CONCLUSIONS

1. The constant b/c was measured as a function of the orientation of the magnetic field relative to the crystal, and is approximately inversely proportional to the static susceptibility.
2. In copper sulfates, an anisotropy was discovered in the spin-lattice relaxation time. In this case, the relaxation time is shortest when the magnetic field is directed parallel to the γ axis.
3. The anisotropy of ρ in crystals of $CuSO_4 \cdot 5H_2O$ is much greater than in the two copper sulfates $Cu(NH_4)_2(SO_4)_2 \cdot 6H_2O$ and $CuK_2(SO_4)_2 \cdot 6H_2O$.
4. In single crystals of $MnSO_4 \cdot 4H_2O$, $Mn(NH_4)_2(SO_4)_2 \cdot 6H_2O$, $Fe(NH_4)(SO_4)_2 \cdot 12H_2O$, $CrK(SO_4)_2 \cdot 12H_2O$, no dependence of the values b/c and ρ on the orientation of the external magnetic field was observed.

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A GROUP-THEORETICAL CONSIDERATION OF THE BASIS OF RELATIVISTIC QUANTUM MECHANICS. I.

THE GENERAL PROPERTIES OF THE INHOMOGENEOUS LORENTZ GROUP

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This paper is the first of a series in which the possibilities of relativistic quantum mechanics are investigated on the basis of group theory, without the use of any specific form of the equations of motion. In the present paper we discuss some of the general properties of group representations as well as some of the specific properties of the representations of the inhomogeneous Lorentz group.

1. INTRODUCTION

THE requirements of invariance of quantum mechanics with respect to some group of transformations of space-time impose quite rigid restrictions on possible wave functions (state vectors): The wave function must transform according to one of the linear representations of the particular group. The most general group that has been treated is the inhomogeneous Lorentz group, consisting of all four-dimensional translations and rotations. A large number of unitary irreducible representations of this group were classified by Wigner.[1] However, the mathematical apparatus used by Wigner is very complicated and difficult for understanding or practical use, so that the results of this important paper have not, with rare exceptions, been used up to the present in physical investigations. In a series, of which the present paper is the first, we shall not only classify all the irreducible representations of the inhomogeneous Lorentz group, both unitary and non-unitary, but will also get them explicitly; for each of these representations we shall give the specific form of the operators for infinitesimal displacements and 4-rotations. We shall use a much simpler mathematical method than that of Wigner.

We shall also find the representations of the improper group, which includes the inversion.

The non-trivial question of time reflection will be subjected to a particular examination. The essential new feature compared to existing treatments of this problem will be the consistent use, on the one hand, of the theory of the representations of the inhomogeneous Lorentz group, and on the other, the use of the concept of a universal covering group with respect to which the spinor representations as well as the vector representations are single valued.[2] As a consequence of this procedure, we are able to obtain various new results. Thus, for example, it turns out that space-time reflections can be introduced into the universal covering group in eight non-equivalent ways. In other words, there are eight non-equivalent groups containing translations, rotations and reflections. The number of mathematically admissible non-equivalent laws of transformation under reflections for, say, the Dirac equation, turns out to be much greater than has been assumed up to now.

At present, the most detailed study available is on the homogeneous Lorentz group, which does not include displacements. The transition from the homogeneous to the inhomogeneous group is not trivial, since rotations and translations do not commute with one another. The importance of treating the inhomogeneous group is clear simply from the fact that its invariants are such fundamental quantities as mass and spin, whereas the invariants of the homogeneous group have no clear physical meaning. The fact that the representations of the homogeneous group (e.g., tensors) have wide use in physics is related to the fact that the homogeneous Lorentz group is homomorphic to the inhomogeneous group. Thus the representations of the homogeneous group are one of the classes of representations of the inhomogeneous group.

Later, by using the technique of representations developed here, we shall give the expansions of the most frequently used wave functions in irreducible components. Unlike Bargmann and Wigner,[3] we shall carry out the expansion not only for the set of solutions but for the whole domain of definition of the wave function. It turns out that contemporary theory does not use all of the unitary representations of the inhomogeneous Lorentz group, but does use a whole series of non-unitary representations. The use of the theory of representations of the inhomogeneous Lorentz group enables us to pose anew the question of possible equations for elementary particles. The unsatisfactory nature of the present scheme (cf., for example, Ref. 4), is obvious simply from the fact that it gives not a single equation which can be used for a completely correct configuration description of a single particle and which leads to positive definite normalization and energy.

