[13]G. Racah, Phys. Rev. **63**, 367 (1943); B. Flowers and A. Edmonds, Proc. Roy. Soc. **A214**, 515 (1952).
[14]S. Yanagawa, J. Phys. Soc. Jap. **8**, 302 (1953).
[15]D. R. Inglis, Rev. Mod, Phys. **25**, 390 (1953).
[16]L. Eisenbud and E. Wigner, Proc. Nat. Acad. Sci. **27**, 281 (1941); L. Rosenfeld, Nuclear Forces, Amsterdam, 1948.
[17]D. Brink and G. Satchler, Nuovo cimento **4**, 549 (1956).
[18]S. Pandya, Phys. Rev. **103**, 956 (1956).

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THE ENERGY DEPENDENCE OF A SCATTERING CROSS SECTION NEAR THE THRESHOLD OF A REACTION

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The energy dependence of the cross section for an elastic scattering process X(aa)X is examined at energies near the threshold for a reaction X(ab)Y (b being a neutron). It is shown that the scattering cross section has a singularity at the threshold. From the singularity, one can obtain the spin and parity of the nucleus Y formed in the reaction, and also simplify the phase analysis of the elastic scattering.

1. INTRODUCTION

In a recently published work[1] it was shown that the cross section for the elastic scattering of protons from Li^7, $Li^7(pp)Li^7$, has a peak at an energy corresponding to the threshold of the reaction $Li^7(pn)Be^7$. The peak is about 40 kev wide and at some angles its height is 20 — 30% of the cross section at the same angles. The existence of such a noticeable anomaly in an elastic scattering cross section makes it worth while to consider in general the behavior of the cross section for the elastic scattering X(aa)X at energies near the threshold E_{thr} of the reaction X(ab)Y. Taking advantage of the fact that the energy dependence of the reaction cross section near its threshold is known, one can use the unitarity of the scattering matrix to determine the energy dependence of the phases for scattering near threshold.

It turns out that the behavior of the elastic scattering near threshold gives information not only on the scattering itself, but also on the reaction. In particular, such experiments can be used to find the spin and parity of the particles formed in the reaction, and also greatly simplify the phase analysis of the elastic scattering near threshold. We consider the simplest case first, that all particles a, X, b, Y have spin zero.

2. SPINLESS PARTICLES

We write down the wave function at energies above threshold ($E \geq E_{thr}$). At such energies, both elastic scattering X(aa)X and the reaction X(ab)Y are possible and the wave function has the asymptotic form

$$\Phi_a\Phi_X\left[e^{i\mathbf{k}_1\mathbf{r}}+\frac{1}{r}e^{ik_1r}\sum_l\frac{2l+1}{2ik_1}(S_l-1)P_l(\cos\theta)\right]-\Phi_b\Phi_Y\frac{e^{ikr}}{r}\sum_l\frac{2l+1}{2\sqrt{k_1k}}M_lP_l(\cos\theta), \tag{2.1}$$

where Φ_a, Φ_b, Φ_X and Φ_Y are the internal wave functions of the particles a, b, X and Y; k_1 and k are the wave vectors describing the relative motion of a + X and b + Y, and the P_ℓ are Legendre polynomials. The first term in (2.1) describes the elastic scattering X(aa)X, while the second one refers to the reaction X(ab)Y. The matrix elements for scattering, S_ℓ, and those for the reaction, M_ℓ, satisfy the relation

$$|S_l|^2 + |M_l|^2 = 1, \tag{2.2}$$

which expresses the conservation of particles. If there is no Coulomb interaction between b and Y, then it is known[2] that as $E \to E_{thr}$, M_ℓ behaves as follows:

$$M_l \to (kR)^{l+1/2}\mathfrak{M}_l, \quad k = (1/h)\sqrt{2\mu(E - E_n)}; \; \mu = m_b m_Y/(m_b + m_Y), \tag{2.3}$$

where R is the reaction radius, and $\mathfrak{M}_\ell$ is a constant, independent of energy. Substituting (2.3) in (2.2), we find that for small kR,

$$|S_l| = 1 - \frac{1}{2}(kR)^{2l+1}|\mathfrak{M}_l|^2 \equiv 1 - (kR)^{2l+1}a_l,$$

i.e.

$$S_l = S_l^{(0)}(1 - (kR)^{2l+1}a_l), \tag{2.4}$$

where $|S_\ell^{(0)}| = 1$ for $E \geq E_{thr}$. The matrix elements S_ℓ are analytic functions of k, and hence formula (2.4) holds also for $E \leq E_{thr}$, at which energies k becomes imaginary, $k = i|k|$. For $E \leq E_{thr}$, there are no inelastic processes, and $|S_\ell| = 1$. k being purely imaginary, it follows from (2.4) that $|S_\ell^{(0)}| = 1$. Hence $|S_\ell^{(0)}| = 1$ for both real and imaginary k.

