

- <sup>9</sup> A. Z. Dolginov and I. N. Toptigin, *Nuclear Physics* **2**, 147 (1956/57); *Izv. Akad. Nauk SSSR, Ser. Fiz.* **20**, 1377 (1956).
- <sup>10</sup> A. Z. Dolginov, *J. Exptl. Theoret. Phys. (U.S.S.R.)* **22**, 668 (1952).
- <sup>11</sup> A. Z. Dolginov, *J. Exptl. Theoret. Phys. (U.S.S.R.)* **22**, 658 (1952).
- <sup>12</sup> L. A. Sliv and B. A. Volchok, Таблицы кулоновских фаз и амплитуд с учетом конечных размеров ядра (Table of Coulomb Phases and Amplitudes with the Finite Dimensions of the Nucleus Taken into Account), AN SSSR, 1956.
- <sup>13</sup> E. Condon and G. Shortley, Theory of Atomic Spectra, Cambridge Univ. Press, 1935.
- <sup>14</sup> H. Jahn, *Proc. Roy. Soc. (London)* **205**, 192 (1951).
- <sup>15</sup> K. Alder, *Helv. Phys. Acta* **25**, 235 (1952).
- <sup>16</sup> H. Matsunobu and H. Takebe, *Progr. Theor. Phys.* **14**, 589 (1955).
- <sup>17</sup> H. Bethe, *Quantenmechanik der Ein- und Zwei-Elektronenprobleme*, Handbuch der Physik XXIV (1), 2nd ed., (1933).
- <sup>18</sup> L. D. Landau and E. M. Lifshitz, Квантовая механика (Quantum Mechanics), ГИИТ, 1948.
- <sup>20</sup> A. Z. Dolginov, *Nuclear Physics* **2**, 723 (1956/57).
- <sup>21</sup> I. S. Shapiro, Report to the 7th All-Soviet Conference on Nuclear Spectroscopy (1957).
- <sup>22</sup> A. Z. Dolginov, *Nuclear Physics* **2**, 723 (1956/57).
- <sup>23</sup> E. P. Wigner, Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren, (Braunschweig, 1931).

Translated by E. J. Saletan

280

SOVIET PHYSICS JETP

VOLUME 6 (33), NUMBER 6

JUNE, 1958

*CONTRIBUTION TO THE DECAY THEORY OF A QUASI-STATIONARY STATE*

L. A. KHALFIN

Submitted to JETP editor April 23, 1957

*J. Exptl. Theoret. Phys. (U.S.S.R.)* **33**, 1371-1382 (December, 1957)

Some general problems relating to the decay theory of a quasi-stationary state are considered. Dispersion relations are derived on the basis of the semi-finiteness of the energy distribution density  $\omega(E)$ . A criterion of physical feasibility in decay theory is formulated and studied on the basis of the Paley and Wiener theorem. It is shown that an exponential decay law cannot hold for all  $\Gamma t/\hbar$ . Corrections to the exponential decay law are computed under the simplest assumptions. Dispersion relations between the modulus and the phase of the function  $p(t)$  are derived and investigated. On the basis of a knowledge of the decay law, these allow us to determine the energy distribution density  $\omega(E)$  analytically. The results obtained are derived from the general laws of quantum mechanics and do not depend on the model of the decaying system.

**I**N the present work we consider several problems of decay theory, in particular, the decay of a quasi-stationary (almost stationary) state.\* As is well known, the theory of the decay of a quasi-stationary state has great significance in the investigation of  $\alpha$ -decay, in the transmission of particles through a potential barrier, in the theory of the nucleus, in the determination of the distribution of energy levels, etc.<sup>1,2</sup> The basic theorem of decay theory of a quasi-stationary state was obtained by Fock and Krylov in

\*A brief account of results that have been obtained is given in Ref. 5.

the study of the uncertainty relation for energy and time.<sup>1</sup> A theoretical proof of the observed exponential decay law was obtained in that paper, and the well-known dispersion formulas for the energy distribution of the decaying system were derived.

The present work is a further study of the decay theory of the quasi-stationary state. The study is based on the theory of the Fourier transform in the complex plane,<sup>3</sup> which has already been used in the study of dispersion relations.<sup>4</sup>

The fundamental assumptions of decay theory of the quasi-stationary state, and the results obtained in the work of Krylov and Fock,<sup>1</sup> are treated briefly in the first section. In the second section, the general dispersion relations in decay theory are obtained. In the third section, a criterion of physical feasibility in decay theory is formulated and investigated. Several fundamental questions are also considered in connection with this fact. Finally, the definitive "dispersion" relations and their detailed investigation are treated in the fourth section.

## 1. BASIC PREMISES OF THE DECAY THEORY OF A QUASI-STATIONARY STATE

Let  $\psi_0 = \psi(x, 0)$  be a wave function describing the state of a physical system at the time  $t = 0$ . Then the state of this system at the time  $t$  is determined by the wave function  $\psi(x, t)$ \*

$$\psi(x, t) = \int e^{-iEt/\hbar} c(E) \psi_E(x) dE, \quad (1.1)$$

with

$$\psi_0 = \psi(x, 0) = \int c(E) \psi_E(x) dE, \quad (1.2)$$

while  $\psi_E(x)$  are the eigenfunctions of the energy operator of the physical system.

Investigation of the behavior of the function  $\psi(x, t)$  in its dependence on the variable  $t$ , which is the most general problem pertaining to the time development of the system, can be carried out by starting from certain general principles and restrictions applied to the coefficients  $c(E)$  and the eigenfunctions  $\psi_E(x)$ .

