NUCLEON ENERGY LEVELS IN SPHEROIDAL NUCLEI

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THE problem of energy levels in a nonspherical nucleus has been treated fully only for an oscillator potential.[1] The case of an ellipsoidal well was treated by perturbation theory with expansion in terms of the deformation parameter (see for example, Ref. 2). Because of poor convergence this treatment cannot be used both for highly excited states and for large deformations. The present note concerns a method by which the spectrum of highly excited states in an ellipsoidal well can be found for arbitrary deformations, except for vanishingly small ones.

For the case of an infinitely deep prolate spheroidal well the problem consists of solving the equation

$$(\hbar^2/2M)\,\Delta\Psi + E\Psi = 0$$

inside the spheroid with zero boundary conditions. We introduce the spheroid coordinates u, θ, and φ:

$$x = f\sinh u \sin\theta\cos\varphi,\quad y = f\sinh u\sin\theta\sin\varphi,\quad z = f\cosh u\cos\theta,$$

where $0 \le u \le u_0$, $0 \le \theta \le \pi$, and 2f is the distance between the foci of the spheroid.

Putting $\Psi = R(u)Y(\theta)\exp(\pm im\varphi)$ we obtain

$$d^2R/du^2 + \coth u\,dR/du + (\gamma^2\cosh^2 u - \Lambda - m^2/\sinh^2 u)R = 0,$$

$$d^2Y/d\theta^2 + \cot\theta\,dY/d\theta + (\Lambda - \gamma^2\cos^2\theta - m^2/\sin^2\theta)\,Y = 0,\quad \gamma^2 = 2f^2ME/\hbar^2,$$

where Λ is the separation constant. Let γ be large. Utilizing the boundedness of Y in the range $0 \le \theta \le \pi$ we can show that Λ can be given in the form[3,4]

$$\Lambda = q\gamma + m^2 - \frac{1}{8}(q^2+5) + \sum_{i=1}^{\infty}\lambda_i(l,m)/\gamma^i,\quad q = 2(l - |m|) + 1,$$

where $l - |m|$ is the number of nodes of the function Y in the interval $0 - \pi$ of θ. The functions $\lambda_i(l, m)$ are given by Meixner[3] and by Sips.[4]

We find the function R(u) by a generalization of the method of Miller and Good[5] and obtain

$$R(u) = [S(u)/\sinh u(dS/du)]^{1/2}J_m(\gamma S(u)), \tag{1}$$

where $J_m(z)$ is the Bessel function, and

$$S(u) = \sum_{k=0}^{\infty}\gamma^{-k}S_k(u),\ S_0(u) = \sinh u,\ S_1(u) = -\frac{q}{2}\cot^{-1}\sinh u,\ S_2(u) = -\frac{q^2+3}{16}\frac{\sinh u}{\cosh^2 u},$$

$$S_3(u) = -\frac{q}{128}(3q^2+41)\frac{\sinh u}{\cosh^2 u} + \frac{q}{64}(q^2+19)\frac{\sinh^3 u}{\cosh^4 u} + \frac{q}{4}\left(m^2 - \frac{1}{4}\right)\left(1 - \frac{\cot^{-1}\sinh u}{\sinh u}\right)\frac{1}{\sinh u}$$

etc.

Since $R(u_0) = 0$, the quantities γ_{nlm} and with it the energies E_{nlm} are obtained from the algebraic equation

$$\gamma S(u_0) = j_{mn}, \tag{2}$$

where j_{mn} is the n-th root of the function $J_m(z)$. Numerical calculations for a prolate spheroid with a ratio of the axes of 1.2 : 1 show that the use of four terms of the expansion of S(u) in (2) allows the determination of all levels with an error ≤ 1%. The results for the lower levels agree with Moszkowski's results.[2]