From the above it follows that $S_\ell^{(0)}$ can be written in the form $\exp\{2i\delta_\ell(k)\}$, with the phase $\delta_\ell(k)$ being real for both real and imaginary k, so that the expansion in powers of k contains only even powers of k.

Near the threshold, $kR \ll 1$, and we may neglect all powers of kR higher than the first, so that we have

$$S_0 = e^{2i\delta_0}(1 - kRa_0); \quad S_l = \exp(2i\delta_l) \quad (l \neq 0), \tag{2.5}$$

where δ_ℓ denotes the phase of the ℓ-th partial wave at $E = E_\pi$ (i.e., k = 0). Hence near the threshold, all the S_ℓ ($\ell \neq 0$) can be taken as energy independent, while S_0 is a linear function of k.

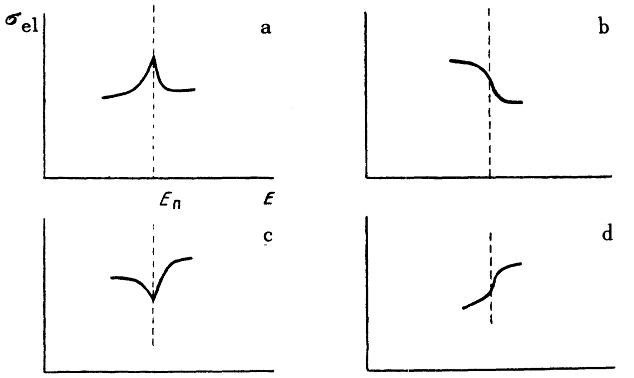

FIG. 1

Knowing the energy dependence of the S_ℓ, we can calculate the energy dependence of the elastic cross section near the threshold:

$$\sigma_{el}(\theta, E) = |f(\theta, E)|^2$$

$$\equiv \left|\frac{1}{2ik_1}\sum_l (2l+1)(S_l - 1)P_l(\cos\theta)\right|^2 = \sigma_{el}(\theta, E_{thr}) \tag{2.6}$$

$$- \frac{k_1}{2\pi}\sqrt{\sigma_{el}(\theta, E_{thr})\sigma_{inel}(|k|)}\begin{cases}\sin(2\delta_0 - \alpha(\theta)) & \text{for } E > E_{thr}\\ \cos(2\delta_0 - \alpha(\theta)) & \text{for } E < E_{thr}.\end{cases}$$

Here $\sigma_{el}(\theta, E_{thr})$ is the differential scattering cross section at $E = E_{thr}$, $\alpha(\theta)$ is the phase of the scattering amplitude at $E = E_{thr}$,

$$f(\theta, E_{thr}) = e^{i\alpha(\theta)}\sqrt{\sigma_{el}(\theta, E_{thr})},$$

and $\sigma_{inel} = 2\pi a_0 |k| R/k_1^2$ is the total reaction cross section for $E > E_{thr}$.

Near the threshold, $\sigma_{inel}(|k|)$ varies with energy as $|k| \sim \sqrt{|E - E_{thr}|}$ (Ref. 2), while all the other quantities in (2.6) can be taken to be energy independent. Hence it follows from (2.6) that if $[2\delta_0 - \alpha(\theta)]$ is in the first quadrant, the elastic scattering cross section has a cusp at the threshold (Fig. 1a), if $[2\delta_0 - \alpha(\theta)]$ lies in the third quadrant, the peak turns into a valley (Fig. 1c), while if $[2\delta_0 - \alpha(\theta)]$ is in the second or fourth quadrants, the cross section exhibits a step (Figs. 1b, 1d).