One of the basic problems which arises in the study of the development of a system according to the Schrödinger equation consists in the consideration of the decay of the initial state. But this problem amounts precisely to the determination of the probability that the system at the time  $t$  will still be in the initial state  $\psi_0 = \psi(x, 0)$ . The desired probability is then the projection of the state vector  $\psi(x, t)$  on the initial state vector  $\psi_0(x, 0)$ , and is consequently equal to the square of the modulus of the scalar product

$$p(t) = (\psi(x, t), \psi(x, 0)) = \int \psi(x, 0) \psi(x, t) dx, \quad (1.3)$$

so that the probability that the physical system at the time  $t$  will still be in the initial state  $\psi_0$  is

$$L(t) = |p(t)|^2, \quad (1.4)$$

where in turn, on the basis of (1.1) and (1.2),

$$p(t) = \int e^{-iEt/\hbar} c^*(E) c(E) dE = \int e^{-iEt/\hbar} \omega(E) dE, \quad (1.5)$$

while  $\omega(E)$  is the energy distribution density for the initial state  $\psi_0$  (and consequently also for the state at the time  $t$ ).

The basic theorem of the decay theory (pointed out above), giving the connection between the decay law of the quasi-stationary state and the energy distribution density in this state, was obtained in Ref. 1.

Inasmuch as the condition for the decay of the initial state  $\psi_0$  is evidently the condition

$$L(t) \rightarrow 0 \text{ for } t \rightarrow \infty, \quad (1.6)$$

\*We shall keep the same notation as used in the paper of Krylov and Fock.<sup>1</sup> In particular, the set of variables in terms of which (excluding the time  $t$ ) the wave function of the system is expressed, is denoted by the letter  $x$ .

then it follows from (1.5) (because of the Fourier theorem and the theorem on characteristic functions) that the necessary and sufficient condition for the decay is the absolute integrability of the energy distribution density  $\omega(E)$ .<sup>1</sup> In particular, it follows from this that if we have a discrete spectrum, then the decay is possible only if all the probabilities pertaining to a discrete spectrum vanish in the initial state; consequently, the  $\omega(E_1)$  also vanish. This principal limitation is of advantage to us in what follows.

We note that the determination of the decay probability as the projection of the state vector  $\psi(x, t)$  on the initial state vector  $\psi_0$  is most natural and general. It is clear that in the determination of the decay it is necessary to point out the state the decay of which is studied, which leads to a natural determination of the initial instant of time  $t = 0$ .

We can arbitrarily define the decay probability as the square of the modulus of the wave function  $\psi(x, t)$  for a certain fixed value of  $x$ . As will be shown, the results obtained in the present research are also applicable for such an "arbitrary" determination of the decay probability.

It was shown in Ref. 1 that relative to the energy distribution density  $\omega(E)$  of the decaying state, we can frequently exhibit more than merely the property of absolute integrability. In particular, in the problem of the emission of a particle from a potential well through a potential energy barrier ( $\alpha$ -decay) the energy distribution density  $\omega(E)$  is a meromorphic function of the complex variable  $E$ .

Then, taking into account only the pair of poles close to the real axis,

$$E = E_0 \pm i\Gamma; E_0 > 0; \Gamma > 0, \tag{1.7}$$

we can, for sufficiently large times  $t$ , obtain the following formula:<sup>1</sup>

$$p(t) = \exp\left\{-\frac{iE_0t}{\hbar} - \frac{\Gamma}{\hbar}|t|\right\}, \tag{1.8}$$

and then

$$L(t) = e^{-2\Gamma|t|/\hbar}, \tag{1.9}$$

which expresses the usual, well-known exponential law of decay of the quasi-stationary state. In this case,  $\omega(E)$  is evidently determined by the expression

$$\omega(E) = \frac{1}{\pi} \Gamma / ((E - E_0)^2 + \Gamma^2), \tag{1.10}$$

i.e., a widely-used dispersion formula for the energy distribution of the quasi-stationary state is obtained for  $\omega(E)$ . In what follows, we shall consider some fine points in the derivation of Eq. (1.8).

Concluding this section, we note that the fundamental theory of Fock and Krylov [Eqs. (1.4) and (1.5)] has very general meaning and is based only on the fundamental definitions and concepts of the quantum theory; at the same time, it is completely independent of the concrete form of the Hamiltonian operator. i.e., of the concrete properties of the decaying system.

## 2. GENERAL "DISPERSION" RELATIONS IN DECAY THEORY

For the following investigation it is necessary to make several basic formulas of the previous section more specific. That is, we must consider (taking into account the note in the previous section) that all the integrals in  $E$  are taken in the region of a continuous positive energy spectrum  $E \in (0, \infty)$ . Then the basic formula (1.5) takes the form

$$p(t) = \int_{-\infty}^{\infty} e^{-iEt/\hbar} \bar{\omega}(E) dE, \tag{2.1}$$

where the function  $\bar{\omega}(E)$  that we have introduced is "semi-finite:"

$$\bar{\omega}(E) = \begin{cases} \omega(E) & E > 0 \\ 0 & E \leq 0. \end{cases} \tag{2.2}$$

From (2.1), we get

$$\bar{\omega}(E) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{iEt/\hbar} p(t) dt, \tag{2.3}$$

so that the knowledge of the complex function  $p(t)$  allows us to find analytically the energy distribution density  $\bar{\omega}(E)$  in accordance with (2.3), and therefore to find the spectrum of the Hamiltonian operator of the decaying system.

Restriction of (2.2) to the admissible distribution density  $\bar{\omega}(E)$  adds definite restrictions to the function  $p(t)$  and consequently to the function  $L(t)$ , which has the direct physical meaning of a decay probability.

