however, that one can make serious objections to an application of this method to these cases. The trouble is that the terms which are neglected in the expansion of the collision integral for $T \ll T_0$ are of the same order of magnitude as the terms which are left in [see Ref. 6, Eq. (4)].

The inapplicability of the Fokker-Planck method to the problem under consideration can clearly be seen from the fact that the directly calculated coefficients A_1 and A_2 (see Ref. 6) do not agree with the values obtained by the author from the condition $S(f_0) = 0$. Apart from that, the author arrives in Ref. 7 at the paradoxical conclusion that the resistivity of a metal must go to infinity as $T \to 0$.

We shall not compare here all our results with the analogous equations of Refs. 6, 7, 8, although several of them agree in order of magnitude with the equations of those papers, in the case of small temperature differences ($\Delta T \ll T$).

The authors would like to use this opportunity to thank I. M. Lifshitz and E. S. Borovik for discussions of the problem considered in this paper.

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SOVIET PHYSICS JETP VOLUME 6 (33), NUMBER 5 MAY, 1958

INVARIANT REPRESENTATIONS OF THE SCATTERING MATRIX

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Submitted to JETP editor May 24, 1957

J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 1264-1267 (November, 1957)

The scattering matrix S for the reaction $a + b \to a' + b'$ is expressed in terms of a finite number of spin operators Q_i, each invariant under rotations and reflections. A method for constructing the Q_i is given, and their number is determined for a reaction with given initial and final spins. The restrictions placed upon the form and number of the Q_i by the condition that the scattering matrix be invariant under time reversal are considered. Examples are given in which S is represented by the Q_i for several reactions.

THE scattering matrix $S(\mathbf{k}', \mathbf{k})$ for the reaction $a + b \to a' + b'$ is an operator in the space defined by the spins of the incident and scattered particles, and is a function of their relative momenta $\mathbf{k}, \mathbf{k}'$. Since S is invariant under rotation, it can be written in the form

$$S(\mathbf{k}', \mathbf{k}) = \sum A_i(\mathbf{k}'\mathbf{k})\, Q_i(\mathbf{k}', \mathbf{k}, \mathbf{T}), \tag{1}$$

where the $Q_i(\mathbf{k}', \mathbf{k}, \mathbf{T})$ are invariant operators that depend on the vectors $\mathbf{k}', \mathbf{k}$ and on the matrices $\mathbf{T}$

(Ref. 1) which take the system from its initial spin state into its final one. The $A_i(\mathbf{k}'\mathbf{k})$ are invariant function of the scalar product $(\mathbf{k}'\mathbf{k})$ and the total energy of the system. The representation (1) is convenient to use in studying the general properties of scattering matrices, investigating reactions with polarized particles, carrying out phase analyses, and similar problems.

In the following we examine the representation (1), find the connection between the operators Q_i and the angular operators $L(\mathbf{k}', \mathbf{k})^2$, give a general method for constructing the operators Q_i, and determine the number of operators for a given reaction.

In Ref. 2, the matrix $S(\mathbf{k}', \mathbf{k})$ was written as a series

$$S(\mathbf{k}', \mathbf{k}) = \sum S^{fi}_{Jl'S'lS} L_{Jl'S'lS}(\mathbf{k}', \mathbf{k}) \tag{2}$$

over invariant angular operators L. The coefficients $S^{fi}_{Jl'S'lS}$ in the expansion are functions of the total energy only. The operators L are matrices whose elements $L(\alpha', \alpha)$ are given by the formula

$$L_{Jl'S'lS}(\mathbf{k}'\alpha', \mathbf{k}\alpha) = \sum_{M\mu\mu'} C^{l'M-\mu';\, S'\mu'}_{JM} C^{lM-\mu;\, S\mu}_{JM} Y_{l'M-\mu'}(\mathbf{k}')\, Y^*_{lM-\mu}(\mathbf{k})\, Q_{S'\mu'}(\alpha')\, Q^*_{S\mu}(\alpha) \tag{3}$$