As a function of $\sqrt{|E - E_{thr}|}$, σ_{el} varies linearly on both sides of the threshold. Knowing the slope before and after the threshold, and the cross section at threshold, we can use (2.6) to find the reaction cross section and the quantity $(2\delta_0 - \alpha)$. It is not difficult to see (cf. Sec. 3) that knowing $(2\delta_0 - \alpha)$ and the modulus of the scattering amplitude $f(\theta, E_{thr})$, we can easily obtain the full scattering amplitude and hence solve the phase analysis problem.

Formula (2.6) is applicable when $kR \ll 1$, which corresponds to an energy range of 20 — 50 kev around the threshold. This is also the width of the peak or step in the differential cross section. In deriving (2.6), it was assumed that there is no Coulomb interaction between b and Y. Hence all our results apply only to the case where one of the reaction products is a neutron.

In the above, the spins of all the particles were taken to be zero. However, the results can be generalized to the more interesting case of non-zero spins. This will be done in the next section.

3. THE CROSS SECTION FOR THE SCATTERING OF α PARTICLES FROM SPIN-ZERO NUCLEI NEAR THE THRESHOLD OF AN (α, n) REACTION

In this case, the spins of the initial particles a and X are zero, the spin of the neutron in $1/2$, and the spin of the nucleus Y formed in the reaction is some half-odd-integer s.

Two situations are possible:

(a) $p(a)\,p(X)\,p(b)\,p(Y) = 1$ [$p(i)$ is the parity of particle i]. Near the threshold the particles b and Y have orbital angular momentum $\ell = 0$, i.e., a total momentum $I = s \pm 1/2$. Hence only that partial wave in the particles $(a + X)$ which has $\ell' = I$ (ℓ' even) can contribute to the reaction. Then the condition (2.2) will link M_0 with $S_{\ell'}$ rather than S_0, and (2.5) will be replaced by

$$S_{l'} = \exp(2i\delta_{l'})(1 - akR); \quad S_l = \exp(2i\delta_l) \quad (l \neq l'). \tag{3.1}$$

The scattering cross section will now be given by the formula

$$\sigma_{\rm el}(\theta, E) = \sigma_{\rm el}(\theta, E_{\rm thr}) - \frac{k_1}{2\pi}\sqrt{\sigma_{\rm el}(\theta, E_n)}\,\sigma_{\rm inel}(|k|)(2l' + 1)P_{l'}(\cos\theta)\begin{cases}\sin(2\delta_{l'} - \alpha(\theta)) & E > E_{\rm thr}\\ \cos(2\delta_{l'} - \alpha(\theta)) & E < E_{\rm thr}\end{cases} \tag{3.2}$$

and we see that the anomaly in the cross section must disappear at angles such that $P_{\ell'}(\cos\theta) = 0$. From this one can find ℓ', and the spin s (up to an additive constant $\pm 1/2$). We note that since ℓ' is even and $P_{2n}(\cos 90°) \neq 0$, the anomaly in the cross section does not vanish at $\theta = 90°$ (in the center of mass system).

(b) $p(a)\,p(X)\,p(b)\,p(Y) = -1$. The only difference between this case and the preceding one is that now ℓ' must be odd (by conservation of parity). The scattering cross section is given as before by (3.2), but since ℓ' is odd the anomaly in the cross section vanishes at $\theta = 90°$.

Hence from the energy dependence of the differential cross section for (α, α) scattering near the threshold of the (α, n) reaction one can find the following: (1) the cross section for the reaction (α, n), $\sigma_{\rm inel}(|k|)$ (see Sec. 2), (2) the parity of the nucleus Y formed in the reaction, and (3) the spin of the nucleus Y (with an accuracy $\pm 1/2$). In addition, one can find the quantity $(2\delta_{\ell'} - \alpha)$ by measuring the slope of the cross section before and after the threshold, and hence find all the elastic scattering phases δ_ℓ at threshold (the modulus $|f(\theta, E_{\rm thr})|$ being known). In order to do this one must expand the experimentally known angular distribution in Legendre polynomials:

$$\exp\{i(\alpha - 2\delta_{l'})\}\,|f(\theta, E_{\rm thr})| = \exp\{-2i\delta_{l'}\}\,f(\theta, E_{\rm thr}) = \exp\{-2i\delta_{l'}\}\frac{1}{2ik_1}\sum_l (2l + 1)(\exp\{2i\delta_l\} - 1)P_l(\cos\theta)$$

The coefficients in the expansion immediately determine the phases.

