

We also note that the pseudospinor treatment of the neutrino leads to the Salam condition⁵ of invariance with respect to the transformation $\psi'_\nu \rightarrow \gamma_5 \psi_\nu$, introduced by him as a postulate. Also, the possibility of applying pseudospinors to other fermions is not excluded.

Note added in proof (Sept. 18, 1957). We should mention the interesting possibility of mixed spinors of the first kind with respect to space (time) reflections and of the second kind with respect to time (space) reflections. We also note that in the case of nonconservation of parity with invariance with respect to the Salam transformation,⁵ a new law of conservation of "neutrino charge" holds, with a current density defined by a pseudovector.

¹T. D. Lee and C. N. Yang, Phys. Rev. **104**, 254 (1956); **105**, 1671 (1957).

²E. Cartan, *Leçons sur la théorie des spineurs*, Paris, 1938.

³I. S. Shapiro, Usp. Fiz. Nauk **53**, 14 (1957).

⁴L. Lederman et al., Phys. Rev. **105**, 1415 (1957). Castagnoli, Frazinetti, and Manfredini, Proc. Avogadro Congress, Turin, 1956.

⁵A. Salam, Nuovo cimento **4**, 1 (1957).

Translated by D. Lieberman

184

SOVIET PHYSICS JETP

VOLUME 6 (33), NUMBER 4

APRIL, 1958

*NONLINEAR THEORY OF PHASE OSCILLATIONS INDUCED IN ELECTRON SYNCHROTRONS
BY QUANTUM RADIATION FLUCTUATIONS*

A. N. MATVEEV

Moscow State University

Submitted to JETP editor March 2, 1957; resubmitted May 8, 1957

J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 913-917 (October, 1957)

A nonlinear theory of phase oscillations induced by radiation fluctuations in electron synchrotrons is considered. It is shown that the nonlinear theory gives essential correction to results of the linear theory. The nonlinear theory predicts greater electron losses and imposes upon synchrotron parameters stronger restrictions than does a linear theory.

IT is well known¹ that the quantum nature of radiation causes phase oscillations of electrons in synchrotrons. The mechanism of exciting these oscillations is similar to that described by Sokolov and Ternov^{2,3} for exciting betatron oscillations.

Sands¹ calculated the mean square deviation of the electron phase oscillations induced by radiation for a weak-focusing synchrotron. The calculation was performed in the linear approximation with a neglect of the existence of limits to the phase oscillations. Such a calculation is valid only for small oscillations. These results have been generalized in the same approximation to the case of strong-focusing synchrotrons.⁴⁻⁶

The linear theory of phase vibrations treated by Sands,¹ Kolomenskii,⁴ and the present author^{5,6} is valid only for small deviations of the electron phase from its equilibrium value. In general, however, this assumption does not resolve the fundamental problem, that of determining the loss of electrons caused by these oscillations. In order that an electron be removed from further acceleration, it is necessary that its phase of oscillation be outside the allowable limits. As is well known, the left limit is φ_s , the negative of the equilibrium phase. The boundary φ_1 on the right is given by

$$\sin \varphi_1 + \sin \varphi_s - (\varphi_1 + \varphi_s) \cos \varphi_s = 0. \quad (1)$$

Thus in order that an electron reach the limits, its phase must change by $2\varphi_S$ to the left or by $2\varphi_1 - \varphi_S$ to the right, and these are relatively large durations. If, for instance, $\varphi_S = 45^\circ$, these deviations must be $2\varphi_S \approx \pi/2$ and $\varphi_1 - \varphi_S \approx \pi/4$, respectively, which can in no sense be considered small. The linear approximation is therefore insufficient for an investigation of the theoretically fundamental problem of loss of electrons due to phase changes. One may thus question the accuracy of the predictions of the linear theory, and it becomes necessary to formulate a nonlinear theory without assuming small deviations of the electron phase from the equilibrium value.

In the present article we shall consider certain problems of the nonlinear theory of phase oscillations induced by quantum radiation fluctuations.