The infinitesimal operators of the inhomogeneous Lorentz group are integral quantities — the total 4-momentum and total four-dimensional angular momentum, which characterize the free motion of the system as a whole. Thus, covariance with respect to this group is sufficient only for the description of free motion, and is not sufficient (though, of course, necessary) for the description of interaction. In the presence of interaction, the important quantities include, in addition to the integral quantities, densities, i.e., quantities defined at a point. In order to introduce these quantities in a covariant fashion and to establish their commutation relations, it is necessary to consider more general groups which include not only transformations of the whole space but also transformations in the neighborhood of a particular point. Questions of this sort[5] will also be treated.

The purpose of general investigations of the type presented here is to ascertain which of the specific features of contemporary theory follow by necessity from one or another of the general assumptions, which features are based on arbitrary assumptions and can be changed, and what are the limitations on these changes. In a period when the causes of the principal difficulties of the theory are not known, such investigations may prove to have more than mere academic interest.

2. DEFINITION OF THE INHOMOGENEOUS LORENTZ GROUP

A relativistic quantum equation must be invariant with respect to the transformations of the inhomogeneous Lorentz group, which includes all possible 4-rotations and 4-translations

$$x_\mu = g(x'_\mu) = a_{\mu\nu}x'_\nu + b_\mu, \tag{1}$$

where

$$a_{\mu\nu}a_{\mu\lambda} = \delta_{\nu\lambda}. \tag{2}$$

Throughout the papers, we set $c = h = 1$; $\mu, \nu, \lambda \ldots = 1, 2, 3, 4$; $x_4 = it$. The transformations (1) include space and time reflections in addition to rotations and translations. In treating the proper inhomogeneous Lorentz group, which does not contain space and time reflections, we must impose on the coefficients $a_{\mu\nu}$ the conditions

$$\det|a_{\mu\nu}| = 1, \tag{3}$$

$$a_{44} > 0. \tag{4}$$

We shall use the symbol G to designate the proper inhomogeneous Lorentz group. If we include space reflections among the admissible transformations, but do not include time reflections (giving us the improper inhomogeneous Lorentz group, which we shall designate as G_S), then only condition (4) should be imposed on the coefficients $a_{\mu\nu}$. Finally, if we consider the full inhomogeneous Lorentz group, including reflections in both space and time, we must drop both conditions (3) and (4). We shall first treat the proper group and then generalize the results to the improper group. The question of the invariance of the equations of quantum mechanics with respect to time reflection requires special treatment.

3. IRREDUCIBLE REPRESENTATIONS OF GROUPS

Under a coordinate transformation (1), any wave function (state vector) Ω of a relativistic quantum theory must be subjected to a non-degenerate linear transformation $U(a, b)$:

$$\Omega = U(a, b)\Omega', \tag{5}$$

which depends on the coefficients $a_{\mu\nu}$, b_μ. The set of such transformations must form a representation of the group G. This means that if the successive application of two transformations g_2, g_1 of type (1) gives the transformation

$$g_2 g_1 = g_3, \tag{6}$$

then the corresponding matrices must satisfy the relation

$$U_2 U_1 = U_3. \tag{7}$$

The set of wave functions Ω can be subjected to an arbitrary linear non-degenerate transformation V

$$\Omega = V\Omega''. \tag{8}$$

When this is done, the matrix U is transformed into

$$U''(a, b) = V^{-1}U(a, b)V. \tag{9}$$

If the set of matrices $U(a, b)$ forms a representation of the group, then the set $U''(a, b)$ also gives a representation of the same group. The representations which are obtained with the matrices U, U″ are physically equivalent, and will not be considered as distinct.

It may happen that for some choice of V all the matrices U″ take on the "block" form

$$U'' = \begin{pmatrix} U^{(1)} & 0 & 0. & \ldots \\ 0 & U^{(2)} & 0 & \ldots \\ 0 & 0 & U^{(3)} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}, \tag{10}$$

where $U^{(1)}$, $U^{(2)} \ldots$ are square matrices. Then the matrices $U^{(1)}$ (and also $U^{(2)}$, $U^{(3)}, \ldots$) also form representations of the group G, since

$$U''(a, b)\,U''(a', b') = \begin{pmatrix} U^{(1)}(a, b)\,U^{(1)}(a', b') & 0 \ldots \\ 0 & U^{(2)}(a, b)\,U^{(2)}(a', b') \ldots \\ \ldots & \ldots \end{pmatrix} \tag{11}$$

If it is impossible to find a transformation V which simultaneously brings all the matrices U(a, b) to "block" form, then the representation is said to be irreducible. Representations which can be brought to the form (11), where $U^{(1)}$, $U^{(2)}$, . . . all belong to irreducible representations are said to be reducible. (For simplicity, we shall not consider representations which are not fully reducible.) It was shown by Wigner[1] that any unitary representation of the group G (i.e., a representation consisting of unitary matrices) can be expanded in irreducible representations. Therefore, at least for the unitary representations, we need only find all the irreducible representations; from them we can find all the representations of the inhomogeneous Lorentz group, i.e., all possible wave functions satisfying the requirements of relativistic invariance.

