SOVIET PHYSICS JETP VOLUME 6 (33), NUMBER 5 MAY, 1958

CONCERNING A MODEL FOR A NON-IDEAL FERMI GAS

A. A. ABRIKOSOV and I. M. KHALATNIKOV

Institute of Physical Problems, Academy of Sciences, U.S.S.R.

Submitted to JETP editor April 24, 1957

J. Exptl. Theoret. Phys. (U.S.S.R.) 33, 1154-1159 (November, 1957)

The energy of the ground state and the effective mass of the excitations are found for a non-ideal Fermi gas with short-range forces, the radii of action of which are small in comparison with the average wavelength.

YANG and Huang[1] and Yang and Lee[2*] have recently investigated the properties of non-ideal Bose and Fermi gases consisting of particles whose dimensions are small in comparison with their average wavelength. The study of such a model is of considerable interest. The method developed by Yang, et al., however, is extremely cumbersome.

In the present work another method will be described which allows the thermodynamic quantities in the model of Yang, et al., to be found reasonably simply for the case of Fermi statistics. The idea for this method is due to Landau.

The model, called a hard-sphere model in the work of Yang et al., has in reality a more general character. It describes a system of particles with arbitrary short-range forces whose radii of action are short compared to the average wavelength.

We first find an expression for the energy of the system correct to terms of order $(a/\lambda)^2$ (where a is the radius and λ the wavelength). It would be possible in principle to determine the terms of order $(a/\lambda)^3$ in an analogous manner; terms of higher order, however, are in principle impossible to find by means of the method presented below, since beginning with terms of the fourth order it is necessary to take into account the contribution of multiple collisions (cf. Ref. 1).

We make use of perturbation theory as applied to the interaction energy of the particles, which we write (taking the volume of the gas as unity, for simplicity)

$$V = 2U \sum_{\substack{n_1 n_2 m_1 m_2 \\ n_1 < n_2}} a^+_{m_1} a^+_{m_2} a_{n_2} a_{n_1}, \tag{1}$$

where a_i^+ and a_i are the creation and annihilation operators for the particle. The summation is performed with conservation of the total momentum taken into account, with the projection of the spin of the state m_1 equal to the projection of the spin of the state n_1, and similarly for m_2 and n_2. The placing of U ahead of the summation sign reflects the fact that the interaction is identical for all pairs of particles, while the scattering amplitude is independent of the angle. The magnitude of U is in the first approximation related to this amplitude by the expression

$$U = 4\pi a \hbar^2 / m \tag{2}$$

(a being the s-scattering amplitude).

The first-order contribution to the energy is equal to the diagonal matrix element of V:

$$E_n^{(1)} = 2U \sum_{n_1 < n_2} N_{n_1} N_{n_2} Q_{n_1 n_2}, \tag{3}$$

where the N_i are the occupation numbers.

The factor Q_{ik} in Eq. (3) takes into account the circumstance that Fermi particles, having an angle-independent scattering amplitude, do not interact in the case of parallel spins. Thus we shall assume that

*We take this opportunity to express our gratitude to Professors Yang, Huang, and Lee for sending us their manuscripts prior to publication.

$$Q_{ik}^{\cdot} = {}^1/_4 - \sigma_i \sigma_k \tag{4}$$

(σ_i being the spin operator for the i-th particle).

Substituting Eqs. (2) and (4) into (3), we obtain

$$E^{(1)} = (2\pi a\hbar^2/m)N^2/2. \tag{5}$$

To find the second-order correction we use the perturbation-theory expression

$$E_n^{(2)} = \sum_{m \neq n} |V_{nm}|^2 / (E_n - E_m). \tag{6}$$

Substituting Eq. (1) into this formula, we obtain the following sum:

$$4U^2 \sum_{\substack{n_1 n_2 m_1 m_2 \\ n_1 < n_2}} \frac{N_{n_1} N_{n_2} (1 - N_{m_1})(1 - N_{m_2}) Q_{n_1 n_2}}{E_{n_1} + E_{n_2} - E_{m_1} - E_{m_2}}, \tag{7'}$$

where the N_i are the equilibrium occupation numbers and the E_i are the particle energies.

