OSCILLATIONS OF A FERMI-LIQUID IN A MAGNETIC FIELD

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A study is made of the spin oscillations of a paramagnetic Fermi-liquid (He^3) placed in a constant magnetic field at low temperatures, where collisions can be ignored.

LANDAU, using a phenomenological theory,[1] has carried out an investigation of the oscillations of a Fermi-liquid[2] by generalizing certain results obtained earlier for a gas with weak interactions[3,4] (also see Ref. 5). In particular, he has shown that it is apparently possible for neutral sound waves to be propagated in actual liquid He^3 at low temperatures, but not for spin waves.

The present paper is devoted to a study of spin oscillations in a Fermi-liquid placed in a magnetic field. In formulating the relevant kinetic equation, Landau[1] ignored the presence of a magnetic field; consequently in the first section of this report we derive a kinetic equation which takes it into account.[6] In Sec. 2 we investigate spin oscillations for the isotropic case and obtain the characteristic frequencies of these oscillations. These frequencies appear to be the limiting values of the spin-wave frequencies when the wavelength goes to infinity. Section 3 is devoted to a study of spin waves. Here it is shown, in contrast with the results of Landau quoted above, that it is possible for spin waves to be propagated in actual liquid He^3 in the presence of a constant magnetic field.

1. THE KINETIC EQUATIONS

To obtain the kinetic equations which describe the quasi-particles of a Fermi-liquid in the presence of a magnetic field, we start from the equation for the density matrix

$$\hbar \frac{\partial}{\partial t} \rho_{mn}(\mathbf{r}', \mathbf{r}; t) = H_{mn'}\rho_{n'n}(\mathbf{r}', \mathbf{r}; t) - H'^{*}_{n'n}\rho_{mn'}(\mathbf{r}', \mathbf{r}; t), \tag{1.1}$$

where ρ is the density matrix, whose subscripts indicate the spin properties of the quasi-particle (we shall actually concern ourselves later on with the case of spin $\frac{1}{2}$). H is the Hamiltonian operator of a quasi-particle. The possibility that such a Hamiltonian exists is tied in with the possibility of being able to speak in general of quasi-particles. Transforming to a mixed representation for the density matrix,

$$n_{mn}(\mathbf{p}, \mathbf{r}; t) = (2\pi)^{-3} \int d\boldsymbol{\tau} e^{-i\boldsymbol{\tau}\mathbf{p}} \rho_{mn}\left(\mathbf{r} - \frac{\hbar\boldsymbol{\tau}}{2}, \mathbf{r} + \frac{\hbar\boldsymbol{\tau}}{2}, t\right), \tag{1.2}$$

we can rewrite equation (1.1) as follows:

$$\begin{aligned}\frac{\partial}{\partial t} n_{mn}(\mathbf{p}, \mathbf{r}, t) = \frac{1}{(2\pi)^6} \frac{i}{\hbar} \int d\boldsymbol{\tau}\, d\mathbf{k}\, d\boldsymbol{\eta}\, d\mathbf{q} \exp\{i[\boldsymbol{\tau}(\boldsymbol{\eta} - \mathbf{p}) + \mathbf{k}(\mathbf{q} - \mathbf{r})]\}, \\ \times \left[\varepsilon_{mn'}\left(\boldsymbol{\eta} + \frac{\hbar\mathbf{k}}{2}, \mathbf{q} - \frac{\hbar\boldsymbol{\tau}}{2}\right) n_{n'n}(\boldsymbol{\eta}, \mathbf{q}, t) - \varepsilon_{n'n}\left(\boldsymbol{\eta} - \frac{\hbar\mathbf{k}}{2}, \mathbf{q} + \frac{\hbar\boldsymbol{\tau}}{2}\right) n_{mn'}(\boldsymbol{\eta}, \mathbf{q}, t)\right].\end{aligned} \tag{1.3}$$

Here $\epsilon_{mn}(\mathbf{p}, \mathbf{r})$ has the same functional dependence on the c-numbers $\mathbf{p}$ and $\mathbf{r}$ as the Hamiltonian H has on the momentum and coordinate operators.