4. THE REPRESENTATIONS U^{-1T}, U^{*T}, U^{*-1}

The wave function Ω is, in general, complex and changes under a transformation of coordinates. It is essential for physical applications that there should exist real bilinear forms in Ω* and Ω which are invariant with respect to coordinate transformations. In order to prove the existence of such real bilinear invariants, we first note that if the matrices U(a, b) form a representation of the group G, then it follows from (7) that

$$U_2^{-1T} U_1^{-1T} = U_3^{-1T}, \quad \text{(I)} \qquad U_2^{*T} U_1^{*T} = U_3^{*T}, \quad \text{(II)} \qquad U_2^{*-1} U_1^{*-1} = U_3^{*-1}. \quad \text{(III)} \tag{12}$$

Here U* is the Hermitian conjugate of U, and U^T is the transpose of U. Thus the sets of matrices U^{-1T}, U^{*T}, U^{*-1} also form representations of the group G and are irreducible if the original representation U was irreducible.

In general all four representations U, U^{-1T}, U^{*T}, U^{*-1} are non-equivalent. We shall show that if the representations U and U^{*-1} are equivalent, we can construct a real bilinear invariant from Ω and Ω*.

5. THE METRIC MATRIX

If the representations U, U^{*-1} are equivalent, then by definition there exists a non-degenerate matrix h* such that

$$U(a, b) = h^{*-1} U^{*-1} h^* \tag{13}$$

for all $a_{\mu\nu}$, b_μ. From the Hermitian conjugate of (13),

$$U^* = h U^{-1} h^{-1}, \tag{14}$$

it follows that the quantity $\langle \Omega^* h \Omega \rangle$ is invariant:

$$\langle \Omega^* h \Omega \rangle = \langle \Omega'^* U^* h U \Omega' \rangle = \langle \Omega'^* h U^{-1} h^{-1} h U \Omega' \rangle = \langle \Omega'^* h \Omega' \rangle = \text{inv.} \tag{15}$$

The quantity

$$\langle \Omega^* h^* \Omega \rangle, \tag{16}$$

which is the complex conjugate of (15), will also be invariant. Adding and subtracting (15), (16), we find that the two quantities

$$J_1 = \left\langle \Omega^* \frac{h + h^*}{2} \Omega \right\rangle = \text{real}, \quad J_2 = \left\langle \Omega^* \frac{h - h^*}{2i} \Omega \right\rangle = \text{real} \tag{17}$$

are real invariants, and at least one of them is not identically equal to zero. It is not difficult to prove that the converse theorem is also true: if a non-degenerate matrix h exists, such that the quantity $\langle \Omega^* h \Omega \rangle$ is invariant, then the representations U, U^{*-1} (and also U^{-1T} and U^{*T}) are equivalent.

We can construct no more than one invariant bilinear in Ω, Ω* from functions which transform according to an irreducible representation. In fact, if we assume that in addition to the invariant (15) there exists an invariant

$$J' = \langle \Omega^* h_1 \Omega \rangle = \langle \Omega^* h \alpha \Omega \rangle,$$

where $\alpha = h^{-1} h_1$, then the relations

$$U^* h U = h, \quad U^* h \alpha U = h \alpha. \tag{18}$$

must be satisfied. Subtracting α times the first equation in (18) from the second, we get $U^*h(U\alpha - \alpha U) = 0$, or, since U^*h is non-degenerate,

$$U\alpha - \alpha U = 0. \tag{19}$$

According to Schur's Lemma, a matrix α which commutes with all the matrices of an irreducible representation can only be the unit matrix, which proves the uniqueness of the invariant for an irreducible representation. In particular, the invariants $\langle \Omega^* h \Omega \rangle$ and $\langle \Omega^* h^* \Omega \rangle$ must coincide, i.e., for an irreducible representation the operator h, which we shall call the "metric matrix," is Hermitian and uniquely determined to within a numerical factor.