4. THE CROSS SECTION FOR THE SCATTERING OF α PARTICLES BY NUCLEI WITH SPIN $1/2$ (OR PROTONS BY NUCLEI WITH SPIN ZERO) NEAR THE THRESHOLD OF THE (α, n) [OR (p, n)] REACTION

In this case, the spin of one of the initial particles (a) is $1/2$, that of the other (X) is 0, the spin of particle b (neutron) is $1/2$, while the spin of the nucleus Y formed in the reaction is some integer s. The non-zero spin of a introduces considerable complications. This is because in general the particles b + Y can have both spin $I = s + 1/2$ and $I = s - 1/2$. For a given ℓ in the incident wave, the particles a + X can have angular momentum $j = \ell \pm 1/2$, and for each momentum I of the particles b + Y, with arbitrary parity, there will always be a value of ℓ, with the appropriate parity, such that $\ell \pm 1/2 = I$. Hence near the threshold there will be two matrix elements depending linearly on k

$$S_{l'}^{s+1/2} = \exp\{2i\delta_{l'}^{s+1/2}\}(1 - a_1 kR); \quad S_{l''}^{s-1/2} = \exp\{2i\delta_{l''}^{s-1/2}\}(1 - a_2 kR); \quad S_l^j = \exp\{2i\delta_l^j\} \quad (j \neq s \pm 1/2), \tag{4.1}$$

If the intrinsic parity of the particles does not change during the reaction, then l' and l'' are even, while if the intrinsic parity does change, then l' and l'' are odd. Knowing the behavior of the S^j_l, we can calculate the energy dependence of the scattering cross section X(aa)X and of the polarization $P(\theta, E)$ of the particle a:

$$\sigma_{el}(\theta, E) = |g(\theta, E)|^2 + |h(\theta, E)|^2; \quad P(\theta, E) = 2\mathrm{Im}(h(\theta, E)g^*(\theta, E))/\sigma_{el}(\theta, E), \tag{4.2}$$

$$g(\theta, E) = \frac{1}{2ik_1}\sum_l [(l+1)(S_l^{l+1/2}-1) + l(S_l^{l-1/2}-1)]P_l; \quad h(\theta, E) = -\frac{1}{2ik_1}\sum_l [(S_l^{l+1/2} - S_l^{l-1/2})P_l^{(1)}. \tag{4.3}$$

If, for example, the intrinsic parity of the particles does not change during the reaction, and s is odd, then the following matrix elements depend linearly on k,

$$S_{l'}^{l'-1/2} = \exp\{2i\delta_{l'}^{l'-1/2}\}(1 - a_1kR) \text{ and } S_{l''}^{l''+1/2} = \exp\{2i\delta_{l''}^{l''+1/2}\}(1 - a_2kR); \quad l' = s+1; \quad l'' = s-1$$

and (4.3) becomes

$$g(\theta, E) = g(\theta, E_{\mathrm{thr}}) + \frac{ik}{2k_1}[a_1 l' \exp\{2i\delta_{l'}^{l'-1/2}\}P_{l'} + a_2(l''+1)\exp\{2i\delta_{l''}^{l''+1/2}\}P_{l''}],$$
$$h(\theta_1 E) = h(\theta, E_{\mathrm{thr}}) - \frac{ik}{2k_1}\left[a_1\exp\{2i\delta_{l'}^{l'-1/2}\}P_{l'}^{(1)} + a_2\exp\{2i\delta_{l''}^{l''+1/2}\}P_{l''}^{(1)}\right], \tag{4.4}$$

where, as before, k_1 is the wave vector of the particle, and $P_l^{(1)}$ is an associated Legendre polynomial. Substituting (4.4) in (4.2), one easily obtains formulas for the cross section and polarization analogous to (3.2). As is to be expected, the scattering cross section and the polarization have singularities at the threshold. Near the threshold, $\sigma_{el}(\theta, E)$ and $P(\theta, E)$ are linear functions of k, so that experiments at a given scattering angle θ allow one to compute the following six quantities: $\sigma_{el}(\theta, E_{thr})$, $P(\theta, E_{thr})$ and the slopes of σ_{el} and P on both sides of the threshold. The analysis of the experimental data is not as simple as in the preceding case, but can still yield as before the spin and parity of the particle Y, the cross section $\sigma_{inel}^{s\pm1/2}$ and also the phases δ^j_l describing the elastic scattering.