Actually, transforming to the complex  $t = t_1 + it_2$  in

$$p_1(t) \equiv p(-t) = \int_{-\infty}^{\infty} e^{iEt/\hbar} \bar{\omega}(E) dE \quad (2.4)$$

we get

$$p_1(t) = \int_{-\infty}^{\infty} e^{iEt_1/\hbar} e^{-Et_2/\hbar} \bar{\omega}(E) dE, \quad (2.5)$$

and it is easy to see that (2.5) defines  $p_1(t)$  as an analytical function in the upper half plane  $\text{Im } t = t_2 > 0$  (Ref. 6), which is the direct consequence of the "semi-finiteness" of  $\bar{\omega}(E)$  [Eq. (2.2)]. The analytic character of  $p_1(t)$  in the upper half plane imposes substantial limitations on the form of the function  $p_1(t)$  in the case of real  $t$ , since, on the basis of the Cauchy formula, we evidently obtain\*

$$\frac{1}{i\pi} P \int_{-\infty}^{\infty} \frac{p_1(t')}{t' - t} dt' = p_1(t) + \sum_i \frac{1}{t_i - t} \text{Res } p_1(t_i), \quad (2.6)$$

where  $p_1(t)$  is an analytic function in the upper half plane  $\text{Im } t > 0$ ,  $t_i$  are simple poles of the function  $p_1(t)$  on the real axis,  $\text{Res } p_1(t_i)$  are the residues at these poles; the symbol  $P$  denotes the integral in the sense of the principal value.

Separating the real and imaginary parts in (2.6), and assuming that  $p_1(t)$  does not have any poles on the real axis (a natural physical assumption), we obtain the so-called dispersion relations:

$$\text{Re } p_1(t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\text{Im } p_1(t')}{t' - t} dt', \quad \text{Im } p_1(t) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\text{Re } p_1(t')}{t' - t} dt'. \quad (2.7)$$

We shall now transform these dispersion relations. By virtue of the real nature of  $\bar{\omega}(E)$  we obtain, obviously,

$$p_1^*(t) = p_1(-t); \quad p(t) = p^*(-t). \quad (2.8)$$

From (2.7) and (2.8) we can, after a number of transformations, obtain the following dispersion relations:

$$\text{Re } p(t) = -\frac{2}{\pi} P \int_0^{\infty} \frac{t' \text{Im } p(t') - t \text{Im } p(t)}{t'^2 - t^2} dt', \quad \text{Im } p(t) = \frac{2t}{\pi} P \int_0^{\infty} \frac{\text{Re } p(t') - \text{Re } p(t)}{t'^2 - t^2} dt'. \quad (2.9)$$

Thus, the fact that the energy distribution density  $\bar{\omega}(E)$  is "semi-finite" leads to the result that the real and imaginary functions of  $p(t)$  are interconnected by integral dispersion relations, so that it is impossible to assign  $\text{Re } p(t)$  and  $\text{Im } p(t)$  separately and arbitrarily. The dispersion relations make it possible to calculate  $\text{Im } p(t)$  from a knowledge of  $\text{Re } p(t)$  alone and vice versa. Inasmuch as, on the basis of (2.3), knowledge of  $p(t)$  completely determines  $\bar{\omega}(E)$ , we can select the value of the dispersion relations (2.9) by the following statement: all the information concerning the energy distribution density  $\bar{\omega}(E)$  is contained either in the real or in the imaginary part of the function  $p(t)$ .

From the mathematical point of view, the dispersion relations (2.9) form a system of explicitly solved integral equations in  $\text{Re } p(t)$  and  $\text{Im } p(t)$ .

The dispersion relations introduced above, as was shown, are the consequence of the "semi-finiteness"

\*In the derivation it is essentially assumed that  $p_1(t)$  falls off sufficiently rapidly as  $|t| \rightarrow \infty$ .

in the sense of (2.2), i.e., when the limit of "semi-finiteness" below is zero. In the problem of the emission of a particle from a potential well through a potential barrier, it is obvious that the limit of the "semi-finiteness" is zero.

Now let us consider the consequence of the general condition of semi-finiteness:

$$\bar{\omega}(E) = \begin{cases} \omega(E) & E > E_1 \\ 0 & E \leq E_1. \end{cases} \quad (2.10)$$

It is easy to see that in this more general case we can introduce the corresponding "dispersion" relations.

Actually, carrying out a change of variable, we obtain

$$p_1(t) = e^{iE_1 t/\hbar} \int_{-\infty}^{\infty} e^{iE't/\hbar} \bar{\omega}_1(E') dE' \equiv e^{iE_1 t/\hbar} \varphi_1(t; E_1), \quad (2.11)$$

where  $\bar{\omega}_1(E')$  is already "semi-finite" in the sense of (2.2). But then, according to what was pointed out above, the "dispersion" relations (2.7) are valid for the analytic function  $\varphi_1(t; E_1)$ .

On the basis of (2.7), (2.9), and (2.11), we then get the following "dispersion" relations:

$$\begin{aligned} & \operatorname{Re} p(t) \cos \frac{E_1}{\hbar} t - \operatorname{Im} p(t) \sin \frac{E_1}{\hbar} t \\ &= -\frac{2}{\pi} P \int_0^{\infty} \left\{ t' \operatorname{Re} p(t') \sin \frac{E_1}{\hbar} t' + t' \operatorname{Im} p(t') \cos \frac{E_1}{\hbar} t' - t \operatorname{Re} p(t) \sin \frac{E_1}{\hbar} t - t \operatorname{Im} p(t) \cos \frac{E_1}{\hbar} t \right\} \frac{dt'}{t'^2 - t^2}, \\ & \operatorname{Re} p(t) \sin \frac{E_1}{\hbar} t + \operatorname{Im} p(t) \cos \frac{E_1}{\hbar} t \\ &= \frac{2t}{\pi} P \int_0^{\infty} \left\{ \operatorname{Re} p(t') \cos \frac{E_1}{\hbar} t' - \operatorname{Im} p(t') \sin \frac{E_1}{\hbar} t' - \operatorname{Re} p(t) \cos \frac{E_1}{\hbar} t + \operatorname{Im} p(t) \sin \frac{E_1}{\hbar} t \right\} \frac{dt'}{t'^2 - t^2}, \end{aligned} \quad (2.12)$$

which represent explicitly unsolved integral equations relative to  $\operatorname{Re} p(t)$  and  $\operatorname{Im} p(t)$ . It is obvious that in the particular case in which  $E_1 = 0$ , the relations (2.12) transform to (2.9).