6. REAL AND COMPLEX, UNITARY AND NON-UNITARY REPRESENTATIONS

Representations in which U and U^{*-1} are equivalent are said to be real representations. For these representations there exists a non-degenerate Hermitian matrix h such that the quantity $\langle \Omega^* h \Omega \rangle$ is a real invariant. Real representations whose metric matrix has both positive and negative eigenvalues (indefinite metric) are said to be real non-unitary.

Real representations for which all the eigenvalues of the metric matrix are positive (definite metric) will be called real unitary, or simply unitary. For unitary representations, the invariant $\langle \Omega^* h \Omega \rangle$ is not only real but also positive definite. In this case which is very important for quantum mechanics, the bilinear invariant possesses all the properties of the scalar product which is defined in linear algebra. By means of a suitable linear transformation, the metric matrix h can be diagonalized, i.e., brought to the form

$$(VhV^{-1})_{\alpha\beta} = h_\alpha \delta_{\alpha\beta}, \tag{20}$$

where h_α are the eigenvalues of the matrix $h_{\alpha\beta}$. For a unitary representation all the h_α are positive, and the transformation

$$\Omega = \Omega'_\alpha / \sqrt{h_\alpha}$$

reduces the diagonalized metric matrix $h_\alpha \delta_{\alpha\beta}$ to the unit matrix. The invariant then acquires the simple form

$$J = \langle \Omega'^* \Omega' \rangle. \tag{21}$$

Representations in which $U(a, b)$ is not equivalent to U^{*-1} will be called complex non-unitary, or simply complex. For such representations there exist no real (or complex) invariants bilinear in Ω^* and Ω. From (7) and (12) it follows that the product of a function Ω transforming according to (5) and a function Ω_I transforming according to U^{-1T}

$$\Omega_I = U^{-1T}\,\Omega'_I \equiv \Omega'_I\, U^{-1},$$

is invariant

$$\langle \Omega_I \Omega \rangle = \mathrm{inv} = J. \tag{22}$$

The quantity

$$J^* = \langle \Omega^* \Omega_I \rangle = \mathrm{inv}. \tag{23}$$

is also invariant. From (22), (23) it follows that the quantities

$$A = J + J^*, \quad B = i\,(J - J^*) \tag{24}$$

are real invariants.

We now introduce a new wave function ψ, ψ^* having twice as many components as Ω:

$$\psi = \begin{pmatrix} \Omega \\ \Omega_I^* \end{pmatrix}, \quad \psi^* = (\Omega^* \Omega_I) \tag{25}$$

and two Hermitian matrices ρ_1 and ρ_2, which act on ψ, ψ^*

$$\rho_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}. \tag{26}$$

It is obvious that the real invariants A and B are bilinear in ψ^* and ψ:

$$A = \langle \psi^* \rho_1 \psi \rangle, \quad B = \langle \psi^* \rho_2 \psi \rangle, \tag{27}$$

i.e. the function ψ transforms according to a real non-unitary representation. Either one of the matrices ρ_1, ρ_2 may serve as the metric matrix, as is easily verified directly.

In fact, ψ is transformed by means of the matrix

$$\begin{pmatrix} U & 0 \\ 0 & U^{*-1} \end{pmatrix},$$

which satisfies condition (14) if h is chosen to be either ρ_1 or ρ_2. For example,

$$\rho_1 \begin{pmatrix} U & 0 \\ 0 & U^{*-1} \end{pmatrix}^{-1} \rho_1^{-1} = \begin{pmatrix} U^* & 0 \\ 0 & U^{-1} \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & U^{*-1} \end{pmatrix}^* \tag{28}$$

Thus if the irreducible representation U (a, b) is complex, the direct sum of the representations U and U^{*-1} is a representation which is real but reducible. For convenience in writing we shall denote representations and their matrices by the same letters.

In conclusion we remark that the results of Secs. 3 — 5 are applicable to any group, since no use was made of the specific properties of the inhomogeneous Lorentz group.

7. AN EXAMPLE

As an example of a complex non-unitary representation, we may mention the two-dimensional spinor representation of the homogeneous proper Lorentz group, which consists of the set of all two-by-two unimodular matrices. The wave function (basis) of this representation is the two-dimensional relativistic spinor ψ_λ. Then the spinor χ^λ with upper index transforms according to U^{-1T}. The spinor $\chi^{\dot\lambda}$ with dotted upper index transforms according to U^{*-1}, and the spinor $\psi_{\dot\lambda}$ with dotted lower index transforms according to U^{*T}. The spinors ψ_λ and $\chi^{\dot\lambda}$ transform according to non-equivalent representations, which also shows that the representation is complex. The four-component function

$$\psi = \begin{pmatrix} \psi_\lambda \\ \chi^{\dot\lambda} \end{pmatrix}$$

transforms according to a real non-unitary representation which is reducible with respect to 4-rotations, but is irreducible with respect to the improper Lorentz group which includes the inversion. The function ψ is equivalent to a Dirac wave function, and its two real invariants $\langle \psi^* \rho_1 \psi \rangle$ and $\langle \psi^* \rho_2 \psi \rangle$ are respectively a scalar and a pseudoscalar constructed from the Dirac matrices.

