where σ is the differential cross-section. To obtain the relations we sum Eq. (2) over the isotopic spin components, keeping one of them fixed, for example m_b. Using the orthogonality and symmetry of the Clebsch–Gordan coefficients[1] we get

$$\sum_{m_a, m_A, m_B} (\langle T \rangle \sigma)_{m_a m_A; m_b m_B} = \sum_j \frac{(2j+1)}{(2j_b+1)} R_j^+ T_k R_j. \qquad (3)$$

The sum on the left side does not depend on m_b. Consequently, equating the values of this sum for various values of m_b, we get relations between the observable quantities. It is obvious that any one of the components can be held fixed. It is clear from the proof that in the case in which the number of particles changes, relations between the observable quantities are obtained in just the same way. For $T_k = 1$ we get relations between the differential cross-sections, and the conclusion so obtained provides a foundation for the rule formulated by Shmushkevich.[2]

Let us consider the scattering of π-mesons by nucleons. Equating the sums (2) for π^+ and π^0-mesons we get besides Heitler's relations for the differential cross-sections, the relation

$$(P\sigma)_{\pi^+p;\,\pi^+p} + (P\sigma)_{\pi^-p;\,\pi^-p} = 2(P\sigma)_{\pi^0p;\,\pi^0p} + (P\sigma)_{\pi^-p;\,\pi^0n}, \qquad (4)$$

which involes the polarization P of the recoil nucleon.

For the production of a π-meson in collision of nucleons with formation of a deuteron we have:

$$\langle T_k \rangle_{pp;\,\pi^+d} = \langle T_k \rangle_{np;\,\pi^0d}. \qquad (5)$$

T_k characterizes the polarization of the deuteron (3 components of a vector and 5 components of a tensor). If the nucleons in the final state are free, we get

$$(\langle T_k\rangle \sigma)_{pp;\,\pi^+np} + (\langle T_k\rangle \sigma)_{pp;\,\pi^+pn} + (\langle T_k\rangle \sigma)_{np;\,\pi^+nn} + (\langle T_k\rangle \sigma)_{np;\,\pi^-pp}$$
$$= 2(\langle T_k\rangle\sigma)_{pp;\,\pi^0pp} + 2(\langle T_k\rangle \sigma)_{np;\,\pi^0np} + 2(\langle T_k\rangle \sigma)_{np;\,\pi^0pn}.$$

In conclusion we remark that the relations (3) are very convenient for expressing the cross-sections and polarizations in states with definite total isotopic spin in terms of the experimentally observable cross-sections and polarizations.

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THERMODYNAMIC FUNCTIONS OF SUPERFLUID HELIUM FILMS

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IT is well known that a calculation of the thermodynamic functions of superfluid helium with an excitation spectrum of the phonon-roton type proposed by Landau gives excellent agreement with experimental data.[1] In liquid helium which has a free surface, however, there exists one more branch of the energy spectrum associated with the presence of surface waves. Its contribution to the thermodynamic functions is proportional to the area of the free surface and consequently it can play a role only for very thin films.

The spectrum of the surface oscillations has the form[2]

$$\omega^2 = \left(\frac{3ak}{l^4} + \frac{\alpha k^3}{\rho}\right) \tanh kl,$$

where ω is the frequency, k the wave vector, ℓ the film thickness, $\alpha \approx 0.35$ the surface tension, $\rho \approx 0.145$ the density, and $a \approx 8 \times 10^{-15}$ a constant connected with the van der Waals interaction between the film and the wall.

Analysis shows that for

$$l \gg a^{1/4} (\rho/\alpha)^{1/12} (\hbar/\kappa T)^{1/3} \approx 6 \cdot 10^{-8}\ \mathrm{cm}$$

we can use the formula $\omega^2 = \alpha k^3/\rho$, with which* we can easily find the free energy and the specific heat associated with the surface per unit area (the coefficient $1/2$ appears in the equation because the problem is two-dimensional):

$$F_{\text{surf}} = -\frac{1}{2}\int n k \frac{\partial \varepsilon}{\partial k} \frac{2\pi k dk}{(2\pi)^2} = -\frac{\Gamma(7/3)\,\zeta(7/3)}{4\pi} \frac{(\kappa T)^{7/3}}{\hbar^{4/3}} \left(\frac{\rho}{\alpha}\right)^{2/3}, \qquad C_{\text{surf}} = -T\frac{\partial^2 F}{\partial T^2} = \frac{7\Gamma(7/3)\,\zeta(7/3)}{9\pi} \kappa \left(\frac{\kappa T}{\hbar}\right)^{4/3} \left(\frac{\rho}{\alpha}\right)^{2/3}$$

$\epsilon = \hbar\omega$, n is the Planck distribution function, and κ is Boltzmann's constant.

We now ascertain when the quantity obtained above becomes equal to the phonon part of the bulk specific heat

$$C_{\text{ph}} = (2\pi^2\kappa/15)(\kappa T/\hbar s)^3 = C_{\text{surf}}/l$$

(where $s = 2.38 \times 10^4$ is the velocity of sound). After substituting numerical values we obtain $\ell \approx 0.7 \times 10^{-6}\, T^{-5/3}$ cm. In addition the wave length of the phonons with energy κT, which give the principal contribution to the specific heat, has to be much smaller than the film thickness

$$\hbar s/\kappa T \sim 1.8 \cdot 10^{-7}/T \ll l \sim 7 \cdot 10^{-7} T^{-5/3}.$$

We see that the condition is satisfied, although by a narrow margin.

The surface oscillations should also make a contribution to the normal density of superfluid helium. Carrying out the calculations with the usual formulas we obtain

$$\rho_{n\,\text{surf}} = -\frac{\hbar^2}{4\pi\kappa T}\int k^2 \frac{dn}{dz} k dk = \frac{5\Gamma(5/3)\,\zeta(5/3)}{18\pi} \frac{(\kappa T)^{5/3}}{\hbar^{2/3}} \left(\frac{\rho}{\alpha}\right)^{4/3}$$

$(z = \hbar\omega/\kappa T)$. The surface part of the normal density becomes equal to the phonon part when the following equality is satisfied:

$$\rho_{n\,\text{ph}} = 2\pi^2 (\kappa T)^4 / 45\hbar^3 s^5 = \rho_{n\,\text{surf}}/l,$$

which gives

$$l \approx 6 \cdot 10^{-6} T^{-7/3}.$$

The influcence of film thickness on thermodynamic properties has been studied by Frederikse.[4] Unfortunately his data refer to film thicknesses of 12 atomic layers and to temperatures near the λ-point, so that it is not possible to make a quantitative comparison with our results. It can be seen from his data at lower temperatures, however, that the specific-heat curve goes above the curve corresponding to liquid helium in bulk. In these curves there can also be seen a displacement of the λ-point to lower temperatures with decreasing film thickness. One can attempt to interpret this as a result of the influence of $\rho_{n\,\text{surf}}$ on the temperature of the λ-transition.

I am grateful to Academician L. D. Landau for a discussion of this work.

*In an analogous manner Atkins[3] has found the temperature dependence of the surface tension of He^4.

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