The equation of the quasi-classical approximation can be obtained by expanding (1.3) in powers of $\hbar$. However, Eq. (1.3) does not take account of effects described by the collision integral in the usual Boltzmann equation. In Eq. (1.1) it is assumed that there is a Hamiltonian operator for each individual quasi-particle, different from the Hamiltonian of a free particle in that there is a certain self-consistent field of the other particles. The Hamiltonian does not depend explicity on the coordinates of these other particles, however, and in this sense does not take into account the interaction between them. It is possible in principle to take such an interaction into consideration. On the other hand, it is clear that the collisions which have been ignored in Eq. (1.3) are the result of such an interaction. We consequently add a collision integral J_{mn} to the terms obtained from (1.3) in the equation of the quasi-classical approximation:

$$\frac{\partial n}{\partial t} - \frac{i}{\hbar}[\varepsilon, n] + \frac{1}{2}\left(\frac{\partial \varepsilon}{\partial \mathbf{p}}\frac{\partial n}{\partial \mathbf{r}} + \frac{\partial n}{\partial \mathbf{r}}\frac{\partial \varepsilon}{\partial \mathbf{p}}\right) - \frac{1}{2}\left(\frac{\partial \varepsilon}{\partial \mathbf{r}}\frac{\partial n}{\partial \mathbf{p}} + \frac{\partial n}{\partial \mathbf{p}}\frac{\partial \varepsilon}{\partial \mathbf{r}}\right) = \hat{J}, \tag{1.4}$$

where $[\epsilon, n]$ is the commutator of ϵ and n, which are matrices with respect to the spin variable.

It is convenient to get rid of the matrices and to introduce instead of n a distribution function f of the particles in phase space and a vector function σ of the spin density in phase space, determined by the relations:

$$f(\mathbf{p}, \mathbf{r}, t) = \mathrm{Sp}_\sigma n; \quad \sigma(\mathbf{p}, \mathbf{r}, t) = \mathrm{Sp}_\sigma \hat{\sigma} n. \tag{1.5}$$

Here the $\hat{\sigma}$ are the Pauli spin matrices.

Then, noting that $\epsilon(\mathbf{p}, \mathbf{r})$ can be written in the form

$$\varepsilon_{mn} = \delta_{mn}\varepsilon_1 + \hat{\sigma}_{mn}\varepsilon_2, \tag{1.6}$$

we obtain the following equations for the functions f and σ:

$$\frac{\partial f}{\partial t} + \frac{\partial \varepsilon_1}{\partial \mathbf{p}}\frac{\partial f}{\partial \mathbf{r}} - \frac{\partial \varepsilon_1}{\partial \mathbf{r}}\frac{\partial f}{\partial \mathbf{p}} + \frac{\partial \varepsilon_2}{\partial p_j}\frac{\partial \sigma}{\partial r_j} - \frac{\partial \varepsilon_2}{\partial r_j}\frac{\partial \sigma}{\partial p_j} = J; \tag{1.7}$$

$$\frac{\partial \boldsymbol{\sigma}}{\partial t} + \left(\frac{\partial \varepsilon_1}{\partial \mathbf{p}}\frac{\partial}{\partial \mathbf{r}}\right)\boldsymbol{\sigma} - \left(\frac{\partial \varepsilon_1}{\partial \mathbf{r}}\frac{\partial}{\partial \mathbf{p}}\right)\boldsymbol{\sigma} + \frac{2}{\hbar}[\boldsymbol{\varepsilon}_2\boldsymbol{\sigma}] + \left(\frac{\partial f}{\partial \mathbf{r}}\frac{\partial}{\partial \mathbf{p}}\right)\boldsymbol{\varepsilon}_2 - \left(\frac{\partial f}{\partial \mathbf{p}}\frac{\partial}{\partial \mathbf{r}}\right)\boldsymbol{\varepsilon}_2 = \mathbf{J}. \tag{1.8}$$

Here the quantities J and $\mathbf{J}$ denote $\mathrm{Sp}_\sigma\hat{J}$ and $\mathrm{Sp}_\sigma\hat{\boldsymbol{\sigma}}\hat{J}$, respectively.