(for notation, see Ref. 2). Upon choosing the x, y, and z axes to be in the directions of the vectors $\mathbf{x} = \frac{((\mathbf{k}\times\mathbf{k}')\times\mathbf{k})}{\sqrt{1-(\mathbf{k}'\mathbf{k})^2}}$, $\mathbf{y} = \frac{\mathbf{k}\times\mathbf{k}'}{\sqrt{1-(\mathbf{k}'\mathbf{k})^2}}$, $\mathbf{z} = \mathbf{k}$ it is not hard to see that (3) becomes

$$L(\mathbf{k}'\alpha', \mathbf{k}\alpha) = \sum_{\mu'\mu} L^{\mu'\mu}_{Jl'S'lS}(\mathbf{k}'\mathbf{k})\, Q^{\mu'\mu}_{S'S}(\alpha', \alpha), \tag{4}$$

where $L^{\mu'\mu}_{Jl'S'lS}(\mathbf{k}'\mathbf{k}) = \sqrt{2l+1}\, C^{l'\mu-\mu';\, S'\mu'}_{J\mu}\, C^{l0;S\mu}_{J\mu}\, \theta_{l'\mu-\mu'}(\mathbf{k}'\mathbf{k})$ are invariant functions of $(\mathbf{k}'\mathbf{k}) \equiv \cos\theta$ and $Q^{\mu'\mu}_{S'S}(\alpha', \alpha) = Q_{S'\mu'}(\alpha')Q^*_{S\mu}(\alpha)$ are invariant spin operators which take the spin of the system from S, μ to S', μ'. These operators (matrices in α', α) are equal to $\delta_{\alpha'\mu'}\delta_{\mu\alpha}$ and can be expressed as invariant functions of the matrices $\mathbf{T}$ (Ref. 1) and the vectors $\mathbf{k}, \mathbf{k}'$. For example if $S = S' =$, then $\mathbf{T} = \boldsymbol{\sigma}$ and $Q^{\frac{1}{2}\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}}(\alpha', \alpha) = \frac{1}{2}(1+\sigma_z)_{\alpha'\alpha} = \frac{1}{2}[1+(\boldsymbol{\sigma}\mathbf{k})]_{\alpha'\alpha}$. In the general case, the operators $Q^{\mu'\mu}_{S'S}$ can be written in terms of Racah's spin-tensor $T^{(n)}_m$ through the relation

$$Q^{\mu'\mu}_{S'S} = \sum_{n=|S'-S|}^{S'+S} (-1)^{n+S'+\mu}(2n+1)^{1/2} C^{S'-\mu';\, S\mu}_{n\mu-\mu'} T^{(n)}_{\mu'-\mu}.$$

From the properties of the $\mathbf{T}$ matrices, it follows that the operators $Q_{S'S}$ are polynomials of degree $S + S'$ in the components of $\mathbf{T}$. Hence, $Q^{\mu'\mu}_{S'S} = Q^{\mu'\mu}_{S'S}(\mathbf{k}', \mathbf{k}, \mathbf{T})$. Each angular operator $L(\mathbf{k}', \mathbf{k})$ can therefore be expanded into $(2S'+1)(2S+1)$ spin operators, with coefficients which are known functions of $(\mathbf{k}'\mathbf{k})$. Actually, however, there are fewer than $(2S'+1)(2S+1)$ spin operators in (4) because parity is conserved in the reaction $a + b \rightarrow a' + b'$.

Indeed, from the symmetry of the Clebsch-Gordan coefficients and the properties of the functions θ_{lm} it follows that

$$L^{-\mu',-\mu}_{Jl'S'lS}(\mathbf{k}'\mathbf{k}) = (-1)^{l'-l+S'-S+\mu'-\mu} L^{\mu'\mu}_{Jl'S'lS}(\mathbf{k}'\mathbf{k}). \tag{5}$$

Hence (4) can be written

$$L_{Jl'S'lS}(\mathbf{k}', \mathbf{k}) = \sum_{\mu'>0,\, \mu} L^{\mu'\mu}[Q^{\mu'\mu} + \xi Q^{-\mu'-\mu}] + \sum_{\mu'=0,\, \mu>0} L^{\mu'\mu}[Q^{\mu'\mu} + \xi Q^{-\mu'-\mu}] + L^{00}Q^{00}, \tag{6}$$

