THE USE OF LEONTOVICH'S BOUNDARY CONDITIONS IN THE THEORY OF CERENKOV RADIATION

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The field of a charge moving above a plane boundary is computed by using Leontovich's approximate boundary condition.

THE computation of the field in the case of the motion of charges or currents near the surface separating a medium from vacuum presents certain difficulties. It is therefore proposed in the present note to use in problems in which only the field in the vacuum is of importance the Leontovich approximate boundary condition[1]

$$\sqrt{\varepsilon'}E_t = \sqrt{\mu'}H_t, \tag{1}$$

where ε' and μ' are the dielectric constant and permeability of the isotropic medium,* while E_t and H_t are the tangential components of $\mathbf{E}$ and $\mathbf{H}$.

We shall consider the case of greatest interest, when the transverse dimensions of the source are of the order of or greater than the dimensions of the source in the direction of motion. An evaluation of the corrections to (1) shows that the application of (1) to the plane boundary of separation is equivalent under these conditions to neglecting terms of the order of $1/\beta^2\mu'\varepsilon'$, corresponding to velocities much higher than the velocity of light in the medium. This case is of particular interest in the generation of microwaves and for the stabilization of moving beams.

It should be noted that the condition (1) is satisfied with the accuracy indicated above far away from, as well as in the neighborhood of, the source. As an illustration we shall give the values of $\sqrt{\varepsilon'}E_t/\sqrt{\mu'}H_t$ taken from two problems which have been solved exactly.

(a) In the case of uniform motion of a current filament along the separation boundary,[2]

$$\sqrt{\varepsilon'}E_t/\sqrt{\mu'}H_t = (1 - 1/\beta^2\varepsilon'\mu')^{-1/2} \approx 1 + 1/2\,\beta^2\varepsilon'\mu'.$$

(b) In the case of uniform motion of a point source normal to the surface of the dielectric[3]

$$\frac{\sqrt{\varepsilon'}E_t}{\sqrt{\mu'}H_t} = \left(1 + \frac{1 - \beta^2}{\beta^2\varepsilon'\mu'}\right)^{-1/2} \approx 1 - \frac{1 - \beta^2}{2\beta^2\varepsilon'\mu'}.$$

The use of condition (1) permits us to investigate problems in which only the field above the medium is of importance and the parameter $\beta^2\varepsilon'\mu'$ is large. As an example we shall give the outline of the computation of the field of a point charge moving uniformly along the z axis at a distance y = h from an arbitrary non-dispersive medium.

The boundary conditions $\Phi = 0$ and $\Psi = 0$ follow from (1) for the components of the potential A_z, A_y, φ where

$$\Phi = \sqrt{\mu'}A_y + \sqrt{\varepsilon'}\varphi, \quad \Psi = \sqrt{\mu'}\frac{\partial A_z}{\partial y} - \sqrt{\varepsilon'}\frac{1}{c}\frac{\partial A_z}{\partial t}. \tag{2}$$

According to Maliuzhinets[4] the functions Φ and Ψ can be found directly by the method of images and therefore determination of A_y, A_z, φ reduces to solving the system of equations (2) to which the Lorentz condition must also be added. Simple calculations yield

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*The case of anisotropic and gyrotropic media will be considered separately.

$$A_z = \frac{e\beta}{r} + e\beta \left[\frac{1-\lambda^2}{1+\lambda^2} \frac{1}{r^*} - 2\frac{\lambda}{\Lambda} W\right], \quad A_y = \frac{e\beta}{\theta}\left[\frac{2}{\varepsilon'-\mu'}\left(\mu'\frac{W_1}{\Lambda_1} - \varepsilon'\frac{W}{\Lambda}\right) + \frac{1}{r^*}\frac{2(\lambda_1+\lambda)}{\Lambda^2\Lambda_1^2}\right],$$

(3)

$$\varphi = e\left(\frac{1}{r} - \frac{1}{r^*}\right) - 2e\lambda_1\left[\frac{1}{r^*}\frac{\lambda_1+\lambda}{\Lambda^2\Lambda_1^2} + \frac{1}{\varepsilon'-\mu'}\left(\frac{\mu'W_1}{\Lambda_1} - \varepsilon'\frac{W}{\Lambda}\right)\right],$$

where

$$\xi = z - vt, \quad \theta^2 = 1 - \beta^2;$$

$$r^2 = x^2 + (y-h)^2 + (z-vt)^2; \quad \lambda = (\beta/\theta)\sqrt{\varepsilon'/\mu'}, \quad \Lambda^2 = 1 + \lambda^2;$$

$$r^{*2} = x^2 + (y+h)^2 + (z-vt)^2; \quad \lambda_1 = (\beta/\theta)\sqrt{\mu'/\varepsilon'}, \quad \Lambda_1^2 = 1 + \lambda_1^2;$$

$$W(\lambda) = \frac{\xi - \lambda(y+h)}{(\xi-\lambda(y+h))^2 + x^2\Lambda^2}\left(1 - \frac{(y+h)+\lambda\xi}{\Lambda r^*}\right); \quad W_1 \equiv W(\lambda_1).$$

The expression for the radiation losses obtained with the aid of (3) agrees for $\beta^2\mu'\varepsilon' \to \infty$ with the exact expression.[5]

In the non-relativistic domain the solution of problems is further simplified by the possibility of expanding the field in powers of $\beta = v/c$.