It is also easy to see that a condition of semi-finiteness of the type

$$\bar{\omega}(E) = \begin{cases} \omega(E) & E < E_2 \\ 0 & E \geq E_2 \end{cases} \quad (2.13)$$

leads to the following "dispersion" relations:

$$\begin{aligned} & \operatorname{Re} p(t) \cos \frac{E_2}{\hbar} t - \operatorname{Im} p(t) \sin \frac{E_2}{\hbar} t \\ &= \frac{2}{\pi} P \int_0^{\infty} \left\{ t' \operatorname{Im} p(t') \cos \frac{E_2}{\hbar} t' + t' \operatorname{Re} p(t') \sin \frac{E_2}{\hbar} t' - t \operatorname{Im} p(t) \cos \frac{E_2}{\hbar} t - t \operatorname{Re} p(t) \sin \frac{E_2}{\hbar} t \right\} \frac{dt'}{t'^2 - t^2}, \\ & \operatorname{Im} p(t) \cos \frac{E_2}{\hbar} t + \operatorname{Re} p(t) \sin \frac{E_2}{\hbar} t \\ &= -\frac{2t}{\pi} P \int_0^{\infty} \left\{ \operatorname{Re} p(t') \cos \frac{E_2}{\hbar} t' - \operatorname{Im} p(t') \sin \frac{E_2}{\hbar} t' - \operatorname{Re} p(t) \cos \frac{E_2}{\hbar} t + \operatorname{Im} p(t) \sin \frac{E_2}{\hbar} t \right\} \frac{dt'}{t'^2 - t^2}. \end{aligned} \quad (2.14)$$

It is now easy to introduce the dispersion relations for the modulus (which has direct physical meaning) and phase of the function  $p(t)$ . Actually, taking it into account that

$$p(t) = M(t) \exp[iN(t)], \quad \operatorname{Re} p(t) = M(t) \cos N(t); \quad \operatorname{Im} p(t) = M(t) \sin N(t), \quad (2.15)$$

it is not difficult to obtain "dispersion" relations on the basis of (2.9), (2.12), and (2.14) that connect  $M(t)$  and  $N(t)$ . For example, from (2.9) we obtain the relation

$$\begin{aligned}
 M(t) \cos N(t) &= -\frac{2}{\pi} P \int_0^{\infty} \frac{t' M(t') \sin N(t') - t M(t) \sin N(t)}{t'^2 - t^2} dt', \\
 M(t) \sin N(t) &= \frac{2t}{\pi} P \int_0^{\infty} \frac{M(t') \cos N(t') - M(t) \cos N(t)}{t'^2 - t^2} dt',
 \end{aligned}
 \tag{2.16}$$

which are explicitly unsolved integral equations relative to  $M(t)$  and  $N(t)$ .

Equations (2.16) permit us, generally speaking, to determine  $N(t)$  as a solution of an integral equation, knowing  $M(t')$  for all  $t'$  (i.e., knowing the behavior of the decay probability), and in the same way to determine  $\bar{\omega}(E)$  analytically on the basis of (2.3). It is true that the solution of the integral equations for  $N(t)$  or  $M(t)$  is not generally unique. A detailed study and criticism of the "dispersion" relations is given in Sec. 4 of the present work.

To conclude this section, we note that the "dispersion" relations obtained above are valid even for the wave function

$$\psi(x, t) \equiv \Psi(x, t) e^{i\varphi(x, t)}, \tag{2.17}$$

so that the "dispersion" relations for  $\psi(x, t)$  are obtained by an obvious substitution in (2.9), (2.12), (2.14), (2.16):

$$\begin{aligned}
 \operatorname{Re} p(t) &\rightarrow \operatorname{Re} \psi(x, t), & \operatorname{Im} p(t) &\rightarrow \operatorname{Im} \psi(x, t), \\
 M(t) &\rightarrow \Psi(x, t), & N(t) &\rightarrow \varphi(x, t).
 \end{aligned}
 \tag{2.18}$$

### 3. CRITERION FOR PHYSICAL FEASIBILITY IN DECAY THEORY

In the previous section, dispersion relations were derived on the basis of the "semi-finiteness" of the energy distribution density  $\bar{\omega}(E)$ . These established the integral connection between the real and imaginary parts of the function  $p(t)$ , and therefore between the modulus  $M(t)$  (which has direct physical meaning) and the phase  $N(t)$ . However, as will be shown, the dispersion relations do not exhaust all the consequences of the condition of "semi-finiteness" of the energy distribution density  $\bar{\omega}(E)$ .

