ELECTRON LOSSES DUE TO PHASE OSCILLATIONS INDUCED BY RADIATION FLUCTUATIONS IN SYNCHROTRONS

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An effective method is given for calculating electron losses due to phase oscillations induced by radiation. An approximate method is given for accounting for nonlinear effects by means of the appropriate boundary conditions in the linear theory.

1. INTRODUCTION

IT is well known that the quantum nature of radiation causes both phase[1] and betatron oscillations.[2,3] These oscillations lead to electron losses when electrons go out of phase with the accelerating voltage and when they collide with the walls of the synchrotron vacuum chamber. In the operation of a synchrotron it is the electron losses due to phase oscillations which are of most importance.

The mean-square deviation of phase oscillations in a weak-focusing synchrotron has been calculated by Sands.[1] Kolomenskii and the present author[4-6] have generalized these results to strong-focusing synchrotrons, and elsewhere the present author[7] has emphasized the importance of nonlinear effects in the theory of synchrotron oscillations induced by radiation. It has been shown in Ref. 6 how to obtain the formula for the smallest losses. By the use of this formula one can merely assert that "the losses are greater than some given quantity," but not how much greater. Because the smallest losses are quite large, a practical investigation of this phenomenon is necessary.

The present article describes an effective method for obtaining a formula which gives the electron losses to any previously determined accuracy, and gives an evaluation of this accuracy.

In addition, it is shown how to formulate the boundary conditions so that the nonlinear effects, as previously derived,[7] can be accounted for.

2. STOCHASTIC EQUATION AND FORMULATION OF THE PROBLEM

To derive stochastic equations for the phase oscillations induced by radiation fluctuations, we shall start with the obvious relation

$$\delta E = E - E_s = \sum eV_0 (\cos\varphi - \cos\varphi_s) + \int_0^t W_s dt - \sum \varepsilon_i. \tag{1}$$

In this equation ϵ_i is the energy of the radiated photons, eV_0 is the high-frequency field amplitude, φ is the phase with which an electron passes through the accelerating gap, and W_S is the equilibrium power radiated. It is clear that in this problem the excitation of energy in betatron oscillations need not be accounted for.

We define the packing factor α of a synchrotron by the relation

$$\delta \langle R \rangle / \langle R \rangle = \alpha \delta E / E. \tag{2}$$

We then obviously have

$$\delta E / E = (\lambda / k\omega\alpha) \dot{\Psi}, \tag{3}$$

where k is the number of the accelerating harmonic of the high-frequency field, $\Psi = \varphi - \varphi_S$ is the deviation of the particle phase from equilibrium, ω is the angular frequency of rotation of a particle in the synchrotron, $\lambda = 1 + L/2\pi R_0$, and L is the total length of the linear sections of the synchrotron per

revolution. Intensive phase oscillations are excited by radiation fluctuations only in the last stage of acceleration. Therefore we can neglect quantities proportional to the variation of the equilibrium energy and the energy in betatron oscillations in obtaining the stochastic equation. With the aid of (3) and well known formulas for the power radiated, Eq. (1) leads to the nonlinear stochastic equation[7]

$$\ddot{\Psi} + \gamma\dot{\Psi} + f^2[\cos\varphi_s - \cos(\varphi_s + \Psi)] = (k\omega\alpha/\lambda E_s)\,[W - \sum_i \varepsilon_i\delta(t - t_i)], \tag{4}$$

where

$$f^2 = (k\omega^2\alpha/2\pi\lambda)\,eV_0/E_s,\ \gamma = (4-\alpha)(2\omega r_0/3R_0)(E_s/m_0c^2)^3,\ \ r_0 = e^2/m_0c^2,\ \ W = (2ce^2/3R^2)(E/m_0c^2)^4.$$

If the phase deviation Ψ is considered small, Eq. (4) can be linearized and leads to the linear stochastic equation[6]

$$\ddot{\Psi} + \gamma\dot{\Psi} + \Omega^2\Psi = (k\omega\alpha/\lambda E_s)\Big[W - \sum_i \varepsilon_i\delta(t - t_i)\Big],\ \ \Omega^2 = f^2\sin\varphi_s. \tag{5}$$

We have already treated[7] the excitation of phase oscillations according to the nonlinear equation (4). We have shown that a particle in the potential well is "heated" basically the same way according to either the linear or the nonlinear theory. The main differences are due to the depth of the potential well. If the maximum allowable phase deviation from equilibrium is taken from the nonlinear theory, the potential well which will give the same deviation in the linear theory must be much greater than that of the nonlinear theory. This indeed is why the linear theory predicts lower electron losses than does the nonlinear one, as was previously shown.[7] We can therefore account for this fundamentally nonlinear effect by means of the linear theory if we proceed as follows. The potential-well depth is taken from the nonlinear theory, and the allowable limits of deviation in the linear theory are obtained from the requirement that the well depth in both theories be equal. This will account for the nonlinear effects.