We note that the relativistic spinor representation U (a) is equivalent to the representation U^{-1T} (a), which permits us to lower and raise spinor indices. This equivalence does not hold for many representations of the inhomogeneous Lorentz group, as we shall discuss when we consider the closely related question of the invariance of quantum theory with respect to time reflection.

8. THE CONDITIONS FOR RELATIVISTIC INVARIANCE OF QUANTUM THEORY

The proper group G (excluding reflections) is continuous, so that we may consider the infinitesimal 4-rotations and translations, thus considerably simplifying our investigation. Under an infinitesimal transformation (1)

$$x_\mu = x'_\mu + \varepsilon_{\mu\nu} x'_\nu + \xi_\mu, \quad \varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}, \tag{29}$$

where $\epsilon_{\mu\nu}, \xi_\nu$ are first order infinitesimals, the wave function undergoes the transformation

$$\Omega = \left(1 + i\xi_\mu p_\mu + \frac{i}{2} \varepsilon_{\mu\nu} M_{\mu\nu}\right) \Omega'. \tag{30}$$

Here p_μ and $M_{\mu\nu}$ are operators which are independent of the coefficients $\epsilon_{\mu\nu}, \xi_\lambda$ of the transformation.

If we demand that the transformations (30) form a representation of the group in an arbitrary infinitesimal neighborhood of the identity, we obtain the well-known commutation relations for the operators of the 4-momentum p_μ and the 4-angular momentum $M_{\mu\nu}$ (cf. for example, Ref. 6), which are the conditions of relativistic invariance of a quantum theory:

$$[M_{\mu\nu}, M_{\lambda\sigma}]_- = i(\delta_{\mu\sigma}M_{\lambda\nu} + \delta_{\mu\lambda}M_{\nu\sigma} + \delta_{\nu\sigma}M_{\mu\lambda} + \delta_{\lambda\nu}M_{\sigma\mu}), \quad [M_{\mu\nu}, p_\lambda]_- = i(p_\nu\delta_{\mu\lambda} - p_\mu\delta_{\nu\lambda}), \quad [p_\mu, p_\nu]_- = 0. \tag{31}$$

Or, in three-dimensional form:

$$p_\mu = (\mathbf{p}, iH). \quad M_{ij} = \varepsilon_{ijk}M_k, \quad M_{i4} = iN_i, \tag{32}$$

$$\left.\begin{array}{l} [p_i, p_j] = 0, \ [p_i, H]_- = 0, \ [M_i, M_j]_- = -[N_i, N_j]_- = i\varepsilon_{ijk}M_k, \\ [M_i, N_j]_- = i\varepsilon_{ijk}N_k, \ [M_i, H]_- = 0, \ [M_i, p_j]_- = i\varepsilon_{ijk}p_k, \\ [N_i, p_j]_- = i\delta_{ij}H, \ [N_i, H]_- = ip_i. \end{array}\right\} \tag{33}$$

The square brackets with the minus sign (plus sign) denote the commutator (anticommutator). Latin indices take on the values 1, 2, 3; ϵ_{ijk} is the three-dimensional unit pseudotensor.

9. CONDITIONS FOR INVARIANCE WITH RESPECT TO SPACE AND TIME REFLECTIONS

The operator I_s for inversion of the space axes must commute with spatial scalars and pseudovectors, and anticommute with pseudoscalars and vectors, so that the relations

$$[I_s, p_i]_+ = 0, \ [I_s, H]_- = 0, \ [I_s, N_i]_+ = 0, \ [I_s, M_i]_- = 0. \tag{34}$$

must be satisfied.