$$E = E^{(0)} + \beta E^{(1)} + \beta^2 E^{(2)} + \dots, \quad H = H^{(0)} + \beta H^{(1)} + \beta^2 H^{(2)} + \dots$$

For the sake of definiteness we shall consider the motion of a charge near a certain boundary over which the condition (1) is satisfied. In this case it may be shown[6] that

$$E_t^{(0)} / E_n^{(0)} \sim \beta\sqrt{\mu'/\varepsilon'} = \lambda_1.$$

(4)

From (4) it follows that for $\lambda_1 \gg 1$ the electric field is "expelled" from the medium, while for $\lambda_1 \ll 1$ it enters the medium essentially at right angles. This permits one to take as the basis the solution of Laplace's equation with the boundary condition $E_n^{(0)} = 0$ or $E_t^{(0)} = 0$ and then by using (1) to find $E^{(1)}$ and $H^{(1)}$, and thus to compute the intensity of radiation. The approximation obtained for $\lambda_1 \gg 1$ will correspond to expanding the intensity of radiation in powers of $1/\lambda_1$, while for $\lambda_1 \ll 1$ the expansion is carried out with respect to $1/\lambda$. With this accuracy one can easily find, for instance, the field of a bunch of charges moving at an arbitrary angle to the surface of a dielectric $(\varepsilon' \gg \mu')$. This field is of the form

$$\varphi = \varphi_0 - \varphi_s - 2\frac{v_y}{c}\sqrt{\mu'/\varepsilon'}\varphi_s, \quad A_x = 0, \quad A_y = -2\sqrt{\mu'/\varepsilon'}\varphi_s + \frac{v_y}{c}(\varphi_0 + \varphi_s) + \sqrt{\frac{\mu'}{\varepsilon'}}v_y\int_{-\infty}^{t}\frac{\partial\varphi_s}{\partial y}dt,$$

$$A_z = \frac{v_z}{c}(\varphi_0 - \varphi_s) - 2v_z\sqrt{\frac{\mu'}{\varepsilon'}}\int_{-\infty}^{t}\frac{\partial\varphi_s}{\partial y}dt,$$

where φ_0 is the potential of the bunch in homogeneous space, while φ_s is the potential of its image taken with opposite sign.

In the case when the motion takes place near a curved surface the error due to the use of (1) will be of the order of

$$h(1/R_1 - 1/R_2)/\beta\sqrt{\mu'\varepsilon'},$$

where R_1 and R_2 are the principal radii of curvature, while h is the distance of the moving source from the surface. In the non-relativistic case one may also use simpler boundary conditions as was done above in the case of the plane boundary. We have calculated by this method the instantaneous value of the flux of the Poynting vector into a dielectric cylinder $(\lambda_1 \ll 1, \lambda \gg 1)$ of radius a when a charged filament parallel to its axis passes over it. This flux turned out to be equal to

$$P = 2c\sqrt{\frac{\mu'}{\varepsilon'}}\rho^2 \frac{\beta^2}{a}\frac{2r^2 + (r^2-1)p^2}{r^2(r^2-1)},$$

where pa is the impact distance, ar $= \sqrt{a^2p^2 + (vt)^2}$ is the distance from the filament to the axis of the cylinder, and ρ is the charge density of the filament.

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[6] A. I. Morozov, Thesis, Moscow State University, (1957).

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THERMODYNAMICS OF THE He I — He II PHASE TRANSITION IN HELIUM ISOTOPE MIXTURES

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A thermodynamic investigation of the phenomena associated with a phase transition of the second order in liquid mixtures is carried out. The results obtained are compared with the experimental data on liquid-vapor equilibrium in the $He^3 - He^4$ system.

IN pure He^4, as is well-known, the He I — He II transition is a phase transition of the second order. The question of the nature of this transition in mixtures of the helium isotopes has been repeatedly submitted to theoretical investigation, as a result of which, in particular, it has been predicted[1] that the λ-transition in such mixtures will be a transition of the first order for sufficiently small concentrations and high temperatures.

Inasmuch as this problem is of fundamental significance, it appears desirable to carry out a detailed thermodynamic investigation of the phenomena associated with the He I — He II transition in helium isotope mixtures, and to compare the results thus obtained with the experimental data (cf. also references 2, 3).

It is necessary first of all to note that a conclusion regarding the nature of the He I — He II phase transition in helium isotope mixtures can be drawn from a study of the dependence of the vapor pressures of these mixtures upon their He^3 content. In the case of a first order phase transition, in fact, there should be observed a separation of the liquid phase into two mixtures of differing He^3 content, which would yield vapor pressures independent of the concentration of the light isotope in the region of separation.

Carefully-conducted experiments[4-6], however, have shown that in the interval $1.35 - 3.0°K$ there is a marked dependence of the vapor pressure upon the He^3 content of the liquid phase, indicating that the hypothesis concerning the occurrence of a first order phase transition in this temperature interval is not borne out.

Thus, as in the case of pure He^4, the He I — He II phase transition in helium isotope mixtures within the temperature range investigated is a phase transition of the second order.*

Inasmuch as this is the only instance of the occurrence of a second order phase transition in liquid

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*It must be noted that Walters and Fairbank[7] have recently observed a separation of helium isotope mixtures into two phases below $0.8°K$.