One remaining direct consequence of the condition of semi-finiteness of  $\bar{\omega}(E)$  is the criterion of physical feasibility in decay theory, which is analogous to the criterion of physical feasibility already used.<sup>4\*</sup>

As was shown in the previous section, the modulus  $M(t)$  and the phase  $N(t)$  of the function  $p(t)$  are connected by the "dispersion" relations, so that it is impossible to give separately and arbitrarily the modulus  $M(t)$  and phase  $N(t)$ . However, since the "dispersion" relations form a single relation, then for its proof it is necessary to know the value of  $M(t)$  and  $N(t)$ , i.e., the entire function  $p(t)$ . At the same time it is observed that on the basis of the condition of "semi-finiteness" of  $\bar{\omega}(E)$ , we can point out a criterion by which we need satisfy only the modulus  $M(t)$  of the function  $p(t)$ , which has direct physical meaning because  $L(t) = M^2(t)$  is the probability that at the moment  $t$  the system is still in the initial state  $\psi_0$ , i.e., it is the decay probability.

The statement of the problem is this: let an arbitrary non-negative function  $M(t)$  be given; can we find such a real function  $N(t)$  that the function  $\bar{\omega}(E)$ , the Fourier transform of which is

$$p(t) = M(t) e^{iN(t)}, \tag{3.1}$$

would be "semi-finite" in the sense

$$\bar{\omega}(E) = \begin{cases} \omega(E) & E > E_0 \\ 0 & E \leq E_0. \end{cases} \tag{3.2}$$

It is shown that arbitrary  $M(t)$  are unattainable, i.e., for certain  $M(t)$  it is impossible to select any real function  $N(t)$  so as to satisfy the "semi-finiteness" condition (3.2). The limitations on the admissible function  $M(t)$  follow from the fundamental theorem of Paley-Wiener.<sup>3</sup>

In decay theory, this theorem can be formulated in the following fashion. In order that  $L(t)$  be the probability of decay of a quasi-stationary state, we must satisfy the criterion

\*The criterion of physical feasibility obtained in the present section is also applicable to the study of  $\psi(x, t)$ , so that it is sufficient to substitute in the final results  $\Psi(x, t)$  for  $M(t)$  and  $\varphi(x, t)$  for  $N(t)$ .

$$\int_{-\infty}^{\infty} \frac{|\log M(t)|}{1+t^2} dt < \infty, \tag{3.3}$$

where  $L(t) = M^2(t)$ .

The criterion (3.3), which the modulus  $M(t)$  of the function  $p(t)$  must necessarily satisfy, we call the criterion of physical feasibility<sup>5</sup> in the decay theory of a quasi-stationary state.

We note that while the "dispersion" relations (see Sec.2) depend essentially on the limits of finiteness, the criterion of physical feasibility (3.3) is generally independent of the limits of finiteness (except that the limits of finiteness  $E_0$  be finite).

Failure to satisfy the condition (3.3), i.e., the criterion of physical feasibility, can be due to the integral (3.3) being improper, either on the upper (lower) limit or owing to singularities of the integrand. In either case,  $M(t)$ , and consequently  $L(t)$ , cannot vanish over a certain interval of the variable  $t$ . Therefore, it does not hold that

$$L(t) = 0 \text{ for } t \geq T_0. \tag{3.4}$$

Inasmuch as the integral (3.3) is absolutely convergent, if it converges at all, then the conditions for the convergence of (3.3) are the Cauchy conditions.<sup>7</sup>

The criterion of physical feasibility (3.3) permits us to make the following remarks relative to the function  $M(t)$ , and consequently relative to  $L(t)$  also.

(a) As  $|t| \rightarrow \infty$ , the convergence of the integral (3.3) is guaranteed by satisfaction of the inequality

$$|\log M(t)| / (1+t^2) < At^{-l}, \tag{3.5}$$

where  $A > 0$ ;  $l > 1$ .

(b) The convergence at certain points  $t_0$ :  $M(t_0) = 0$ ;  $M(t_0) = \infty$  is guaranteed by fulfillment of the following inequality:

$$|\log M(t)| / (1+t^2) < A_1(t-t_0)^{-l_1}, \tag{3.6}$$

where  $A_1 > 0$ ;  $l_1 < 1$ .

We note that if  $M(t)$  satisfies the conditions of the Paily-Wiener theorem, i.e., the criterion (3.3), then the function obtained by polynomial change of  $M(t)$  satisfies this same criterion.

The inequalities (3.5) and (3.6) are of fundamental value in the decay theory of the quasi-stationary state, especially the inequality (3.5).

Actually, the inequality (3.5) means that for  $|t| \rightarrow \infty$ , the inequality

$$L(t) \geq A \exp(-\gamma|t|^q), \tag{3.7}$$

( $A > 0$ ;  $\gamma > 0$ , and  $q < 1$ ), is certainly satisfied in all cases, i.e., there can be in principle no exponential decay law for all  $t \in (0, \infty)$ . It should be noted that the principal conclusion obtained above is the direct consequence of the "semi-finiteness" of the energy distribution density  $\bar{\omega}(E)$  and is completely independent of the specific nature of the decaying system.

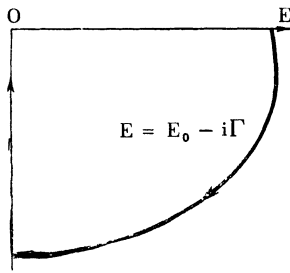
The conclusion thus obtained paradoxically contradicts the well-known experimental exponential decay law. However, as will be shown, in spite of the fact that there can be, in principle, no exponential decay law for all  $t \in (0, \infty)$ , the exponential decay law is satisfied with sufficient accuracy in a rather large middle region of the values of  $t$ .