For time reflection (without the nonlinear transformation involving a change from Ω to Ω^*), the energy H and the vector N_i change sign, while the quantities p_i and M_i remain unchanged, which leads to the relations:

$$[I_t, p_i]_- = 0, \ [I_t, M_i]_- = 0, \ [I_t, H]_+ = 0, \ [I_t, N_i]_+ = 0. \tag{35}$$

From (34) and (35) it follows that the invariant operator for reflection of all four axes, $I_{st} = I_s I_t$ satisfies the invariant commutation relations

$$[I_{st}, p_\mu]_+ = 0, \ [I_{st}, M_{\mu\nu}]_- = 0. \tag{36}$$

10. THE OPERATORS OF INTRINSIC ANGULAR MOMENTUM AND CENTER OF INERTIA

In place of the tensor $M_{\mu\nu}$ it is convenient to consider the vector g_μ:

$$g_\mu = M_{\mu\nu}p_\nu \tag{37}$$

and the pseudovector Γ_σ:

$$\Gamma_\sigma = (1/2i)\,\varepsilon_{\sigma\mu\nu\lambda}M_{\mu\nu}p_\lambda, \tag{38}$$

which satisfy the identities

$$g_\mu p_\mu \equiv 0, \ \Gamma_\sigma p_\sigma \equiv 0. \tag{39}$$

In (38), $\epsilon_{\sigma\mu\nu\lambda}$ is the completely antisymmetric unit pseudotensor. Geometrically, the pseudovector Γ_σ gives the 4-rotations in the plane perpendicular to p_μ, while the vector g_μ gives the remaining 4-rotations. In Ref. 7 it was shown that g_μ is the operator of the center of inertia, while Γ_σ is the intrinsic angular momentum of the isolated quantum mechanical system.

From (37), (38) we find that

$$M_{\mu\nu} = (g_\mu p_\nu - g_\nu p_\mu + i\varepsilon_{\sigma\mu\nu\lambda}\Gamma_\sigma p_\lambda)(p_\xi^2)^{-1}, \tag{40}$$

i.e., $M_{\mu\nu}$ and the pair of quantities g_μ, Γ_σ, determine one another uniquely. Expression (40) becomes meaningless for $p_\xi^2 = 0$, so the case of zero rest mass will be treated separately.

From (31), (37), (38), we can get the commutation relations

$$\left.\begin{array}{l} [p_\mu, \Gamma_\sigma]_- = 0, \ [\Gamma_\sigma, \Gamma_\xi]_- = p_\lambda\Gamma_\mu\varepsilon_{\mu\sigma\xi\lambda}, \ [p_\mu, p_\nu]_- = 0, \\ [g_\mu, \Gamma_\sigma]_- = -i\Gamma_\mu p_\sigma, \ [g_\mu, p_\nu]_- = -i(p_\mu p_\nu - \delta_{\mu\nu}p_\lambda^2), \\ [g_\mu g_\nu]_- = -i(g_\mu p_\nu - g_\nu p_\mu + i\varepsilon_{\sigma\mu\nu\lambda}\Gamma_\sigma p_\lambda) = -iM_{\mu\nu}p_\lambda^2 \end{array}\right\} \tag{41}$$

$$[I_s, p_i]_+ = 0, \ [I_s, p_0]_- = 0, \ [I_s, \Gamma_i]_- = 0, \ [I_s, \Gamma_4]_+ = 0, \ [I_s, g_i]_+ = 0, \ [I_s, g_4]_- = 0, \tag{42}$$

$$[I_{st}, p_\mu]_+ = 0, \ [I_{st}, \Gamma_\sigma]_+ = 0, \ [I_{st}, g_\mu]_+ = 0. \tag{43}$$

11. THE INFINITESIMAL GROUP AND THE GROUP IN THE LARGE

Every irreducible representation of a continuous group will also be a representation in the neighborhood of the identity. The converse statement is not true in general. Thus, if we find all the irreducible representations of the group G in the neighborhood of the identity, i.e., if we find operators $M_{\mu\nu}$, p_λ satisfying (31), then we have not overlooked any of the irreducible representations of the whole group. However, representations may occur which are continuous only in the neighborhood of the identity, but not over the whole group. Such representations must be found and discarded.

When the improper rotations are included in the group, we must include the commutation relations of the operators $M_{\mu\nu}$, p_λ, or the set p_μ, g_ν, Γ_σ with the reflection operators.

12. INVARIANTS OF A GROUP AND THEIR CONNECTION WITH IRREDUCIBLE REPRESENTATIONS

From the well-known Schur lemma[2] in the theory of group representations, it follows that the necessary and sufficient condition for a representation to be irreducible is that the only operator which commutes with all the matrices of the representation is the unit operator. From this it follows that if the group contains an operator which commutes with all the elements of the group,* then the representation can be irreducible only if all the functions which appear in it belong to the same eigenvalue of this operator. In fact, if this is not the case, the representation will be reducible, since it will contain an operator different from the unit operator and commuting with all the elements of the group.