Since our objective is an expansion of the energy in powers of a, we must recall that the relation (2) between U and the scattering amplitude is not exact, but correct only to terms of the first order. Taking second-order terms into account we obtain in place of relation (2) the following:

$$2U + 4U^2 \sum_{m_1 m_2} Q_{n_1 n_2} / (E_{n_1} + E_{n_2} - E_{m_1} - E_{m_2}) = 8\pi a\hbar^2 / m. \tag{2'}$$

If from this we express U in terms of a and substitute the result into Eq. (3), then the expression for $E^{(1)}$ will contain terms proportional to a^2, which are naturally to be referred to the second-order correction. Taking this into account, we obtain the following value for the second approximation to the energy:

$$E^{(2)} = 2U^2 \sum_{n_1 n_2 m_1 m_2} \{N_{n_1} N_{n_2} (1 - N_{m_1})(1 - N_{m_2}) Q_{n_1 n_2} / (E_{n_1} + E_{n_2} - E_{m_1} - E_{m_2}) - N_{n_1} N_{n_2} Q_{n_1 n_2} / (E_{n_1} + E_{n_2} - E_{m_1} - E_{m_2})\}. \tag{7}$$

In view of the fact that the expression in brackets is symmetrical in n_1 and n_2, we replace the limit $n_1 > n_2$ by the factor $\frac{1}{2}$.

The reason for the operation we have just performed is that the expansion in powers of U does not actually take place. The presence of the constant U would simply have led to an infinite value for the energy, as can be seen directly from Eq. (7′). In the present case the essential circumstance is that the scattering amplitude a has a finite, and moreover a small, value, which makes possible the expansion in terms of this quantity.

In the first part of Eq. (7) the term containing the product of four N_i's is equal to zero, since the denominator is antisymmetric relative to the interchange $n_1 n_2 \to m_1 m_2$, while the numerator is symmetrical and the total range of summation is the same. The remaining two terms containing the products of three N_i's are identical. Thus we obtain, finally

$$E^{(2)} = -4U^2 \sum_{n_1 n_2 m_1 m_2} N_{n_1} N_{n_2} N_{m_1} Q_{n_1 n_2} / (E_{n_1} + E_{n_2} - E_{m_1} - E_{m_2}). \tag{8}$$

This is the fundamental expression for the energy, which is correct so long as the assumption ($a/\lambda \ll 1$) is fulfilled.

Our goal is to obtain the characteristics of a degenerate Fermi gas. From Eq. (8) we find for the energy of the ground state

$$E^{(2)} = -\frac{4U^2}{(2\pi\hbar)^9} \int\limits_{|\mathbf{p}_1| < p_0} d\mathbf{p}_1 \int\limits_{|\mathbf{p}_2| < p_0} d\mathbf{p}_2 \int\limits_{|\mathbf{p}_3| < p_0} d\mathbf{p}_3 \int d\mathbf{p}_4 \frac{\delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)}{(\mathbf{p}_1^2 + \mathbf{p}_2^2 - \mathbf{p}_3^2 - \mathbf{p}_4^2)/2m}, \tag{9}$$

where p_0 is the limiting momentum, equal to $\hbar(3\pi^2 N)^{1/3}$.

In accordance with the work of Landau on the theory of a Fermi liquid,[3] the excitation energy is determined by the relation

$$\varepsilon_i = \delta E / \delta N_i. \tag{10}$$

Taking the variation of Eqs. (3) and (8) with respect to N_i gives

$$\varepsilon(\mathbf{p}) = \frac{\mathbf{p}^2}{2m} + \frac{UN}{2} + \frac{2U^2}{(2\pi\hbar)^6}\int\limits_{|\mathbf{p}_1|<p_0} d\mathbf{p}_1 \int\limits_{|\mathbf{p}_2|<p_0} d\mathbf{p}_2 \int d\mathbf{p}_3 \left[\frac{\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}-\mathbf{p}_3)}{(\mathbf{p}^2+\mathbf{p}_3^2-\mathbf{p}_1^2-\mathbf{p}_2^2)/2m} - 2\frac{\delta(\mathbf{p}_1+\mathbf{p}-\mathbf{p}_2-\mathbf{p}_3)}{(\mathbf{p}^2+\mathbf{p}_1^2-\mathbf{p}_2^2-\mathbf{p}_3^2)/2m}\right]. \qquad (11)$$