where $\xi = (-1)^{l'-l+S'-S+\mu'-\mu}$. Since parity is conserved, $l'-l$ must be even or odd according to whether the intrinsic parity of the system changes or not during the reaction. Hence ξ and the operators shown in brackets in (6) do not depend on l, l'. We choose these operators to be the $Q_i(\mathbf{k}', \mathbf{k}, \mathbf{T})$. From (6) it follows that if the spins S, S' are half-integral, then the terms with $\mu' = 0$ or $\mu = 0$ in (6) are absent and $L(\mathbf{k}', \mathbf{k})$ is expressed in terms of $\frac{1}{2}(2S'+1)(2S+1)$ operators Q_i. If S, S' are integers, then all terms are present in (6). In this case, if $l'-l+S'-S$ is odd, from (5) it follows that $L^{00} = 0$ and $L(\mathbf{k}', \mathbf{k}_0$ is expressed in terms of $2S'S + S' + S$ operators Q_i. If, on the other hand, $l'-l+S'-S$ is even, then $L^{00} \neq 0$ and $L(\mathbf{k}', \mathbf{k})$ is expressed in terms of $2S'S + S' + S + 1$ operators Q_i.

In this way, the angular operators $L(\mathbf{k}', \mathbf{k})$, and with them the scattering matrix $S(\mathbf{k}', \mathbf{k})$, are written in terms of a finite number r of independent operators $Q_i(\mathbf{k}', \mathbf{k}, \mathbf{T})$. For half-integral spins, $r = \frac{1}{2}(2S' + 1)(2S + 1)$, while for integer spins $r = 2S'S + S' + S + 1$ or $r = 2S'S + S' + S$ depending on whether the number $\ell' - \ell + S' - S$ is even or odd.

It is clear that instead of choosing the r independent operators Q_i to be as given by formula (6), one could equally well have chosen any linear combination of these (with functions of $(\mathbf{k}'\mathbf{k})$ as coefficients), in particular the first r angular operators $L(\mathbf{k}', \mathbf{k})$.

In addition to being invariant under rotations and reflections, the matrix S must be invariant under time reversal, i.e., the matrix elements between states $\psi_{S\mu\mathbf{k}}$ and $\psi_{S'\mu'\mathbf{k}'}$ must be equal to the matrix elements between the time-reflected states $\psi_{S'-\mu'-\mathbf{k}'}$, $\psi_{S-\mu-\mathbf{k}}$. This leads to the condition $KSK^{-1} = S^+$ (Ref. 3), or in detail

$$(KSK^{-1})_{\mathbf{k}'\alpha', \mathbf{k}\alpha} = S^*(\mathbf{k}\,\alpha, \mathbf{k}'\,\alpha'), \tag{7}$$

where K is the Wigner time reversal operator,[4] including complex conjugation. The angular operators L must satisfy relation (7), which now becomes

$$(KL_{Jl'S'lS}K^{-1})_{\mathbf{k}'\alpha', \mathbf{k}\alpha} = L^*_{JlSl'S'}(\mathbf{k}\,\alpha, \mathbf{k}'\,\alpha'). \tag{8}$$

Hence (7) establishes a connection between the coefficients $S^{fi}_{J\ell'S'\ell S}$ and $S^{if}_{J\ell S\ell'S'}$ in the expansion (2), i.e., between the direct and time-reversed reactions.

$$S^{fi}_{Jl'S'lS} = S^{if}_{JlSl'S'}. \tag{9}$$

For inelastic processes ($i \neq f$), or processes involving spin change ($S' \neq S$), (9) does not restrict the matrix S. However, for elastic scattering ($i = f$, $S' = S$), (9) becomes a relation between the coefficients of one and the same reaction. Hence the operators Q_i now appear in the matrix S only in the combination $Q_i(\mathbf{k}', \mathbf{k}, \mathbf{T}) + Q_i^+(\mathbf{k}, \mathbf{k}', \mathbf{T})$, and the number of independent operators $Q_i(\mathbf{k}', \mathbf{k}, \mathbf{T})$ decreases by the number t of relations $S^{ii}_{J\ell S\ell' S} = S^{ii}_{J\ell' S\ell S}$ where $\ell' \neq \ell$. For half-integral spin, $t = S^2 - 1/4$, while for integer spin, $t = S^2$. Hence for purely elastic reactions, the number of spin invariants is $(S + 1)^2 - 1/4$ or $(S + 1)^2$, depending on whether the spin is half-integral or integral.

As an example, we write out the matrices $S(\mathbf{k}', \mathbf{k})$ in terms of the operators Q_i for several reactions, the + and — signs indicating whether the intrinsic parity is conserved or not.

