + \frac{1}{2(A-B)}\Delta(A+B)\right]^2;\quad \frac{q_{a2}}{q_{a0}} = 1 + O[(\Delta c)^2],$$

$$\frac{q_{b1}}{q_{a0}} = \frac{s\beta}{u\alpha}\,\frac{(\Delta A)^2}{4(A-B)^2} + \frac{\Delta A\cdot\Delta B}{2(A-B)^2},$$

$$\frac{q_{b2}}{q_{a0}} = \frac{s\beta}{u\alpha}\,\frac{(\Delta A)^2}{4(A-B)^2} - \frac{\Delta A\cdot\Delta B}{2(A-B)^2}.$$

(b) Incidence of second sound:

$$\frac{q_{b1}}{q_{b0}} = \left[\frac{1}{2u\alpha}\Delta u\alpha - \frac{1}{2(A-B)}\Delta(A+B)\right]^2;\quad \frac{q_{b2}}{q_{b0}} = 1 + O[(\Delta c)^2],$$

$$\frac{q_{a1}}{q_{b0}} = \frac{s\beta}{u\alpha}\,\frac{(\Delta A)^2}{4(A-B)} + \frac{\Delta A\cdot\Delta B}{2(A-B)^2},$$

$$\frac{q_{a2}}{q_{b0}} = \frac{s\beta}{u\alpha}\,\frac{(\Delta A)^2}{4(A-B)^2} - \frac{\Delta A\cdot\Delta A}{2(A-B)^2}.$$

3. Second liquid nonsuperfluid

(a) Incidence of first sound $\left(z_1 = \left(1 - \frac{\alpha_1}{\beta_1} + \frac{\rho_1 s_1}{\rho_2 s_2}\right)^{-2}\right)$;

$$\frac{q_{a1}}{q_{a0}} = \left(1 - \frac{\alpha_1}{\beta_1} - \frac{\rho_1 s_1}{\rho_2 s_2}\right)^2 z_1,$$

$$\frac{q_{a2}}{q_{a0}} = 4\,\frac{\rho_1 s_1}{\rho_2 s_2}\left(1 - \frac{\alpha_1}{\beta_1}\right)^2 z_1,$$

$$\frac{q_{b1}}{q_{a0}} = 4\,\frac{u_1\alpha_1}{s_1\beta_1}\left(\frac{\rho_1 s_1}{\rho_2 s_2}\right)^2 z_1.$$

(b) Incidence of second sound $z_2 = \left(1 + \frac{\rho_2 s_2}{\rho_1 s_1}\,\frac{\beta_1 - \alpha_1}{\beta_1}\right)^{-1}$:

$$\frac{q_{b1}}{q_{b0}} = 1 - 4\,\frac{u_1\alpha_1}{s_1\beta_1} z_2,$$

$$\frac{q_{a1}}{q_{b0}} = 4\,\frac{u_1\alpha_1}{s_1\beta_1} z_2^2,$$

$$\frac{q_{a2}}{q_{b0}} = 4\,\frac{u_1\alpha_1}{s_2\beta_2}\left(1 + \frac{\rho_1 s_1}{\rho_2 s_2}\,\frac{\beta_1}{\beta_1 - \alpha_1}\right)^{-1} z_2.$$

4. First liquid nonsuperfluid $(\beta_1 = \infty)$ $z_3 = \left(1 + \frac{\rho_2 s_2}{\rho_1 s_1}\,\frac{\beta}{\beta_2 - \alpha_2}\right)^{-2}$:

$$\frac{q_{a1}}{q_{a0}} = \left(1 - \frac{\rho_2 s_2}{\rho_1 s_1}\,\frac{\beta_2}{\beta_2 - \alpha_1}\right)^2 z_3,$$

$$\frac{q_{a2}}{q_{a0}} = 4\,\frac{\rho_2 s_2}{\rho_1 s_1}\,\frac{\beta_2}{\beta_2 - \alpha_2} z_3,$$

$$\frac{q_{b2}}{q_{a0}} = 4\,\frac{u_2\alpha_2\rho_2}{s_1(\beta_2 - \alpha_2)\rho_1} z_3.$$

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SCATTERING OF ELECTRONS BY PROTONS

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It is shown that under some very general assumptions, scattering of electrons by protons is determined by two real functions $a(q^2)$ and $b(q^2)$ of the invariant $q^2 = (p_1 - p_2)^2$ (p_1 and p_2 are the four-momenta of the electron before and after the collision). Experiments with polarized electrons and protons are considered which permit one in principle to determine the particular form of the functions $a(q^2)$ and $b(q^2)$.

1. In the region of not too high energies, where the proton recoil may be neglected, the scattering of electrons by protons is considered as the scattering of a Dirac particle in an external field. At high energies both particles are relativistic, which leads to a formula of the Møller type for the scattering cross section. Here, we must take account of the anomalous magnetic moment of the proton, which may be done by introducing a suitable term into the Hamiltonian of the interaction of nucleons with the electromagnetic field. But if the nucleon recoil momentum $q \gtrsim \mu c$, where μ is the mass of the π-meson, then the nucleon may not be considered to be a point, and a more detailed analysis is necessary, taking account of the interaction of the nucleon with the meson vacuum. It is clear, from semi-intuitive considerations, that this leads to the appearance of two form factors characterizing the distributions of charge and anomalous magnetic moment of the nucleon. However, a more rigorous investigation of this question is appropriate, particularly in connection with the interpretation Hofstadter's[1] experimental results.

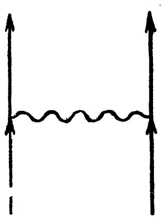

FIG. 1

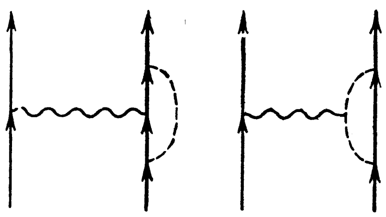

FIG. 2

2. The simplest Feynman diagram corresponding to e–p scattering is shown in Fig. 1 (the heavy line corresponds to the nucleon). The corresponding element of the S-matrix is*

$$S_{i\to f} = -i(2\pi)^4 e^2 \mathfrak{M}_{i\to f}\,\delta(p_1 + P_1 - p_2 - P_2), \quad (1)$$

*We use the system of units in which $\hbar = c = 1$ and $\alpha = e^2/4\pi = 1/137$.