Thus the problem of calculating the energy of the ground state and the effective mass of the excitations reduces to that of computing the integrals in (9) and (11). The integration is quite involved, due to the high multiplicity of the integrals and the inconvenience of the regions of integration (the computation of the integral in (9) will be carried out in the Appendix). It is possible, instead, to use a simpler method based on the relations obtained by Landau. If we introduce the function

$$f_{ik} = \delta^2 E/\delta N_i \delta N_k, \qquad (12)$$

which depends upon the momenta and spins of the i-th and k-th particles, then, in accordance with Ref. 3,

$$\frac{1}{m} = \frac{1}{m^*} + \frac{p_0}{2(2\pi\hbar)^3}\mathrm{Sp}_{\sigma,\sigma'}\int f(\mathbf{p},\sigma;\mathbf{p}'\sigma')\cos\theta\, d\Omega; \qquad (13)$$

$$c^2 = \frac{p_0^2}{3m^2} + \frac{1}{6m}\left(\frac{p_0^2}{2\pi\hbar}\right)^3 \mathrm{Sp}_{\sigma,\sigma'}\int f(\mathbf{p},\sigma;\mathbf{p}',\sigma')(1-\cos\theta)\, d\Omega, \qquad (14)$$

where c is the velocity of sound, m^* is the effective mass, the two vectors $\mathbf{p}$ and $\mathbf{p}'$ in f are taken equal in absolute magnitude to p_0, and θ is the angle between these vectors. From Eq. (13) we find at once the effective mass, and from Eq. (14), after appropriate integration, the energy of the ground state.

The problem thus reduces to that of determining the quantity f. Taking the variation of Eqs. (3) and (8) with respect to N_i and then to N_k, we find

$$f = 2UQ_{\sigma,\sigma'} - \frac{8U^2}{(2\pi\hbar)^3}\int\limits_{|\mathbf{p}_1|<p_0} d\mathbf{p}_1 \int d\mathbf{p}_2 \left[Q_{\sigma,\sigma'}\frac{\delta(\mathbf{p}+\mathbf{p}'-\mathbf{p}_1-\mathbf{p}_2)}{(\mathbf{p}^2+\mathbf{p}'^2-\mathbf{p}_1^2-\mathbf{p}_2^2)/2m} + \frac{1}{4}\frac{\delta(\mathbf{p}+\mathbf{p}_1-\mathbf{p}'-\mathbf{p}_2)}{(\mathbf{p}^2+\mathbf{p}_1^2-\mathbf{p}'^2-\mathbf{p}_2^2)/2m} + \frac{1}{4}\frac{\delta(\mathbf{p}'+\mathbf{p}_1-\mathbf{p}-\mathbf{p}_2)}{(\mathbf{p}'^2+\mathbf{p}_1^2-\mathbf{p}^2-\mathbf{p}_2^2)/2m}\right]. \qquad (15)$$

In this calculation we shall at once set $|\mathbf{p}| = |\mathbf{p}'| = p_0$. Integration over the second term in f is considerably simpler than the integrations in (9) and (11). We find as a result

$$f = \frac{2\pi a\hbar^2}{m}\left[1 + 2\left(\frac{3}{\pi}\right)^{1/3} aN^{1/3}\left(2 + \frac{\cos\theta}{2\sin(\theta/2)}\ln\frac{1+\sin(\theta/2)}{1-\sin(\theta/2)}\right)\right]$$
$$- \frac{8\pi a\hbar^2}{m}(\sigma_1\sigma_2)\left[1 + 2\left(\frac{3}{\pi}\right)^{1/3} aN^{1/3}\left(1 - \frac{\sin(\theta/2)}{2}\ln\frac{1+\sin(\theta/2)}{1-\sin(\theta/2)}\right)\right]. \qquad (16)$$

A peculiarity of this expression deserves some attention. For angles θ near π the function f has a logarithmic singularity

$$f(\theta) \sim \left(\frac{1}{4} - \sigma_1\sigma_2\right)\ln\frac{1}{\pi-\theta}. \qquad (17)$$

It is clear that the approximation we have used is not applicable, strictly speaking, in this case. Analysis of the subsequent approximations shows that f does not go to infinity at the point $\theta = \pi$, but remains finite; i.e., this singularity does not actually occur.†

---

†Correction added in proof (September 15, 1957). The singularity in the function $f(\theta)$ at $\theta = \pi$ reflects the singularity in the scattering amplitude for excitations colliding at the angle π. The corresponding expression, obtained by summing the principal terms in the perturbation theory, is proportional to

$$\left[1 + 2\left(\frac{3}{\pi}\right)^{1/3} aN^{1/3}\ln\frac{p_0^2}{\varepsilon}\right]^{-1},$$

where $\epsilon = p^2 + p'^2 - p_0^2$.

For the case in which a is positive, this expression goes to 0 at $p^2 = p'^2 = p_0^2$.