The functions ϵ_1 and $\boldsymbol{\epsilon}_2$ depend on the space coordinates because of the fact that they are functionals of $n(\mathbf{p}, \mathbf{r})$. Actually, if there is no magnetic field, then according to Landau

$$\delta\varepsilon_{mn}(\mathbf{p}, \mathbf{r}) = \mathrm{Sp}_{\sigma'}\int \hat{f}_{mn'}(\mathbf{p}, \mathbf{p}')\,\delta n_{n'm}(\mathbf{p}', \mathbf{r})\,d\mathbf{p}'. \tag{1.9}$$

If we ignore the small spin-orbit coupling in the expression for $\epsilon(\mathbf{p}, \mathbf{r})$, then in the presence of a magnetic field

$$\delta\varepsilon(\mathbf{p}, \mathbf{r}) = -\frac{1}{2}\beta(\hat{\sigma}\mathbf{H}) + \mathrm{Sp}_{\sigma'}\int\{\varphi(\mathbf{p}, \mathbf{p}') + (\sigma\sigma')\psi(\mathbf{p}, \mathbf{p}')\}\,\delta n(\mathbf{p}', \mathbf{r})\,d\mathbf{p}', \tag{1.10}$$

which, according to (1.6), permits us to write down the following expressions:

$$\delta\varepsilon_1(\mathbf{p}, \mathbf{r}) = \int \varphi(\mathbf{p}, \mathbf{p}')\,\delta f(\mathbf{p}', \mathbf{r})\,d\mathbf{p}', \tag{1.11}$$

$$\delta\boldsymbol{\varepsilon}_2(\mathbf{p}, \mathbf{r}) = -\frac{1}{2}\beta\mathbf{H} + \mathrm{Sp}_{\sigma'}\int \psi(\mathbf{p}, \mathbf{p}')\,\delta\sigma(\mathbf{p}', \mathbf{r})\,d\mathbf{p}'. \tag{1.12}$$

Furthermore, because of the fact that

$$\frac{\partial \varepsilon}{\partial \mathbf{p}} = \frac{\mathbf{p}}{m} + \mathrm{Sp}_{\sigma'}\int \hat{f}(\mathbf{p}, \mathbf{p}')\,\frac{\partial n(\mathbf{p}', \mathbf{r})}{\partial \mathbf{p}'}\,d\mathbf{p}', \tag{1.13}$$

we have

$$\frac{\partial \varepsilon_1}{\partial \mathbf{p}} = \frac{\mathbf{p}}{m} + \int \varphi(\mathbf{p}, \mathbf{p}')\,\frac{\partial f(\mathbf{p}', \mathbf{r})}{\partial \mathbf{p}'}\,d\mathbf{p}', \tag{1.14}$$

$$\frac{\partial \varepsilon_2}{\partial p} = \int \psi(\mathbf{p}, \mathbf{p}')\,\frac{\partial \sigma(\mathbf{p}', \mathbf{r})}{\partial p_j'}\,d\mathbf{p}'. \tag{1.15}$$

In Eq. (1.8) it is necessary to know not only the derivatives of the functions ϵ_1 and ϵ_2 but the entire function ϵ_2. Regarding the magnetic field as negligibly small, or , to be more precise, considering that

$$\beta(\sigma\mathbf{H}) \ll \varepsilon(\mathbf{p}, \mathbf{r}), \tag{1.16}$$

we can also consider the quantity $\boldsymbol{\sigma}(\mathbf{p}, \mathbf{r})$ to be small, and therefore drop the δ operation in Eq. (1.12). At the same time we ignore quantities of second order in expansions in powers of the magnetic field (H/ϵ).