With this aim, let us consider in detail the derivation of Eq. (1.8), obtained in Ref. 1. In the case in which  $\bar{\omega}(E)$  has only poles for  $E = E_0 \pm i\Gamma$ ,  $p(t)$  is defined by the integral

$$p(t) = \frac{1}{\pi} \int_0^{\infty} e^{-iEt/\hbar} \frac{\Gamma}{(E-E_0)^2 + \Gamma^2} dE. \tag{3.8}$$

If we consider that  $\Gamma \ll E_0$  and  $E_0 \gg 0$ , then it seems natural to substitute the lower limit in (3.8) by  $-\infty$ , i.e., approximately

$$p(t) \approx \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-iEt/\hbar} \frac{\Gamma}{(E-E_0)^2 + \Gamma^2} dE. \tag{3.9}$$



The integral of (3.9) is easily computed by residues and gives, as in Ref. 1,

$$\rho(t) \approx e^{-iE_0 t/\hbar} e^{-\Gamma t/\hbar}, \quad (3.10)$$

i.e., the exponential decay law

$$L(t) \approx e^{-2\Gamma t/\hbar}. \quad (3.11)$$

However, the result (3.10) contradicts the criterion (3.3) which is evidently produced by the approximation made in the computation of (3.8). We consider the more accurate calculation of this integral.

Traversing the closed contour in the lower half plane  $\text{Im } E \leq 0$ , shown in the picture, we obviously obtain, from the method of residues:

$$\rho(t) = \exp\{-iE_0 t/\hbar - \Gamma t/\hbar\} - \frac{i}{\pi} \int_0^\infty e^{-Et/\hbar} \frac{\Gamma}{(E_0 + iE)^2 + \Gamma^2} dE. \quad (3.12)$$

The first term in (3.9) gives an approximate solution of (3.10). We compute the second term in (3.9). We have

$$A \equiv \frac{i}{\pi} \int_0^\infty e^{-Et/\hbar} \frac{\Gamma}{(E_0 + iE)^2 + \Gamma^2} dE = \frac{1}{2\pi} \left\{ \int_0^\infty e^{-Et/\hbar} \frac{dE}{E_0 + iE - i\Gamma} - \int_0^\infty e^{-Et/\hbar} \frac{dE}{E_0 + iE + i\Gamma} \right\}. \quad (3.13)$$

In the first integral we make the substitution of variables  $E_0 + iE - i\Gamma \equiv z$ , and in the second integral,  $E_0 + iE + i\Gamma \equiv z$ ; we obtain from (3.13)

$$A = \frac{i}{2\pi} [f(y_1) - f(y_2)], \quad (3.14)$$

where

$$f(y) = e^{-iy} \int_y^{i\infty} e^{ix} \frac{dx}{x}, \quad (3.15)$$

$$y_1 = (E_0 - i\Gamma)t/\hbar, \quad y_2 = (E_0 + i\Gamma)t/\hbar. \quad (3.16)$$

Integrating by parts in (3.15) and assuming that

$$(t/\hbar) \sqrt{E_0^2 + \Gamma^2} \gg 1, \quad (3.17)$$

we get

$$f(y) \approx i/y, \quad (3.18)$$

and then, substituting in (3.14), we obtain

$$A = \frac{i}{\pi} \frac{\Gamma\hbar}{(E_0^2 + \Gamma^2)t}. \quad (3.19)$$

Thus, we finally get for values of  $t$  that satisfy (3.17)

$$\rho(t) = \exp\{-iE_0 t/\hbar - \Gamma t/\hbar\} - \frac{i}{\pi} \frac{\Gamma\hbar}{(E_0^2 + \Gamma^2)t}. \quad (3.20)$$

The presence of the second, non-exponential term in (3.20) once again supports the criterion of physical feasibility (3.3), (3.7), since it follows in every case from (3.20) that for  $t \rightarrow \infty$ ,  $|\rho(t)|$  vanishes more slowly than exponentially. It also follows from (3.20) that the exponential law of decay is valid only in the finite region of values of  $\Gamma t/\hbar$ .

$$(\Gamma t/\hbar) e^{-\Gamma t/\hbar} \gg (\Gamma/E_0)^2. \quad (3.21)$$

We can show that similar results are obtained if  $\bar{\omega}(E)$  differs from (1.10) by the factor  $\sqrt{E}$ . For brevity, we omit the calculations and write down the final result. The additional non-exponential term in this case is proportional to  $t^{-3/2}$  while the condition (3.21) is replaced by



$$(\Gamma/\hbar)^{3/2} e^{-\Gamma t/\hbar} \gg (\Gamma/E_0)^{3/2}. \tag{3.22}$$

It is evident from the estimates obtained that the difference between the decay laws and the exponential can be observed only for very large or rather small values of  $t$ . Here the presence of non-exponential terms naturally leads to the necessity of fixing the initial instant of time  $t = 0$ . In the following section, we shall touch on the experiments possible in this sense.

The criterion of physical feasibility that has been obtained, in addition to its principal value, is important for the investigation of "dispersion" relations in which the integrals generally extend over infinite limits

To conclude this section, we give (without proof) another interesting consequence of the finiteness of  $\bar{\omega}(E)$ . If  $\bar{\omega}(E)$  is finite in the sense

$$\bar{\omega}(E) = \begin{cases} 0 & E \geq E_2 \\ \omega(E) & E \in (E_1, E_2), \\ 0 & E \leq E_1 \end{cases} \tag{3.23}$$

then the function  $p(t)$  is shown to be "quantized:"

$$p(t) = \exp\left\{-\frac{i}{\hbar} \frac{E_1 + E_2}{2} t\right\} \sum_{n=-\infty}^{\infty} p\left(\frac{2\pi n \hbar}{E_2 - E_1}\right) \frac{\sin[(E_2 - E_1)t/2\hbar + n\pi]}{(E_2 - E_1)t/2\hbar + n\pi}, \tag{3.24}$$

that is, the value of the decay probability  $L(t)$  is completely determined by its values at the points  $t_n = 2\pi n \hbar / (E_2 - E_1)$ .