If, however, $a < 0$ (this is possible for a Fermi system), the scattering amplitude has a pole near the Fermi boundary. This corresponds to the possibility of formation of associated pairs from excitations having opposite momenta, which has recently been noted by Cooper,[5] and is evidently the principal reason for the occurrence of superconductivity in metals.[6]

Thus the expression found for f is invalid at angles near to π. In view, however, of the fact that the singularity is a logarithmic one, it is manifested only in the immediate vicinity of the singular point; and, since into the equations in which we are interested there enter only integrals of f with regular functions, the logarithmic singularity in the function f is not important.

Substituting Eq. (16) into (13), we find for the value of the effective mass

$$m/m^* = 1 - (8/15)(3/\pi)^{2/3}(7\ln 2 - 1)a^2N^{2/3}. \qquad (18)$$

We note that if the value $m^* = 1.43$ m, corresponding to liquid He^3 (cf. Ref. 4), is substituted here, the resulting value for a is found to 1.6×10^{-8} cm, i.e., it is of the same order as the gas-kinetic diameter of the helium atom. Such a comparison has, of course, no strict significance. The model under consideration cannot describe liquid He^3. This is already evident from the fact that the quantity $(m^* - m)/m^*$, which should be a second-order quantity with respect to a in accordance with theory, is equal to $1/3$ for the case of He^3.

Setting the formula for f into the expression for the velocity of sound, we obtain

$$c^2 = \frac{\pi^{4/3}}{3^{1/3}} N^{2/3} \frac{\hbar^2}{m^2} + 2\frac{\pi a\hbar^2}{m^2} N\left[1 + \frac{4}{15}\left(\frac{3}{\pi}\right)^{1/3} aN^{1/3}(11 - 2\ln 2)\right]. \qquad (19)$$

From the value found for c^2 it is not difficult to obtain the energy of the ground state for a Fermi liquid. For this we use the relation[3]

$$c^2 = (N/m)(\partial\mu/\partial N). \qquad (20)$$

From this we obtain

$$E = \int \mu dN = E^{(0)} + \frac{\pi a\hbar^2}{m} N^2\left[1 + \frac{6}{35}\left(\frac{3}{\pi}\right)^{1/3} aN^{1/3}(11 - 2\ln 2)\right]. \qquad (21)$$

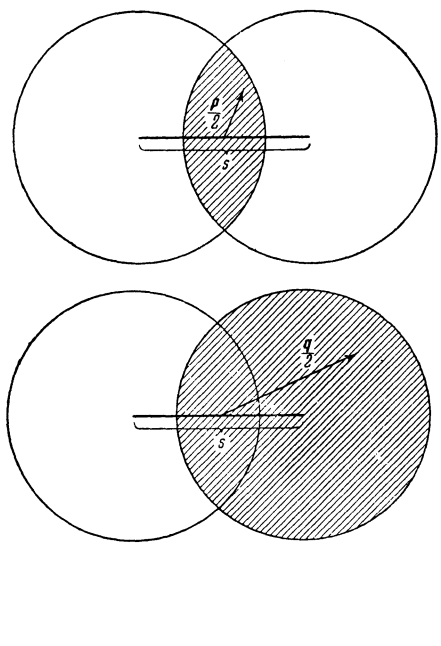
Equation (21) agrees with the result of Lee and Yang.[2] The same result may be obtained by direct integration in Eq. (9); this is done in the Appendix.

In conclusion, the authors express their deep gratitude to Academician L. D. Landau for his valued advice and criticism of the results of this work, to V. Galitskii for his helpful discussions, and, finally, to L. Pogodina for her aid in the preparation of the manuscript for publication.

APPENDIX

For the computation of the integral in Eq. (9) it is convenient to introduce the new variables

$$\mathbf{p} = \mathbf{p}_1 - \mathbf{p}_2, \quad \mathbf{q} = \mathbf{p}_3 - \mathbf{p}_4, \quad \mathbf{s} = \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4.$$