2. SPIN OSCILLATIONS FOR AN ISOTROPIC DISTRIBUTION

When there is no spatial anisotropy, then, according to the results of the preceding section, it is possible to write (1.8) in the following form:

$$\frac{\partial \sigma}{\partial t} + \frac{2}{\hbar}\int d\mathbf{p}'\, \psi(\mathbf{p}, \mathbf{p}')\,[\sigma(\mathbf{p}')\,\sigma(\mathbf{p})] + \frac{\beta}{\hbar}[\sigma \times \mathbf{H}] = \mathbf{J}. \tag{2.1}$$

In particular, we can obtain from (2.1) an equation describing the time rate of change of the magnetization density

$$\mathbf{M} = (\beta/2)\int \sigma\, d\mathbf{p}. \tag{2.2}$$

Integrating (2.1) over $d\mathbf{p}$ and noting that $\psi(\mathbf{p}, \mathbf{p}')$ is a symmetric function of $\mathbf{p}$ and $\mathbf{p}'$, we obtain

$$\partial \mathbf{M}/\partial t + (\beta/\hbar)[\mathbf{M}\times\mathbf{H}] = (\beta/2)\int \mathbf{J}\, d\mathbf{p}, \tag{2.3}$$

which for the appropriate approximation of the collision integral agrees with the ordinary Bloch equation. Thus, given a certain constant magnetic field $\mathbf{H}_0$, we find that if we disregard collisions, which, generally speaking, lead to attenuation, the principal frequency of oscillations of the magnetization turns out to be

$$\Omega_0 = \beta H_0/\hbar. \tag{2.4}$$

However, in contrast with the corresponding results for a gas, the frequency Ω_0 here will not be the only frequency characterizing spin oscillations.

We consider a state, slightly out of equilibrium, in which there is a constant uniform field $\mathbf{H}_0$. Then

$$\sigma = \sigma_0(p) + \sigma_1(\mathbf{p}, t); \quad \mathbf{H} = \mathbf{H}_0 + \mathbf{H}_1(t); \quad \sigma_0 \parallel \mathbf{H}_0; \quad H_0 \gg H_1; \quad \sigma_0 \gg \sigma_1. \tag{2.5}$$

Resolving σ_1 into components parallel $(\sigma_\parallel)$ and perpendicular $(\sigma_\perp = \sigma_x\mathbf{i} + \sigma_y\mathbf{j})$ to $\mathbf{H}_0$, we obtain in the linear approximation the following equations for $\sigma_\parallel$ and for $\sigma^{(\pm)} = \sigma_x \pm i\sigma_y$:

$$\partial \sigma_\parallel/\partial t = J_\parallel, \tag{2.6}$$

$$\frac{\partial}{\partial t}\sigma^{(\pm)} \mp i\Omega_0\sigma^{(\pm)} \mp i\,\frac{2}{\hbar}\int d\mathbf{p}'\,\{\sigma_0(p)\,\sigma^{(\pm)}(\mathbf{p}') - \sigma_0(p')\,\sigma^{(\pm)}(\mathbf{p})\} \pm i\,\frac{\beta}{\hbar}\,\sigma_0(p)(H_{1x} \pm iH_{1y}) = J^{(\pm)}, \tag{2.7}$$

from which it is clear that only the component of σ lying in a plane perpendicular to $\mathbf{H}_0$ can oscillate.