On the other hand, if we find all the independent invariants of a group and form a representation all of whose functions are eigenfunctions belonging to the same eigenvalue of each of the invariants of the group, the representation will be irreducible, since each of the invariants is the unit operator in this representation and, by definition, there are no other operators which commute with all the elements of the group. In other words, to each complete set of eigenfvalues of all the invariants of the group there corresponds one and only one irreducible representation. Thus the problem of classifying the irreducible representations of a group reduces to finding the eigenvalue spectra of the invariants of the group.

In seeking the invariants of the group, we note first that only a scalar can commute with the operator $M_{\mu\nu}$. (Here a scalar is a quantity which is invariant under the transformations of the homogeneous Lorentz group.) For example, for any vector operator A,

$$[M_{\mu\nu}A_\lambda]_- = i\,(A_\nu\delta_{\mu\lambda} - A_\mu\delta_{\nu\lambda}). \tag{44}$$

As a consequence of (39), (40), there are four independent scalars in the proper inhomogeneous Lorentz group:

$$p_\mu^2,\ \Gamma_\sigma^2,\ g_\mu^2,\ g_\mu\Gamma_\mu. \tag{45}$$

All others scalars, for example, $M^2{}_{\mu\nu}$, $M_{\mu\nu}M_{\nu\lambda}M_{\lambda\mu}$, $M_{\mu\nu}M_{\nu\lambda}M_{\lambda\sigma}p_\mu p_\sigma$, etc. can be expressed in terms of the quantities (45) by using (40) and (41).

Of the four scalars in (45), only two, p_μ^2 and Γ_σ^2, commute not only with $M_{\mu\nu}$ but also with p_λ, i.e., are invariants of the group G. The scalars g_μ^2 and $g_\mu\Gamma_\mu$, and any combination of them, do not commute with p_μ and are not invariants of the group G.

We emphasize that the quantities $M_{\mu\nu}^2$ and $M_{\mu\nu}M_{\lambda\sigma}\epsilon_{\mu\nu\lambda\sigma}$, which are the invariants of the homogeneous Lorentz group (not including translations), do not commute with p_λ and are therefore not suitable for classification of the states of relativistic quantum mechanical systems. Other invariants in addition p_μ^2 and Γ_σ^2 are possible for particular classes of representations. Thus, relation (41) is not contradicted by the equality

$$\Gamma_\sigma = \Sigma p_\sigma, \tag{46}$$

where Σ is a number. We then get from (39),

$$p_\sigma^2 = 0. \tag{47}$$

*In mathematics, such operators are said to be central. The use of the term invariant is more appropriate for physics, since any central operator which has a physical meaning always refers to an invariant physical quantity.

Thus if the first invariant (the square of the mass) is zero, we can have proportionality between the vectors Γ_σ and p_σ, and the proportionality factor is an invariant. (We remind the reader that we have not yet considered reflections, so that we do not distinguish between scalars and pseudoscalars.) For time-like (c-number) 4-vectors, the sign of the fourth component is an invariant. For certain representations of the group G, the signs of the fourth components of p_μ and also of Γ_σ turn out to be invariant operators. Finally, the relations (31) are not contradicted by the equality

$$p_\mu = 0. \tag{48}$$

For these representations the invariants are the quantities $M^2_{\mu\nu}$ and $M_{\mu\nu}M_{\lambda\sigma}\epsilon_{\mu\nu\lambda\sigma}$, which are the invariants of the homogeneous Lorentz group.

13. THE CONDITION FOR REALITY OF THE EIGENVALUES OF THE INVARIANTS

The purpose of the present section is to prove the following theorem:

An irreducible representation of the group G is real in the sense of Sec. 6 if and only if the eigenvalues of all the invariants of the group are real for this representation. For real irreducible representations, the eigenvalues of all the components of the operators $M_{\mu\nu}$, p_λ, Γ_σ are also real.