We now take into account the finite depth of the potential well and the spin-orbit force. The operator for the latter[6] has in spheroidal coordinates the form

$$\hat{H}_{s0} = -\frac{\lambda \hbar^2}{2iM^2c^2} \frac{1}{f^2(\cosh^2 u - \cos^2\theta)} \frac{\partial V}{\partial u}\left[-\hat{\sigma}_1\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right) + \hat{\sigma}_2\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right) + \hat{\sigma}_3\coth u\frac{\partial}{\partial\varphi}\right], \quad (3)$$

where λ is the force constant, $\hat{\sigma}_i$ the Pauli spin matrices, and V the potential, which equals zero for $u < u_0$ and V_0 for $u > u_0$. From the symmetry of the problem it follows that only the third term of (3) contributes to the energy. To obtain the energy spectrum one has merely to solve the transcendental equation for γ, which follows from the matching conditions of the wave function at $u = u_0$:

$$\left[\gamma\frac{J_{m+1}(\gamma S)}{J_m(\gamma S)}\frac{dS}{du} - \frac{m+1/2}{S}\frac{dS}{du} + \frac{1}{2}\left(\frac{dS}{du}\right)^{-1}\frac{d^2S}{du^2} - g\chi - \frac{1}{2}\frac{1}{\chi}\frac{d\chi}{du}\right]_{u=u_0} = -A,$$

where

$$g = f\sqrt{\frac{2M}{\hbar^2}(V_0 - E)},$$

$$\chi(u) = \sinh u + \frac{1}{g}\frac{p}{\sinh u} - \frac{p^2 - m^2 + 1}{2g^2}\left(\frac{1}{2\sinh u} + \frac{1}{\sinh^3 u}\right) - \frac{p}{2g^3}\left(\frac{p^2 - m^2 + 1}{8\sinh u} - \frac{p^2 - m^2 + 7}{2\sinh^3 u} - \frac{p^2 - m^2 + 5}{\sinh^5 u}\right) + \ldots,$$

$$A = \pm\lambda m V_0 \coth u_0 / (Mc^2),$$

$p = \ell + 1$ for even $\ell - |m|$ and $p = \ell$ for odd $\ell - |m|$, and the two signs of A correspond to the two possible orientations of the nucleon spin with respect to the z axis.

The case of the oblate spheroid can be treated in an analogous manner.

Numerical computations of the energy levels of nucleons in ellipsoidal wells are being carried out at the present time for different deformations.

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DIFFRACTION SCATTERING OF HIGH-ENERGY PROTONS BY PROTONS

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FOWLER et al.[1] attempted a theoretical analysis of the diffraction p—p scattering, starting, however, with the usual spherical nucleon model with distinct boundaries and a definite transparency. Of greater interest is Belenkii's analysis[2] of the diffraction scattering of high-energy pions by nucleons, based on a general theory involving no specific nucleon model. We considered similarly the collisions of high-energy protons and analyzed the known experimental data, making the following assumptions: (1) the spin dependence of the nuclear forces can be neglected at high energies, (2) the imaginary part of the scattering amplitude is much greater than the real part (since it is known from the experimental data of Ref. 1 that the elastic scattering cross section is on the same order as the inelastic one at high energies and that both are on the order of the geometric cross section).

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ERRATA TO VOLUME 6

Page	Line	Reads	Should Read
643	16 from bottom	where $\kappa = \pi a^2 \Omega - \ldots$	where $\kappa = \pi a^2 \Omega \varphi - \ldots$
690	8 from bottom	$\ldots \sin [- \ldots$	$\ldots \sin \vartheta [- \ldots$
	5 from bottom	$\ldots \sin 2\vartheta \sqrt{\frac{1}{3}} \ldots$	$\ldots \sin 2\vartheta \left[\sqrt{\frac{1}{3}} \ldots \right.$
809	9 from top	$\ldots \left(\frac{1}{2 \sinh u} + \ldots \right.$	$\ldots \left(\frac{1}{\sinh u} + \ldots \right.$
973	unnumbered equation	$\ldots C^{S'-\mu';\, S\mu}_{n\mu-\mu'} T^{(n)}_{\mu'-\mu'}.$	$\ldots C^{S'-\mu';\, S\mu}_{n\mu-\mu'} \langle S' \Vert T^{(n)} \Vert S^{-1} \rangle \times T^{(n)}_{\mu'-\mu}.$
975	5 from bottom	...of a particle by a nucleus ...	...of a particle in state a by a nucleus ...
992	Eq. (18)	$\ldots \tau_1 \tau_2^{-2} / 2\hbar^1 \ldots$	$\ldots \tau_1 \tau_2^{-1} / 2\hbar^2 \ldots$