Noting that

$$\sigma_0(p) = {}^1/_2\,\beta H_0\,\partial f_0/\partial\varepsilon, \tag{2.8}$$

where f_0 is the Fermi distribution function, so that σ_0 has the form of a δ-function, we can express the solution of (2.7) in the form

$$\sigma^{(\pm)}(\mathbf{p}) = (\partial f_0/\partial\varepsilon)\,\zeta^{(\pm)}(\theta, \varphi, t). \tag{2.9}$$

Then, introducing the notation

$$\Psi(\cos\chi) = 4\pi\,\frac{2}{(2\pi\hbar)^3}\,\frac{p_0^2}{v_0}\,\psi(\mathbf{p}_0, \mathbf{p}_0'), \tag{2.10}$$

where p_0 and v_0 are the momentum and velocity of the particles on the Fermi surface and χ is the angle between the vectors $\mathbf{p}_0$ and $\mathbf{p}_0'$, we obtain from (2.7) the following equation for $\zeta^{(\pm)}$:

$$\frac{\partial\zeta^{(\pm)}}{\partial t} \mp i\Omega_0\zeta^{(\pm)} \pm i\Omega_0\int\frac{do'}{4\pi}\,\Psi(\cos\chi)\,\{\zeta^{(\pm)}(\theta', \varphi') - \zeta^{(\pm)}(\theta, \varphi)\} \pm i\Omega_0 H_\pm = \bar{J}_\pm\,. \tag{2.11}$$

The solution of (2.11) can be written as an expansion in spherical harmonics:

$$\zeta^{(\pm)}(\theta, \varphi, t) = \sum_{n,\, m} C_{n,\, m}^{(\pm)}(t)\,\frac{e^{im\varphi}}{\sqrt{2\pi}}\,P_n^m(\cos\theta)\left(\frac{2n+1}{2}\,\frac{(n-m)!}{(n+m)!}\right)^{1/2}. \tag{2.12}$$

If we now represent the function $\Psi(\cos\chi)$ as a series of Legendre polynomials

$$\Psi(\cos\chi) = \sum_{n=0}^{\infty}\alpha_n P_n(\cos\chi), \tag{2.13}$$

we obtain from (2.11) the following equations:

$$\text{for } n=0: \quad \frac{\partial}{\partial t} C_{0,0}^{(\pm)} \mp i\Omega_0 C_{0,0}^{(\pm)} \pm i\Omega_0 H_\pm \sqrt{2\pi} = J_{0,0}^{(\pm)}, \tag{2.14}$$

$$\text{for } n\neq 0: \quad \frac{\partial}{\partial t} C_{n,m}^{(\pm)} \mp i\Omega_0 C_{n,m}^{(\pm)}\left(1+\alpha_0-\frac{2\alpha_n}{2n+1}\right) = J_{n,m}^{(\pm)}. \tag{2.15}$$

Here $J_{n,m}^{\pm}$ denote the appropriate coefficients of the expansion in spherical harmonics.

It is easy to see that (2.14) corresponds to the Bloch equation and agrees with the linearized equation (2.3). In the absence of an external field and with no allowance made for collisions, this equation describes spin oscillations with frequencies $\pm\Omega_0$. If collisions are ignored in Eq. (2.15), then

$$C_{n,m}^{\pm} \sim \exp(\pm i\omega_n t), \text{ where } \omega_n = \Omega_0\left(1+\alpha_0-\frac{2\alpha_n}{2n+1}\right). \tag{2.16}$$

The frequency of all such oscillations coincides with Ω_0 for an ideal gas. One can say that the presence of a self-consistent field in which the quasi-particles move about removes such a degeneracy.

3. SPIN WAVES

In Fermi-liquids, because of the presence of exchange interactions, the propagation of spin waves is possible in principle.[2,4] However, a detailed analysis of this possibility carried out by Landau for He^3, has shown that such waves can apparently not be propagated in real helium.[2] This result was obtained under the assumption that there was no magnetic field. Below we shall study the problem of the propagation of spin waves in a Fermi-liquid placed in a magnetic field.