By definition, the condition for reality of a representation is the equivalence of the representations

$$U(a, b) \text{ and } U^{-1*}(a, b).$$

The infinitesimal transformation of type (30) for function Ω_{III}, which transforms according to the representation U^{*-1}, has the form

$$\Omega_{III} = (1 + i\xi_\mu p^*_\mu + \frac{i}{2}\varepsilon_{\mu\nu}M^*_{\mu\nu})\,\Omega'_{III}. \tag{49}$$

We mention that by the vector p^*_μ we mean the vector with components ($\mathbf{p}^*$, ip^*_0), and by the tensor $M^*_{\mu\nu}$ we mean the tensor with components ($\mathbf{M}^*$, $i\mathbf{N}^*$). This point has no basic significance. The use of covariant and contravariant components with the usual definition of the adjoint will give the same results. From (39) and (40) it follows that the representations U, U^{*-1} are equivalent if the operators $M_{\mu\nu}$, p_λ are equivalent to the operators $M_{\mu\nu}$, p_λ. By taking the Hermitian conjugate of (31), we find that the operators $M_{\mu\nu}$, p_λ always form a representation of the group G, since they satisfy the same commutation relations as the original operators $M_{\mu\nu}$, p_λ. The invariants of this new representation are the complex conjugates of the invariants of the original representation. Thus if any of the eigenvalues of the invariants are complex for the irreducible representation U, the representations U and U^{*-1} are equivalent. But if all the eigenvalues of the invariants are real for the irreducible representation U, then U is equivalent U^{*-1}.

If any one of the components $M_{\mu\nu}$, p_λ is equivalent to its adjoint, the two must coincide when brought to diagonal form, i.e., they must be real. This completes the proof of the theorem. We note that the proof makes essential use of the irreducibility of the representation U. Thus, real irreducible representations are characterized by real eigenvalues of the invariants and the operators of infinitesimal transformations.

According to Sec. 5, for irreducible representations there exists a Hermitian metric matrix h such that $\langle\Omega^* h\Omega\rangle = \text{inv}$. From the invariance of this quantity with respect to infinitesimal transformations, it follows that

$$hM_{\mu\nu} - M^*_{\mu\nu}h = 0, \; hp_\lambda - p^*_\lambda h = 0. \tag{50}$$

From (50), we easily find that

$$h\Gamma_\sigma - \Gamma^*_\sigma h = 0, \tag{51}$$

i.e., the operators $hM_{\mu\nu}$, hp_λ, and $h\Gamma_\sigma$ are Hermitian. For unitary representations, h = 1 and the operators $M_{\mu\nu}$, p_λ, and Γ_σ are Hermitian. In the general case, the transformation matrices are non-unitary and the operators $M_{\mu\nu}$, p_λ and Γ_σ are, in general, non-Hermitian even if their eigenvalues are real.

For a real irreducible representation which is chosen in the form where all the operators of some complete set are diagonal (for example, p_1, p_2, p_3, p_0, Γ^2_σ, Γ_3), the relations (50) and (51) become

$$[p_\lambda, h]_- = 0, \ [\Gamma_\sigma^2, h]_- = 0, \ [\Gamma_3, h]_- = 0. \tag{52}$$

Then the metric matrix h, which commutes with all the operators of the complete set, must also be diagonal.

It may appear strange that, for example, the operators p_λ and h commute in one but not in another of two equivalent representations. This occurs because the transformation for h is different from that for all the other operators:

$$\Omega = V\Omega', \ \Omega^* = \Omega'^* V^*, \qquad \langle \Omega^* h p_\lambda \Omega \rangle = \langle \Omega'^* V^* h V V^{-1} p_\lambda V \Omega' \rangle = \langle \Omega'^* h' p'_\lambda \Omega \rangle; \quad h' = V^* h V; \ p'_\lambda = V^{-1} p_\lambda V. \tag{53}$$

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ELECTRICAL CONDUCTIVITY OF FERROMAGNETIC SEMICONDUCTORS (FERRITES)

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A ferrite is considered as a lattice of classical magnetic dipoles submerged in a dielectric continuum. An analysis of the electron conductivity of such a model shows that the line ln $\lambda \sim T^{-1}$ (λ is the electrical conductivity) must have a break at the Curie point, in agreement with experiment, as the activation energy in the ferromagnetic region decreases.

KOMAR and Kliushin[1] detected a break in the line ln $\lambda = f(T^{-1})$ (λ is electrical conductivity) for ferrites in the transition through the Curie point, where the activation energy in the ferromagnetic region is less than in the paramagnetic region. The fact that the break is observed at precisely the Curie point indicates a connection between this phenomenon and the presence of spontaneous magnetization. We show that the existence of this break finds a simple explanation on the basis of a theory that takes into account the interaction of the conduction electrons and the electrons of the unoccupied shells of the magnetic ions.[2]

Ferrites have an electron conductivity due to the stoichiometric excess of metal[3] or to the presence of impurities.[4] The problem of electron motion in a lattice of nonmagnetic ionic crystals has already been solved under the assumption that the ion lattice can be replaced by a dielectric continuum (polaron theory, Refs. 5 and 6). It is natural to use this method for ferrites, which essentially are also ionic crystals.