#### 4. INVESTIGATION OF THE "DISPERSION" RELATIONS

In the second section we obtained "dispersion" relations, in particular, Eq. (2.16), which relate  $M(t)$  and  $N(t)$ , i.e., the modulus and phase of the function  $p(t)$  by integral equations. If one were to succeed in solving these equations, then, knowing  $M(t)$  (from an experimental determination of the decay), we would be able to compute  $N(t)$ , and in the same way we could, on the basis of (2.3), determine the energy distribution density  $\bar{\omega}(E)$  of the decaying system, without making any assumptions as to its nature, i.e., as to the Hamiltonian operator. In this case we could speak of the following inverse problem: knowing the decay law  $L(t)$ , to determine analytically the energy distribution density  $\bar{\omega}(E)$  of the decaying system.

With the aim of solving this inverse problem, let us study the dispersion relations (2.16). We consider the following function:

$$g(t) \equiv \log p_1(t) = \log M(t) - iN(t). \tag{4.1}$$

According to what was pointed out earlier,  $p_1(t)$  is analytic in the upper half plane  $\text{Im } t > 0$ , and consequently it can have only isolated poles in the upper half plane.

Assuming that all the poles of the function  $p_1(t)$  lie in a semicircle of finite radius, on the basis of the theorem on the number of poles of an analytic function, we get<sup>8</sup>

$$N(\infty) = n\pi, \tag{4.2}$$

where  $n$  is the number of poles.

Initially, let us assume that  $p_1(t)$ , which is analytic in the upper half plane, has no poles there. Then  $g(t)$  will also be an analytic function in the upper half plane  $\text{Im } t > 0$ . However, it is not possible to apply Cauchy's formula to  $g(t)$  directly, since  $p_1(t) \rightarrow 0$ , and consequently  $g(t) \rightarrow \infty$  for  $t \rightarrow \infty$ . However, on the basis of the criterion of physical feasibility and (4.2), it is easy to estimate the order of magnitude of  $g(t)$ . Actually, from (3.3), (3.7), and (4.2),

$$g(t) \sim t^q, \quad t \rightarrow \infty; \quad q < 1. \tag{4.3}$$

Consequently, if we consider the function

$$g_n(t) \equiv t^{-n} g(t) = t^{-n} \log M(t) - it^{-n} N(t), \quad n \geq 2, \tag{4.4}$$

then, being an analytic function for  $\text{Im } t > 0$ , it satisfies at the same time the conditions of applicability of the Cauchy formula, because  $g_n(t)$  approaches zero sufficiently rapidly for  $t \rightarrow \infty$ .

Making use of the more general Cauchy formula in that case in which the analytic function  $g_n(t)$

has poles of  $n$ -th order\*

$$\frac{t^n}{i\pi} P \int_{-\infty}^{\infty} \frac{g(t')}{t'^n (t' - t)} dt' = g(t) - g(0) - \frac{t}{1!} \frac{dg(0)}{dt} - \dots - \frac{t^{n-1}}{(n-1)!} \frac{d^{n-1}g(0)}{dt^{n-1}}, \tag{4.5}$$

and taking it into account that

$$\operatorname{Re} \frac{d^{2m+1}g(0)}{dt^{2m+1}} = 0; \operatorname{Im} \frac{d^{2m}g(0)}{dt^{2m}} = 0, \quad g(0) = 0, \tag{4.6}$$

we get [on the basis of (4.5)] the following "dispersion" relations:

$$\begin{aligned} \operatorname{Re} g(t) &= \frac{t^2}{2!} \frac{d^2g(0)}{dt^2} + \dots + \frac{t^{2m}}{(2m)!} \frac{d^{2m}g(0)}{dt^{2m}} + \frac{t^{2m+1}}{\pi} P \int_{-\infty}^{\infty} \frac{\operatorname{Im} g(t')}{t'^{2m+1} (t' - t)} dt', \\ \operatorname{Im} g(t) &= \frac{t}{1!} \frac{dg(0)}{dt} + \dots + \frac{t^{2m-1}}{(2m-1)!} \frac{d^{2m-1}g(0)}{dt^{2m-1}} - \frac{t^{2m+1}}{\pi} P \int_{-\infty}^{\infty} \frac{\operatorname{Re} g(t')}{t'^{2m+1} (t' - t)} dt', \end{aligned} \tag{4.7}$$

or

$$\begin{aligned} \operatorname{Re} g(t) &= \frac{t^2}{2!} \frac{d^2g(0)}{dt^2} + \dots + \frac{t^{2m-2}}{(2m-2)!} \frac{d^{2m-2}g(0)}{dt^{2m-2}} + \frac{t^{2m}}{\pi} P \int_{-\infty}^{\infty} \frac{\operatorname{Im} g(t')}{t'^{2m} (t' - t)} dt', \\ \operatorname{Im} g(t) &= \frac{t}{1!} \frac{dg(0)}{dt} + \dots + \frac{t^{2m-1}}{(2m-1)!} \frac{d^{2m-1}g(0)}{dt^{2m-1}} - \frac{t^{2m}}{\pi} P \int_{-\infty}^{\infty} \frac{\operatorname{Re} g(t')}{t'^{2m} (t' - t)} dt'. \end{aligned} \tag{4.8}$$

We note that the terms outside the integral sign in these "dispersion" relations contain the moments of the distribution density  $\bar{\omega}(E)$ :

$$\omega_k \equiv \int_0^{\infty} E^k \bar{\omega}(E) dE. \tag{4.9}$$

In the special case when  $m = 1$ , i.e.,  $n = 2$ , we get, after simple transformations, the following "dispersion" relations:

$$\log M(t) = -\frac{2t^2}{\pi} P \int_0^{\infty} \frac{1}{t'^2 - t^2} \left\{ \frac{N(t')}{t'} - \frac{N(t)}{t} \right\} dt', \quad N(t) = -t \frac{dp_1(0)}{dt} - \frac{2t^3}{\pi} P \int_0^{\infty} \frac{1}{t'^2 - t^2} \left\{ \frac{\log M(t')}{t'^2} - \frac{\log M(t)}{t^2} \right\} dt', \tag{4.10}$$

which permit us to determine  $M(t)$  uniquely from the known values of  $N(t')$ ; to determine  $N(t)$  from known values of  $M(t')$ , we must also know  $dp_1(0)/dt$ , i.e., the actual mean value of the energy.