We are interested in states which depart slightly from equilibrium, and for which the magnetic field is constant and uniform. In the linear approximation Eqs. (1.7) and (1.5) can be written in the following way (neglecting collisions):

$$\frac{\partial f_1}{\partial t} + \left(\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right) f_1 - \frac{\partial f_0}{\partial \mathbf{p}}\int d\mathbf{p}'\, \varphi(\mathbf{p},\mathbf{p}')\frac{\partial f_1(\mathbf{p}',\mathbf{r})}{\partial \mathbf{r}} + v_{i,j}\frac{\partial \sigma_{1i}}{\partial r_j} - \frac{\partial \sigma_{0i}}{\partial p_j}\frac{\partial}{\partial r_j}\left\{\int d\mathbf{p}'\,\psi(\mathbf{p},\mathbf{p}')\,\sigma_{1i}(\mathbf{p}',\mathbf{r}) - \frac{1}{2}\beta H_{1i}(\mathbf{r})\right\} = 0; \tag{3.1}$$

$$\begin{aligned}\frac{\partial \sigma_{1i}}{\partial t} + \left(\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)\sigma_{i1} - \int d\mathbf{p}'\,\varphi(\mathbf{p},\mathbf{p}')\left(\frac{\partial f_1(\mathbf{p}',\mathbf{r})}{\partial \mathbf{r}}\frac{\partial}{\partial \mathbf{p}}\right)\sigma_{0i} + v_{i,j}\frac{\partial f_1}{\partial r_j} + \frac{2}{\hbar}\int d\mathbf{p}'\,\psi(\mathbf{p},\mathbf{p}')\{[\boldsymbol{\sigma}_0(p')\times\boldsymbol{\sigma}_1(\mathbf{p},\mathbf{r})]_i + [\boldsymbol{\sigma}_1(\mathbf{p}',\mathbf{r})\times\boldsymbol{\sigma}_0(p)]_i\} \\ - \left(\frac{\partial f_0}{\partial \mathbf{p}}\frac{\partial}{\partial \mathbf{r}}\right)\left\{\int d\mathbf{p}'\,\psi(\mathbf{p},\mathbf{p}')\,\sigma_{1i}(\mathbf{p}',\mathbf{r}) - \frac{1}{2}\beta H_{1i}(\mathbf{r})\right\} - \frac{\beta}{\hbar}[\mathbf{H}_0\times\boldsymbol{\sigma}_1]_i - \frac{\beta}{\hbar}[\mathbf{H}_1\times\boldsymbol{\sigma}_0]_i = 0,\end{aligned} \tag{3.2}$$

where

$$\mathbf{v} = \frac{\partial \varepsilon_{10}}{\partial \mathbf{p}} = \frac{\mathbf{p}}{m} + \int \varphi(\mathbf{p},\mathbf{p}')\frac{\partial f_0(\mathbf{p}')}{\partial \mathbf{p}'}d\mathbf{p}', \tag{3.3}$$

$$v_{i,j} = \int \psi(\mathbf{p},\mathbf{p}')\frac{\partial \sigma_{0i}(\mathbf{p}')}{\partial p_j'}d\mathbf{p}'. \tag{3.4}$$

Noting that $\boldsymbol{\sigma}_0$ is parallel to $\mathbf{H}_0$, we can represent $\boldsymbol{\sigma}_1$ as the sum of components parallel and perpendicular to the field $\mathbf{H}_0$, as we did in Sec. 2. Introducing the notation

$$\sigma^{(\pm)} = \sigma_x \pm i\sigma_y = (\partial f_0/\partial\varepsilon)\,\zeta^{(\pm)}(\mathbf{r};\theta,\varphi,t),$$

we have the following equation for the perpendicular part:

$$\begin{aligned}\frac{\partial}{\partial t}\zeta^{(\pm)} + \left(\mathbf{v}_0\frac{\partial}{\partial \mathbf{r}}\right)\zeta \mp i\Omega_0\zeta^{(\pm)} \pm i\Omega_0 H_\pm \mp i\Omega_0\int\frac{do'}{4\pi}\Psi(\cos\chi)\{\zeta^{(\pm)}(\mathbf{r},\theta',\varphi') - \zeta^{(\pm)}(\mathbf{r},\theta,\varphi)\} \\ = -\left(\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)\left\{\frac{\beta}{2}H_\pm + \int\frac{do'}{4\pi}\Psi(\cos\chi)\,\zeta^{(\pm)}(\mathbf{r},\theta',\varphi')\right\},\end{aligned} \tag{3.5}$$

where $\mathbf{v}_0$ is the velocity at the Fermi surface.

It follows from Eq. (3.5) that the transverse spin waves are not connected with acoustic oscillations. The longitudinal spin waves are, however, related to sound waves; but, as we can easily convince ourselves, a constant magnetic field leads only to a change in the sound velocity, and thus in the velocity of propagation of the longitudinal spin waves, which in order of magnitude is $\sim H_0^2$. Therefore in what follows we limit our study to transverse spin waves, corresponding to oscillations of $\boldsymbol{\sigma}$ in a plane perpendicular to $\mathbf{H}_0$ and described by Eq. (3.5).

If we suppose that there is no varying external field, we can set the solution of (3.5) proportional to $\exp(-i\omega t + i\mathbf{k}\cdot\mathbf{r})$. Equation (3.5) can then be rewritten

$$(kv_0\cos\theta-\omega)\,\zeta^{(\pm)}(\theta,\varphi)\mp\Omega_0\Big(1-\int\frac{do'}{4\pi}\Psi(\cos\chi)\Big)\zeta^{(\pm)}(\theta,\varphi)+(kv_0\cos\theta\mp\Omega_0)\int\frac{do'}{4\pi}\Psi(\cos\chi)\,\zeta^{(\pm)}(\theta',\varphi')=0. \quad (3.6)$$

We now investigate the solution of (3.6) under the assumption that $\Psi(\cos\chi)$ is a constant. If such an assumption is made for actual liquid He^3, the constant appears to be negative. For our case we obtain from (3.6), with Ψ constant, the following relation between the frequency ω and the wave vector k of the spin waves:

$$1+\frac{1}{2}\Psi\int_0^\pi d\theta\sin\theta\,\frac{kv_0\cos\theta\mp\Omega_0}{kv_0\cos\theta\mp\Omega_0(1-\Psi)-\omega}=0. \quad (3.7)$$

Carrying out the integration, we obtain the following result:

$$1+\frac{\omega\mp\Psi\Omega_0}{2kv_0}\frac{\Psi}{1+\Psi}\ln\frac{\omega\pm\Omega_0(1-\Psi)-kv_0}{\omega\pm\Omega_0(1-\Psi)+kv_0}=0. \quad (3.8)$$

If the constant magnetic field is zero, then (3.8) takes the form[2]

$$1+\frac{\omega}{2kv_0}\frac{\Psi}{1+\Psi}\ln\frac{\omega-kv_0}{\omega+kv_0}=0. \quad (3.9)$$

This last expression admits of real values for the ratio ω/kv_0 only for positive Ψ. Since Ψ is not positive for He^3, there can be no spin waves propagated in the absence of a magnetic field.

If the magnetic field is not zero, but the frequency of the spin waves is much greater than Ω_0, (3.8) agrees approximately with (3.9), so one can say that spin waves cannot be propagated at such high frequencies as well.