That such an analysis can be carried out follows from the fact that at each angle the six quantities above depend only on the following six parameters: a_1, a_2, the phases and the moduli of the amplitudes $g(\theta, E_{thr})$ and $h(\theta, E_{thr})$. By their definitions, the quantities a_1 and a_2 do not depend on the scattering angle, so that the following method can be used to find the previously unknown orbital angular momenta l' and l''. Choosing values of l' and l'', we can solve, at each angle, the six equations given by the experiment for a_1, a_2 and the amplitudes g and h. If the values of a_1 and a_2 so obtained depend on the angle, new l' and l'' must be chosen and the process repeated until the true values of l' and l'' are found.

5. CONCLUDING REMARKS

With few changes, our method can be extended to cases where the scattered particles have higher spins. The resulting formulas are too complicated to write down in general, but all the qualitative results derived above still hold: the physical quantities characterizing the scattering (cross section, polarization, the correlation between polarizations) are singular at the threshold. Near the threshold, these quantities are linear functions of k. The energy dependence, near the threshold, of the scattering cross section and of the polarization give the spin and parity of the particle Y and also simplify the phase analysis considerably. To see how much the phase analysis is simplified, one need only note that experiments near the threshold give the phase shifts for the elastic scattering even when these are complex (i.e., there is absorption over and above the reaction whose threshold is considered); in general this is not possible to do without considering all the possible channels.

The reason that so much information can be gained from experiments near the threshold is that there one measures three quantities instead of only one (such as cross section, polarization, etc.). The three quantities are the absolute magnitude and its derivatives with respect to k on both sides of the threshold. This gives three times as many equations as usual for determining the unknowns.

In the preceding we have assumed that there is only one inelastic process X(ab) Y in addition to elastic scattering X(aa)X. Let us see what changes will be introduced if, for $E < E_{thr}$ there is a third reaction X(ac)Z in addition to elastic scattering X(aa)X. If the third reaction is described by matrix elements D_l (for simplicity we take the spins to be zero), the conservation of particles for $E > E_{thr}$ can be written in the form

$$|S_l|^2+|D_l|^2+|M_l|^2=1 \tag{5.1}$$

near the threshold S_0 and D_0 will be linear functions of k:

$$S_0=S_0^{(0)}(1-a_1kR);\quad D_0=D_0^{(0)}(1-a_2kR). \tag{5.2}$$

It is easy to see that the reaction X(ac) Z decreases the size of the coefficient a_1 in the formula for S_0.

The singularity in the cross section which is associated with the threshold "distributes itself," in the case (5.1), over the scattering cross section for X(aa)X and the reaction cross section for X(ac)Z. Two conclusions follow: (1) near the threshold E_{thr} for the reaction X(ab)Y there are anomalies not only in the scattering cross section X(aa)X, but also in the cross sections for all the reactions X(ac)Z whose thresholds E_c lie below E_{thr}; (2) as the number of competing reactions X(ac)Z increases, the anomaly in the scattering cross section must decrease. How much it decreases depends on the details of the situation, but the decrease need not be small. However, the nature of the singularity (break in the cross section) does not change, and whether or not it can be discovered depends only on the precision of the experiment. This applies only when the threshold E_{thr} is sharp. If the state that Y is formed in is wide ($\Gamma > 50$ kev), then the anomaly in the cross section will be smeared out and it will not be possible to observe it.

How big should these effects be? From (2.6) it follows that the height of the peak (or step) in the scattering cross section is, in order of magnitude, equal to the total reaction cross section $\sigma_{inel}(k = 1/R)$ at $k = 1/R \approx 2 \times 10^{12}$ cm, i.e., it is about 0.02 — 0.1 barns. Taking the differential cross section to be about 0.2 barns, we conclude that the height of the anomaly can be 10 — 50% of the cross section. This estimate agrees with the results quoted in Ref. 1 on the Li^7 (pp) Li^7 scattering cross section.

All of the above results refer to the case that there is no Coulomb interaction between the particles b and Y. Only then is $M_0 \sim (kR)^{1/2}$. If b and Y repel each other electrostatically, then M_0 decreases exponentially as k goes to zero and no anomaly will be observed in the scattering cross section.

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[1] P. R. Malmberg, Phys. Rev. 101, 114 (1956).
[2] E. P. Wigner, Phys. Rev. 73, 1002 (1948).

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