The method for deriving the nonlinear stochastic equation describing phase oscillations induced by radiation is entirely analogous to that for obtaining the linear stochastic expression, Eq. (2) of one of the author's previous works.⁶ This nonlinear stochastic equation is

$$\ddot{\psi} + \gamma\dot{\psi} + f^2 [\cos \varphi_s - \cos (\varphi_s + \psi)] = \frac{k\omega\alpha}{\lambda E_s} \left[W_s - \sum_i \varepsilon_i \delta (t - t_i) \right], \quad (2)$$

where $\psi = \varphi - \varphi_S$, φ is the phase with which the electron passes through the high-frequency field, φ_S is the equilibrium value of this phase, ε_1 is the energy of the photon emitted at the instant i , k is the number of the accelerating harmonic of the high-frequency field, R_0 is the radius of curvature of the curved sections of the synchrotron, $\lambda = 1 + L/2\pi R_0$, L is the total length of the linear sections of the synchrotron per revolution,

$$\omega = c/R_0\lambda, \quad f^2 = (k\omega^2\alpha/2\pi\lambda)(ev_0/E_s), \quad \gamma = (4 - \alpha)(2\omega r_0/3R_0)(E_s/mc^2)^3, \\ r_0 = e^2/mc^2, \quad \delta \langle R \rangle / \langle R \rangle = \alpha\delta E/E, \quad W_s = (2ce^2/3R_0^2)(E_s/mc^2)^4,$$

and ev_0 is the amplitude of the high-frequency field.

If ψ is assumed small, Eq. (2) can be linearized and leads to Eq. (2) of the work cited.⁶

As is well known, the problem is generally solved in the following way. One first obtains a solution to the homogeneous Eq. (2) in the form of a Fourier series whose coefficients and frequencies depend on some parameter. The right side of the equation is then treated as a small perturbation, and the effect of this perturbation on the small parameter of the homogeneous solutions is investigated to find its effect on the solutions obtained. Accounting for the chance nature of the small perturbation makes it possible to clarify the statistical properties of the desired solutions.

This method of solution is hardly practical, however, in view of the many calculations necessary, even with computers. We shall therefore use a somewhat different method which is sufficient to answer the most important questions.

We first change to a new independent variable ξ and to a new function z in Eq. (2), using

$$\xi = \int^t f dt, \quad \psi = uz, \quad u = \exp\left(-\frac{1}{2} \int^{\xi} q d\xi\right), \quad q = \frac{f'}{f} + \frac{\gamma}{f}, \quad (3)$$

where the primes indicate differentiation with respect to ξ . Then Eq. (2) becomes

$$z'' + \frac{1}{u} [\cos \varphi_s - \cos (\varphi_s + uz)] - \left(\frac{1}{4} q^2 + \frac{1}{2} q'\right) z = \frac{k\omega\alpha}{\lambda E_s u f} \left[W_{1s} - \sum_i \varepsilon_i \delta (\xi - \xi_i) \right], \quad (4)$$

where

$$W_{1s} = W_s/f.$$

In this equation the term

$$-(q^2/4 + q'/2) z \quad (5)$$

can always be neglected. Indeed, denoting $\Delta_c E_S$ the synchronous energy change per period of synchrotron oscillation, we obviously obtain

$$\frac{f'}{f} \sim \frac{\Delta_c E_S}{E_s} \ll 1, \quad \frac{\gamma}{f} \sim \left[\frac{r_0}{R_0} \left(\frac{E}{mc^2} \right)^3 \right]^{1/2} \ll 1. \quad (6)$$

Therefore $|q| \ll 1$. Similarly, $|q'| \ll 1$.