To obtain "dispersion" relations which permit us to compute  $N(t)$  from known values of  $M(t')$ , we consider the function

$$g_-(t) \equiv g(t) - g(\infty). \tag{4.11}$$

Making use of (4.2), we then get the desired "dispersion" relation on the basis of the Cauchy formula (2.6):

$$N(t) = \frac{2t}{\pi} P \int_0^{\infty} \frac{\log M(t') - \log M(t)}{t'^2 - t^2} dt'. \tag{4.12}$$

It was assumed above that  $p_1(t)$  has no poles in the upper half plane. In this case we can determine  $N(t)$  uniquely in terms of the known values of  $M(t')$  and solve the inverse problem. If  $p_1(t)$  has a pole in the half plane  $\operatorname{Im} t > 0$ , then the connection between  $M(t)$  and  $N(t)$  becomes ambiguous, but the resulting arbitrariness can be exactly identified.

\*The symbol  $P$  in (4.5) and below denotes an integral in the sense of the principal value both at the singular points  $t' = t$  and at  $t' = 0$ .

Let  $p_1(t)$  have a simple pole in the half plane  $\text{Im } t > 0$ , say at  $t_n = \alpha_n + i\beta_n$ ;  $\beta_n > 0$ , i.e.,

$$p_1(t) = \tilde{p}_1(t)(t - \alpha_n - i\beta_n), \tag{4.13}$$

where  $\tilde{p}(t)$  is already an analytic function in the upper half plane  $\text{Im } t > 0$ , having no poles there. We consider the function

$$\bar{p}_1(t) \equiv p_1(t)(t - \alpha_n + i\beta_n)/(t - \alpha_n - i\beta_n). \tag{4.14}$$

It is evident from (4.14) that the function  $\bar{p}_1(t)$ , which is analytic in the upper half plane, has no poles there (its poles are in the lower half plane), and has the same modulus as the original function  $p(t)$ . Applying the dispersion relation (4.12) to  $p(t)$ , we see that  $N(t)$  is given by the following expression:

$$N(t) = \frac{2(t - \alpha_n)\beta_n}{(t - \alpha_n)^2 - \beta_n^2} + \frac{2t}{\pi} P \int_0^\infty \frac{\log M(t') - \log M(t)}{t'^2 - t^2} dt', \tag{4.15}$$

because

$$\arg \bar{p}_1(t) = \arg p_1(t) - \arg \frac{t - \alpha_n + i\beta_n}{t - \alpha_n - i\beta_n}. \tag{4.16}$$

Thus, finally, knowing the modulus of the function  $p(t) - M(t)$  [which is connected with the experimentally determined decay law  $L(t) = M^2(t)$ ], and the poles of the function  $p_1(t)$  in the upper half plane  $\text{Im } t > 0$ , we can uniquely determine the phase of the function  $p(t) - M(t)$  and also, from (2.3), we can determine analytically the energy distribution function  $\bar{\omega}(E)$ .

Some remarks relative to the experimental proof. As is easily understood from the conclusions reached above, it is necessary to observe the decay, beginning with the initial instant of time  $t = 0$ . For this purpose we could use an experiment with the formation of an artificial, short-lived isotope, fixing the initial moment of time  $t = 0$  by having a "pulse" of particles bombard a stable nucleus, a product of which is an artificial short-lived isotope. In this same experiment, we could observe the departure of the decay law from the exponential.

In conclusion, I express my gratitude to V. A. Fock for his discussion of the work and for valuable comments, and also to the colleagues of the seminar of the Department of Theoretical Physics, Leningrad State University for discussion of the work.

<sup>1</sup>N. S. Krylov and V. A. Fock, J. Exptl. Theoret. Phys. (U.S.S.R.) 17, 93 (1947).

<sup>2</sup>J. Blatt and V. Weisskopf, Theoretical Nuclear Physics, (John Wiley and Sons, Inc., New York, 1952).

<sup>3</sup>R. E. Paily and N. Wiener, American Mathematical Society Colloquium Publication 19, 1934.

<sup>4</sup>L. A. Khal'fin, Dokl. Akad. Nauk SSSR 111, 345 (1956), Soviet Phys. "Doklady" 1, 671 (1956).

<sup>5</sup>L. A. Khal'fin, Dokl. Akad. Nauk SSSR 115, 277 (1957), Soviet Phys. "Doklady" 2, 340 (1957).

<sup>6</sup>E. Titchmarsh, Introduction to the Theory of Fourier Integrals, (Oxford, 1937).

<sup>7</sup>I. G. Fikhtengol'ts, Курс дифференциального и интегрального исчисления (Course of Differential and Integral Calculus), Vol. 2, GITTL, 1947.

<sup>8</sup>V. I. Smirnov, Курс высшей математики (Course of Higher Mathematics), Vol. 3, Part II, GITTL, 1948.

Translated by R. T. Beyer