1. $S = S' = 1/2$, $+.\ S(\mathbf{k}', \mathbf{k}) = A + i(\sigma\mathbf{n})B$,
   $-.\ S(\mathbf{k}', \mathbf{k}) = (\sigma\mathbf{k}')A + (\sigma\mathbf{k})B$.
2. $S = S' = 1$, $+.\ S(\mathbf{k}', \mathbf{k}) = A + i(\mathbf{S}\mathbf{n})B + [(\mathbf{S}\mathbf{k}')(\mathbf{S}\mathbf{k}) + (\mathbf{S}\mathbf{k})(\mathbf{S}\mathbf{k}')]C + (\mathbf{S}\mathbf{k}')^2 D + (\mathbf{S}\mathbf{k})^2 E$, for purely elastic processes, $E = D$
   $-.\ S(\mathbf{k}', \mathbf{k}) = (\mathbf{S}\mathbf{k}')A + (\mathbf{S}\mathbf{k})B + i(\mathbf{S}\mathbf{k}')(\mathbf{S}\mathbf{n})C + i(\mathbf{S}\mathbf{k})(\mathbf{S}\mathbf{n})D$.
3. $S = 1$, $S' = 0$ and $S = 0$, $S' = 1$. $+.\ S(\mathbf{k}', \mathbf{k}) = i(\mathbf{T}\mathbf{n})A$,
   $-.\ S(\mathbf{k}', \mathbf{k}) = (\mathbf{T}\mathbf{k}')A + (\mathbf{T}\mathbf{k})B$.
4. $\gamma + b \to a' + b'$, $S = S' = 1/2$.
   $S(\mathbf{k}', \mathbf{k}) = i(\mathbf{n}\mathbf{e})A + (\sigma\mathbf{e})B + (\sigma\mathbf{k}')(\mathbf{k}'\mathbf{e})C + (\sigma\mathbf{k})(\mathbf{k}'\mathbf{e})D$.
5. $\gamma + b \to \gamma' + b'$, $S = S' = 1/2$.
   $S(\mathbf{k}', \mathbf{k}) = (\mathbf{e}'\mathbf{e})A + (\mathbf{s}'\mathbf{s})B + i(\sigma[\mathbf{e}'\mathbf{e}])C + i(\sigma[\mathbf{s}'\mathbf{s}])D + i(\sigma\mathbf{k})(\mathbf{s}'\mathbf{e})E + i(\sigma\mathbf{k}')(\mathbf{e}'\mathbf{s})F + i(\sigma\mathbf{k}')(\mathbf{s}'\mathbf{e})G + i(\sigma\mathbf{k})(\mathbf{e}'\mathbf{s})H$,

for purely elastic processes, $F = -E$, $H = -G$. In these formulas $\mathbf{n} = \mathbf{k}' \times \mathbf{k}$, $\mathbf{s} = \mathbf{k} \times \mathbf{e}$, A, B, ..., H are functions of $(\mathbf{k}'\mathbf{k})$ and the energy of the system. As mentioned above, invariants other than those given above could have been chosen, for example, a corresponding number of angular operators L^2.

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ERRATA TO VOLUME 6

Page	Line	Reads	Should Read
643	16 from bottom	where $\kappa = \pi a^2\Omega - \ldots$	where $\kappa = \pi a^2\Omega\varphi - \ldots$
690	8 from bottom	$\ldots \sin [- \ldots$	$\ldots \sin \vartheta [- \ldots$
	5 from bottom	$\ldots \sin 2\vartheta \sqrt{\frac{1}{3}} \ldots$	$\ldots \sin 2\vartheta \left[\sqrt{\frac{1}{3}} \ldots \right.$
809	9 from top	$\ldots \left(\frac{1}{2 \sinh u} + \ldots \right.$	$\ldots \left(\frac{1}{\sinh u} + \ldots \right.$
973	unnumbered equation	$\ldots C^{S'-\mu';\, S\mu}_{n\mu-\mu'} T^{(n)}_{\mu'-\mu'}.$	$\ldots C^{S'-\mu';\, S\mu}_{n\mu-\mu'} \langle S' \Vert T^{(n)} \Vert S^{-1} \rangle \times T^{(n)}_{\mu'-\mu}.$
975	5 from bottom	...of a particle by a nucleus ...	...of a particle in state a by a nucleus ...
992	Eq. (18)	$\ldots \tau_1 \tau_2^{-2}/2\hbar^1 \ldots$	$\ldots \tau_1 \tau_2^{-1}/2\hbar^2 \ldots$