Let us investigate this question in some detail. We set

$$\omega=\mp\Omega_0\rho,\quad u_0=v_0/\Omega_0; \quad (3.10)$$

and obtain from (3.8)

$$1=\frac{\rho+\Psi}{2ku_0}\frac{\Psi}{1+\Psi}\ln\frac{\rho-1+\Psi+ku_0}{\rho-1+\Psi-ku_0}. \quad (3.11)$$

In the region of long waves, a solution of (3.11) can be obtained from an expansion in powers of (ku_0), from which we obtain

$$\rho=1+(1+\Psi^{-1})(ku_0)^2,\quad ku_0\ll 1. \quad (3.12)$$

Thus for positive Ψ the absolute value of the frequency increases with increasing k. The same thing occurs for negative Ψ provided that $1 + 1/\Psi > 0$. However if $1 + 1/\Psi < 0$, as might conceivably occur for actual liquid He^3, the absolute value of the frequency decreases with an increase in the wave vector of the spin waves.

It is easy to show that for $\Psi < 0$ there is no real value of ρ, corresponding to unattenuated spin waves, if

$$\rho>1-\Psi-ku_0\quad(ku_0>1). \quad (3.13)$$

Let us explain under what conditions the inequality (3.13) sets the limit for ρ. To do this we set

$$\rho=1-\Psi-ku_0-\Delta,\ \text{where } 1\gg\Delta>0. \quad (3.14)$$

From Eq. (3.11) we obtain

$$\Delta=2ku_0\exp\Big(\frac{1+\Psi}{\Psi}\frac{2ku_0}{1-ku_0}\Big). \quad (3.15)$$

From this it follows that a solution of the form (3.14) is possible if

$$-(1+\Psi^{-1})\,2ku_0/(1-ku_0)\gg 1. \quad (3.16)$$

For the case $1 + 1/\Psi < 0$ this condition is satisfied when

$$u_0^{-1}-k\ll-2k(1+\Psi)/\Psi. \quad (3.17)$$

Thus one can say that as the wave vector approaches the value u_0^{-1} the absolute value of the frequency of the spin waves decreases and

$$\rho = \rho_{\mathrm{lim}} = -\Psi \quad \text{for} \quad k = k_{\mathrm{lim}} = u_0^{-1} = \Omega_0 / v_0. \tag{3.18}$$

Thus for $1 + 1/\Psi < 0$ the absolute value of the frequency of the spin waves decreases with increasing value of the wave vector in the region of small k, in accordance with (3.12), but for values of the wave vector equal to Ω_0/v_0, it attains a minimum value equal to $|\Psi\Omega_0|$.

We note that for the case $1 + 1/\Psi > 0$ the condition (3.16) also leads to (3.18), but only if the wave vector approaches Ω_0/v_0 from the side of large values.

When the function $\Psi(\cos\chi)$ does not reduce to a constant, Eq. (3.6) leads to a whole series of spin waves whose frequencies coincide, in the limit of infinite wave length, with the corresponding frequencies obtained in the preceding section. It is necessary to point out that for spin oscillations corresponding to isotropic distributions, it follows from Eqs. (2.14) and (2.15) that an external magnetic field can excite oscillations of the magnetization only at a frequency Ω_0. Conversely, for spin waves corresponding to arbitrary functions $\Psi(\cos\chi)$, the individual spherical harmonics [see (2.12)] no longer appear to be solutions of the problem; indeed, the solutions of (3.5) appear to be determined by superpositions of the different spherical functions, among them the one corresponding to $n = m = 0$ [see (2.12)]. Upon integration over all angles, only this spherical harmonic gives a non-zero result, and consequently it alone corresponds to a magnetization. It is clear that an external field leads to a change of just the magnetization. Consequently, in the case of spin waves, an external magnetic field in any solution of (3.5) can lead to changes in the zero-order term of the expansion in spherical harmonics. Furthermore, in view of the fact that the relation between all the terms of the expansion is completely determined, the magnetic field leads to a change of the entire solution, or in other words to the excitation of spin waves. It is clear that as the wavelength of the spin waves increases, the amplitude of forced oscillation of all the waves except those going over into oscillations of the magnetization (2.14) must decrease. Conversely, in the region of wavelengths of the order of v_0/Ω_0, the amplitudes of forced oscillation of all the possible spin waves will have the same order of magnitude.

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