In terms of these variables, $E^{(2)}$ takes on the form

$$E^{(2)} = -\frac{mU^2}{4(2\pi\hbar)^9}\int d\mathbf{s}\int d\mathbf{p}\int d\mathbf{q}\frac{1}{\mathbf{p}^2 - \mathbf{q}^2},$$

where the region of integration for the vector **s** is $0 < |\mathbf{s}| < 2p_0$ and the range of values for **p** and **q** is shown in the figure. If we introduce the variables $x_q = \cos(\mathbf{q}, \mathbf{s})$ and $x_p = \cos(\mathbf{p}, \mathbf{s})$, $E^{(2)}$ becomes

$$E^{(2)} = -\frac{8mU^2\pi^3}{(2\pi\hbar)^9}\int_0^{2p_0} s^2\,ds\int_0^1 dx_p\int_0^{z(x_p)} p^2\,dp\int_{-1}^1 dx_q\int_0^{z(x_z)} q^2\,dq\,\frac{1}{p^2 - q^2},$$

where $z(x)$ satisfies the relation

$$z^2 + 2zsx + s^2 = 4p_0^2.$$

From this, by means of a series of transformations and partial integrations over dx_p and dx_q we find

$$E^{(2)} = \frac{2m\,U^2 p_0^7}{\pi^6 \hbar^9} \int_0^1 s^2\,ds \left[\int_0^{1+s} p^2\,dp \int_0^{1-s} q^2\,dq + \frac{1}{4s^2} \int_{\sqrt{1-s^2}}^{1+s} p\,dp\,(1-p^2-s^2) \int_{\sqrt{1-s^2}}^{1-s} q\,dq\,(1-q^2-s^2) \right] \frac{1}{p^2-q^2}.$$

Integrating further by parts over s and then carrying out the remaining integration, we obtain

$$E^{(2)} = (^6/_{35})\,(3/\pi)^{1/3}\,(11-2\ln 2)\,aN^{1/3}E^{(1)}.$$

Here we have expressed U in accordance with Eq. (2) and set $p_0 = \hbar(3\pi^2 N)^{1/3}$. The result thus obtained is identical with the second-order term in Eq. (21).

[1] K. Huang and C. N. Yang, Phys. Rev. 105, 767 (1957).

[2] T. D. Lee and C. N. Yang, Phys. Rev. 105, 1119 (1957).

[3] L. D. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) 30, 1058 (1956), Soviet Phys. JETP 3, 920 (1957).

[4] I. M. Khalatnikov and A. A. Abrikosov, J. Exptl. Theoret. Phys. (U.S.S.R.) 32, 915 (1957), Soviet Phys. JETP 5, 745 (1957).

[5] L. N. Cooper, Phys. Rev. 104, 1189 (1956).

[6] Bardeen, Cooper, and Schrieffer, Phys. Rev. 106, 162 (1957); N. N. Bogoliubov, J. Exptl. Theoret. Phys. (U.S.S.R.) 34, 58 (1958), Soviet Phys. JETP 7 (in press).

Translated by S. D. Elliott
239

SOVIET PHYSICS JETP VOLUME 6 (33), NUMBER 5 MAY, 1958

NUCLEAR REACTIONS IN Li^7 AND C^{12} INDUCED BY N^{14} IONS

D. G. ALKHAZOV, Iu. P. GANGRSKII, and I. Kh. LEMBERG

Leningrad Physico-Technical Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor June 3, 1957

J. Exptl. Theoret. Phys. (U.S.S.R.) 33, 1160-1162 (November, 1957)

In an investigation of the reaction products induced by the bombardment of Li^7 by 15.6-Mev nitrogen ions, activities associated with F^{18}, Ne^{19}, N^{16}, and O^{15} have been found; similarly, an activity associated with Al^{25} has been found in the bombardment of carbon. The production cross sections for the above-mentioned products have been determined. On the basis of an examination of the F^{18}-production cross sections in light elements bombarded by nitrogen ions and the α-particle binding energy in these same nuclei, it is proposed that the F^{18} is formed by capture of an α-particle from the nucleus by the incoming N^{14} nucleus.

NUCLEAR reactions induced in light elements by N^{14} ions have been studied by a number of authors.[1-6] However, in all this work only nuclides with half-lives T greater than 1 min were investigated. The products resulting from the bombardment of Li^7 by N^{14} have not been studied at all.

In the present work we have measured yields for nuclides with $T > 1$ sec produced by bombardment of Li^7 and C^{12} by N^{14} ions. The experiments were carried out with a beam of triply-charged, 15.6-Mev N^{14} ions from a cyclotron; the beam was focussed by two magnetic-quadrupole lenses. The target was placed at the end of a Faraday cylinder. The electric charge deposited by the beam was measured by electronic integration. In these experiments the ion-beam intensity was $4-7 \times 10^{10}$ ions/sec.

The lithium bombardment was carried out with a target consisting of a LiCl layer 70μ thick precipitated from an aqueous solution enriched in Li^7 (the Li^7 content was approximately 99 percent). The