The homogeneous equation corresponding to (4) can be written

$$z'' + u^{-1} [\cos \varphi_s - \cos (\varphi_s + uz)] = 0. \quad (7)$$

We note further that $u' = -qu/2 \ll u$. We therefore multiply (7) by z' and integrate the product, bearing in mind the above inequality, thus obtaining a constant of the motion in the form

$$Q = \frac{1}{2} z'^2 + u^{-2} [uz \cos \varphi_s - \sin (\varphi_s + uz)]. \quad (8)$$

It follows from (8) that bound states will exist only when the constant of the motion satisfies

$$-u^{-2} \sin \varphi_s = Q_{\min} \leq Q \leq Q_{\max} = u^{-2} (-2\varphi_s \cos \varphi_s + \sin \varphi_s). \quad (9)$$

If, however, Q for any reason leaves the interval given in (9), the particle moves to infinity; in other words it leaves the accelerating process and is lost.

Let us now return to Eq. (4). It is seen from this equation that the mechanism of exciting phase oscillations consists of discretely changing, by emission of photons, the rate at which the phase of a particle varies. It is also seen that emission of a photon of energy ϵ produces a change $\Delta z'$ of magnitude

$$\Delta z' = -(k\omega\alpha/\lambda E_s u f) \epsilon. \quad (10)$$

Equation (8), on the other hand, shows that the mean change that chance discontinuities $\Delta z'$ cause in the constant of motion is given by

$$\langle \Delta Q \rangle = \langle z' \Delta z' \rangle + \frac{1}{2} \langle (\Delta z')^2 \rangle = \frac{1}{2} \langle (\Delta z')^2 \rangle, \quad (11)$$

which can be rewritten, using (10):

$$\langle \Delta Q \rangle = (k^2 \omega^2 \alpha^2 / 2 \lambda^2 E_s^2 u^2 f^2) \langle \epsilon^2 \rangle. \quad (12)$$

It follows that if at $\xi = 0$ the electron is at rest on the bottom of the potential well given by (9), the mean value of Q for other values of ξ will be

$$\langle Q \rangle = -\frac{1}{u^2} \sin \varphi_s + \frac{k^2 \omega^2 \alpha^2}{2 \lambda^2} \int_0^\xi \frac{\langle \epsilon^2 \rangle}{E_s^2 u^2 f^2} d\xi. \quad (13)$$

In order that radiation-induced phase oscillations cause only small losses, we must have $\langle Q \rangle \ll Q_{\max}$. According to Eq. (13) this can be written

$$\frac{k^2 \omega^2 \alpha^2 u^2}{4 \lambda^2} \int_0^\xi \frac{\langle \epsilon^2 \rangle}{E_s^2 f^2 u^2} d\xi \ll \sin \varphi_s - \varphi_s \cos \varphi_s. \quad (14)$$

The stronger the inequality (14), the lower the radiation-induced electron losses. It is clear that for $\langle Q \rangle = Q_{\max}$ practically all the electrons will be lost.

Performing the calculations indicated in (14), we have

$$\frac{55\pi}{48\sqrt{3}} \frac{k\alpha\omega}{\lambda} \left(\frac{r_0}{R_0}\right)^2 \frac{hc}{e^2} \sqrt{\frac{E_s}{ev_0}} \int_0^t \exp\left(-\int_\tau^t \gamma d\tau\right) \left(\frac{E_s}{mc^2}\right)^5 \left(\frac{E_s}{ev_0}\right)^{1/2} d\tau \ll \sin \varphi_s - \varphi_s \cos \varphi_s. \quad (15)$$

If we now note that $1/\gamma$ is much greater than the period of oscillations of the system we are treating and is much larger than the period of acceleration, we can show that

$$\int_0^t \exp\left(-\int_\tau^t \gamma dt\right) \left(\frac{E_s}{mc^2}\right)^5 \left(\frac{E_s}{ev_0}\right)^{1/2} d\tau \ll \frac{3R_0}{2(4-\alpha)r_0\omega} \sqrt{\frac{E_s}{ev_0}} \left(\frac{E_s}{mc^2}\right)^2, \quad (16)$$

in which equality is achieved only if (a) the left side increases monotonically, in which case the equality occurs only asymptotically at infinity, or (b) if the left side has a maximum, in which case the equality is attained at this maximum. We note that in actual accelerators it is case (a) which is realized. The proof of (16) and the corollary assertions follow immediately from the identity

$$\frac{d}{dx} \int_0^x \exp\left(-\int_\tau^x \gamma d\tau\right) f(\tau) d\tau = f(x) - \gamma \int_0^x \exp\left(-\int_\tau^x \gamma d\tau\right) f(\tau) d\tau. \quad (17)$$