Thus in solving the problem we shall proceed from the linear stochastic equation (5) and shall account for nonlinearity only in the boundary conditions.

It is convenient first to change in (5) to a new independent variable ζ defined by

$$\zeta = \int_0^t \Omega dt.$$

Then (5) becomes

$$\Psi'' + \beta\Psi' + \Psi = (k\omega\alpha/\lambda E_s\,\Omega)\,[W_\zeta - \sum \varepsilon_i\delta(\zeta - \zeta_i)],\quad \beta = \gamma/\Omega + \Omega'/\Omega, \tag{6}$$

where the primes denote differentiation with respect to ζ.

Let $W(\Psi, \Psi', \zeta)$ be the coordinate and velocity distribution function in the (Ψ, Ψ') phase plane. Then by using standard methods we obtain the equation

$$\frac{\partial W}{\partial\zeta} + \Psi'\frac{\partial W}{\partial\Psi} - \Psi\frac{\partial W}{\partial\Psi'} = \frac{\beta}{2}\left(\Psi\frac{\partial W}{\partial\Psi} + \Psi'\frac{\partial W}{\partial\Psi'}\right) + \beta W + q\left(\frac{\partial^2 W}{\partial\Psi^2} + \frac{\partial^2 W}{\partial\Psi'^2}\right),\ \ q = (k\omega\alpha/2\lambda E_s\Omega)^2\langle\varepsilon^2\rangle_\zeta.$$

for this distribution function. This equation can be greatly simplified by the following considerations.

It is seen from (6) that the decay time of phase oscillations is much greater than the oscillation period, and that the extent to which the oscillations are excited in a single period is insignificant. This means that in the (Ψ, Ψ') phase plane the distribution is axially symmetric to a good approximation, or that it depends only on $r = \sqrt{\Psi^2 + \Psi'^2}$. Therefore the equation for the distribution function $W(r, \zeta)$ becomes

$$\frac{\partial W}{\partial\zeta} = \frac{\beta}{2}r\frac{\partial W}{\partial r} + \beta W + q\left(\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r}\right). \tag{7a}$$

The initial distribution for $\zeta = 0$ can obviously also be taken to be axially symmetric, so that

$$W(r, 0) = W_0(r) \tag{7b}$$

with the normalization

$$\int_0^{a_0} W_0(r)\,r\,dr = 1, \tag{7c}$$

where a_0 is the maximum allowable oscillation amplitude at $\zeta = 0$. As for the boundary condition, we have already indicated how it can be used to account for the nonlinear effects. The corresponding analytic expression will be given below. At present we shall assume only that the maximum allowable amplitude at time ζ is given by $a(\zeta)$. This function depends on ζ because the equilibrium phase changes during an acceleration cycle, so that the maximum allowable amplitude changes also. On the basis of the above, and accounting for the fact that when a particle attains the boundary it leaves the accelerating cycle, we obtain the boundary condition

$$W(a(\zeta), \zeta) = 0. \tag{7d}$$

for Eq. (7a). Solution of (7a) with boundary conditions (7b) and (7d) gives the fraction $N(\zeta)/N(0) = F(\zeta)$ of the particles that are not lost by dropping out of phase. From the normalization condition (7c), the function $F(\zeta)$ is given by

$$F(\zeta) = \int_0^{a(\zeta)} W(r, \zeta)\, r dr. \tag{8}$$

The problem of the theory is to perform the calculation indicated in (8).

An exact analytic solution of (7a) with boundary conditions (7b) and (7d) would seem to be impossible. Furthermore, this exact analytic solution is impossible even if one sets $\beta = 0$, thus neglecting damping, since (7d) is given on a moving boundary.

We shall describe a method for finding a majorizing solution, which is an analytical method for effectively solving the problem to any arbitrary prescribed accuracy, and which evaluates the magnitude of the error.

3. MAJORIZING SOLUTION

Let us change in (7a) to a new independent variable ξ and to a new function Φ according to

$$\xi = r \exp\left(\frac{1}{2}\int_0^{\zeta} \beta\, d\zeta\right), \quad \Phi = W \exp\left(-\int_0^{\zeta} \beta\, d\zeta\right). \tag{9}$$

Then (7a) and boundary conditions (7b) and (7d) become

$$\frac{\partial \Phi}{\partial \zeta} = \varphi\left(\frac{\partial^2 \Phi}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \Phi}{\partial \xi}\right), \quad \varphi = q \exp\left[\int_0^{\zeta} \beta\, d\zeta\right]; \tag{10}$$

$$\Phi(\xi, 0) = W_0(\xi), \quad \Phi(A(\zeta), \zeta) = 0, \quad A(\zeta) = a(\zeta) \exp\left(\frac{1}{2}\int_0^{\zeta} \beta\, d\zeta\right). \tag{10a}$$

Note that

$$F(\zeta) = \int_0^{a(\zeta)} W r\, dr = \int_0^{A(\zeta)} \Phi \xi\, d\xi. \tag{11}$$

Let the functions Φ_i be the solutions of (10) with identical initial conditions

$$\Phi_i(\xi, 0) = W_0(\xi)$$

and with homogeneous boundary conditions on the different boundaries $A_i(\zeta)$, i.e.,

$$\Phi_i(A_i(\zeta), \zeta) = 0.$$

Let, further,

$$F_i(\zeta) = \int_0^{A_i(\zeta)} \Phi_i \xi\, d\xi.$$

Under these conditions it can be shown that if for all ζ we have

$$A_i(\zeta) \leqslant A(\zeta) \leqslant A_j(\zeta), \tag{12}$$

then it follows that

$$F_i(\zeta) \leqslant F(\zeta) \leqslant F_j(\zeta), \tag{13}$$

and the equalities in (13) are attained if and only if the corresponding equalities hold in (12) identically for all ζ.

We prove this assertion as follows. Let $P(A_i(\zeta), \xi_0, \zeta) = P_i$ be the probability that, starting a stochastic process from the point ξ_0 at $\zeta = 0$, a particle will not reach the boundary $A_i(\zeta)$ during the time interval $(0, \zeta)$. If the $A_i(\zeta)$ satisfy (12), we may immediately write the obvious relation

$$P_i \leqslant P \leqslant P_j, \tag{14}$$

which is valid for arbitrary equal initial positions ξ_0 of the particle.

Actually these relations express the obvious fact that a particle can reach boundary $A_i(\zeta)$ without reaching $A(\zeta)$, but cannot reach $A(\xi)$ without reaching $A_i(\xi)$. A similar statement holds for the relation between the boundaries $A(\zeta)$ and $A_j(\zeta)$. This is true for any initial position ξ_0. But this means that (13) is simply another way of stating (14), or that the validity of (13) follows immediately from that of (14). This proves relation (13).

Let us divide the interval $(0, \zeta)$, with which we are dealing, into several segments at the points ζ_n (where $n = 0, 1, 2, \ldots$). Further, let us set

$$A_n^{(+)} = A(\zeta_{n+1}), \quad A_n^{(-)} = A(\zeta_n).$$

To be specific let us consider the case in which

$$A_n^{(-)} \leqslant A(\zeta) \leqslant A_n^{(+)}, \quad \zeta_n \leqslant \zeta \leqslant \zeta_{n+1}. \tag{15}$$

Let $\Phi^{(\pm)}_{n,\,n+1}$ be the function Φ that satisfies Eq. (10), but whose boundary conditions are given in the intervals (ζ_n, ζ_{n+1}) on the broken lines $A_n^{(\pm)}$, which means that this function is defined for

$$\zeta_n \leqslant \zeta \leqslant \zeta_{n+1}, \quad 0 \leqslant \xi \leqslant A_n^{(\pm)}. \tag{16}$$

Solving Eq. (10) for these conditions, we obtain

$$\Phi^{(\pm)}_{n,\,n+1}(\xi, \zeta) = 2\,(A_n^{(\pm)})^{-2} \sum_i \frac{J_0(\lambda^{(\pm)}_{n,\,i}\,\xi)}{J_1^2(\lambda_i)}\, b^{(\pm)}_{n,\,i} \exp\left\{-(\lambda^{(\pm)}_{n,\,i})^2 \int_{\zeta_n}^{\zeta} \varphi\, d\zeta\right\}; \tag{17}$$

$$b^{(\pm)}_{n,\,i} = \int_0^{A^{(\pm)}_{n-1}} \Phi^{(\pm)}_{n-1,\,n}(\xi, \zeta_n)\, J_0(\lambda^{(\pm)}_{n,\,i}\,\xi)\, \xi\, d\xi, \quad \lambda^{(\pm)}_{n,\,i} = \lambda_i / A_n^{(\pm)}, \tag{17a}$$

where J is the Bessel function of order one, and the λ_i are the roots of the equation $J_0(\lambda_i) = 0$. Since the $(\pm)$ superscripts enter the formulas symmetrically, we may suppress them for simplicity.