On the basis of (16), Eq. (15) can be written in the convenient form

$$F(E) \ll f_1(\varphi_s), \tag{18}$$

where

$$F(E) = \frac{55\pi}{16\sqrt{3}} \frac{k\alpha}{\lambda(4-\alpha)} \frac{r_0}{R_0} \frac{hc}{e^2} \frac{mc^2}{ev_0} \left(\frac{E_s}{mc^2}\right)^3, \tag{19}$$

$$f_1(\varphi_s) = 2(\sin \varphi_s - \varphi_s \cos \varphi_s). \tag{20}$$

This is the fundamental requirement which must be fulfilled according to the nonlinear theory if the electron losses due to phase oscillations induced by quantum radiation fluctuations are to be small.

Let us compare the linear and nonlinear criteria. According to Eq. (8) of the previously cited work,⁶ the criterion of the linear theory can be written

$$F(E) \ll (\ell^2/8) \sin \varphi_s = f_2(\varphi_s), \tag{21}$$

where ℓ denotes the region of permissible phase oscillations. For various values of φ_s , f_1 and f_2 are

φ_s	15°	30°	45°	60°
$f_1(\varphi_s)$	0.011	0.09	0.3	0.7
$f_2(\varphi_s)$	0.018	0.15	0.5	1.2

One thus obtain, approximately,

$$f_2/f_1 \approx 1.7. \tag{22}$$

This means that (18), the requirement given by the nonlinear theory, is stronger than (21), that of the linear theory.

As an example let us consider an unsegmented weak-focusing synchrotron whose parameters are $n = 0.6$, $R_0 = 400$ cm, $ev_0 = 100$ kev, and $k = 4$. If we then express E in Bev, we have $F(E) = 0.16 E^3$. To be specific let us take the equilibrium phase to be $\varphi_s = 45^\circ$. One then sees that for the linear theory the left side of (21) will be about half the right side at 1.3 Bev. The nonlinear theory, on the other hand, makes the left side of (18) about half the right side at 1 Bev. The linear theory thus leads to the conclusion that under the above conditions the synchrotron will cease operating somewhere in the neighborhood of 1.3 Bev. The nonlinear theory, however, leads to the conclusion that the synchrotron will actually stop working at an energy near 1 Bev.

The nonlinear theory thus gives essential corrections to the linear theory. We note that the nonlinear theory of phase oscillations induced by quantum radiation fluctuations verifies the conclusions⁵⁻⁷ that strong focusing is unavoidable at electron energies of several Bev.

¹M. Sands, Phys. Rev. **97**, 470 (1955).

²A. A. Sokolov and I. M. Ternov, Dokl. Akad. Nauk SSSR **92**, 537 (1953); J. Exptl. Theoret. Phys. (U.S.S.R.) **25**, 698 (1953).

³A. S. Sokolov and I. M. Ternov, Dokl. Akad. Nauk SSSR **97**, 823 (1954); J. Exptl. Theoret. Phys. (U.S.S.R.) **28**, 432 (1955); Soviet Phys. JETP **1**, 227 (1957).

⁴A. A. Kolomenskii, J. Exptl. Theoret. Phys. (U.S.S.R.) **30**, 207 (1956); Soviet Phys. JETP **3**, 132 (1956).

⁵A. N. Matveev, Dokl. Akad. Nauk SSSR **108**, 432 (1956); Soviet Phys. "Doklady" **1**, 321 (1956).

⁶A. N. Matveev, Dokl. Akad. Nauk SSSR **109**, 495 (1956); Soviet Phys. "Doklady" **1**, 443 (1956).

⁷A. N. Matveev, Dokl. Akad. Nauk SSSR **107**, 671 (1956); Soviet Phys. "Doklady" **1**, 211 (1956).