Inserting (17) into (17a) and performing the integration, we obtain the following formula for $b_{n,\,i}$:

$$b_{n,\,i} = \frac{2J_0(\lambda_{n,\,i} A_{n-1})}{A_{n-1}} \sum_j [J_0'(\lambda_j)]^{-1}\, b_{n-1,\,j} \exp\left(-\lambda^2_{n-1,\,j} \int_{\zeta_{n-1}}^{\zeta_n} \varphi\, d\zeta\right) \frac{\lambda_{n-1,\,j}}{\lambda^2_{n,\,i} - \lambda^2_{n-1,\,j}}. \tag{18}$$

From (17) and (11), we see that the desired quantity $F_{n,\,n+1}$ can be written in the form

$$F_{n,\,n+1} = \int_0^{A_n} \Phi_{n,\,n+1}\,\xi\, d\xi = 2 \sum_i [b_{n,\,i} / \lambda_i J_1(\lambda_i)] \exp\left(-\lambda^2_{n,\,i} \int_{\zeta_n}^{\zeta} \varphi\, d\zeta\right). \tag{19}$$

From (15) and from (12) and (13) we have

$$F^{(-)}_{n,\,n+1}(\zeta) \leqslant F(\zeta) \leqslant F^{(+)}_{n,\,n+1}(\zeta), \quad \zeta_n \leqslant \zeta \leqslant \zeta_{n+1}, \quad n = 0, 1, 2, \ldots \tag{20}$$

Thus the $F^{\pm}_{n,\,n+1}$ are majorizing solutions for the exact solution $F(\zeta)$. By choosing the points ζ_n at which we break up the line so as to make the majorizing solutions sufficiently close to each other, we find the exact solution $F(\zeta)$ to the desired accuracy and know the magnitude of the error involved. This

makes it possible to obtain analytic solutions for the problem to any arbitrary prescribed accuracy. The effectiveness of this analytic method can be traced to two causes. First, the series in the recursion formulas (18) converge extremely rapidly, a small number of terms is sufficient for the calculations, depending, of course, on the accuracy desired. Second, in view of the fact that energetic oscillations are excited only towards the end of the accelerating cycle, a solution which is sufficiently accurate for all practical purposes can be obtained by breaking up the $(0, \zeta)$ interval into only a small number of segments. Thus a few simple calculations based on the above formulas yield all the necessary results to the desired accuracy, together with an accurate evaluation of the error admitted. This indeed makes the method of majorizing solutions effective in our case.

4. BOUNDARY CONDITIONS

To have a complete solution of the given problem, we must have an explicit expression for the function q of Eq. (7a), as well as for $a(\zeta)$ the function used to specify the boundary conditions.

To obtain an explicit expression for q we must calculate $\langle\epsilon\rangle_\zeta^2$. Noting that

$$\langle\varepsilon^2\rangle_\zeta = \Omega^{-1}\langle\varepsilon^2\rangle_t \tag{21}$$

and making use of well known formulas for the probability of emitting a photon of energy ϵ, we obtain the expression

$$\langle\varepsilon^2\rangle_t = (55/24\sqrt{3})(hc^2e^2/\lambda R^3)(E/m_0c^2)^7. \tag{22}$$

Together with (21), this gives the explicit expression for q in (7a).

To set up the boundary conditions for the present problem we must know the maximum allowable phase deviation from equilibrium. These maximum allowable deviations are not the same in both directions. Within the framework of the linear theory, however, they must be equal. This condition alone makes it impossible to introduce the boundary conditions exactly and uniquely in the given problem, so that some arbitrariness remains. For arbitrary boundary conditions, however, the above method of majorizing solutions is equally applicable to all cases.

We shall now show how to state the boundary conditions in a way which would seem to take fullest account of the nonlinear effects. In so doing we make use of the previous results.[7]

We have already noted above that the main difference between the results of the linear and nonlinear theories is due to the different potential-well depths. If the well depth is taken from the nonlinear theory and the maximum allowable deviation in the linear theory is chosen as that corresponding to this well depth, then the linear theory will have accounted for the fundamental nonlinear effect. On the basis of this concept, Eqs. (4) and (5) give the following expression for the quantity entering boundary condition (7d):

$$a(\zeta) = 2(1 - \varphi_s \cot\varphi_s)^{1/2}. \tag{23}$$

5. SUMMARY

The formulas presented above give an explicit analytic solution of the problem under consideration. It is not necessary to illustrate the application of these formulas to any particular example chosen at random. We make only two remarks. First, the equations of the present work again emphasize the practical importance of the phenomenon under consideration even for electron energies of the order of 1 Bev. Second, they show the unavoidability of strong focusing at electron energies of several Bev.

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254