# EQUATIONS OF MOTION FOR ROTATING MASSES IN THE GENERAL THEORY OF RELATIVITY

A. P. RIABUSHKO

Belorussian State University

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The systems to be considered consist of n spherically symmetrical objects of infinitesimal size, which possess angular momentum and which interact only through gravitational forces. For such systems, it is relatively simple to derive the relativistic equations of translation and rotation from the Einstein gravitational equations. This derivation uses L. Infeld's idea of introducing Dirac delta-functions into the energy-momentum tensor.

THE motions of masses in their own gravitational fields is a well-known and important problem. In particular, the important question: "Are the equations of motion for the masses contained in the equations for their gravitational field?" was answered affirmatively by Einstein.[1,2] Methods for deriving detailed equations of motion from the field equations of general relativity theory have recently been developed along two lines, represented on the one hand by the fundamental work of Einstein, Infeld, and their co-workers[3-5] and on the other by Fock and his co-workers[6,7] and Papapetrou.[8] Fock[9] uses his approximational method to derive non-Newtonian equations and their integrals for the translation of finite rotating masses.

The aim of the present paper is to derive non-Newtonian equations for the translation and rotation of infinitesimally small rotating, spherically symmetrical objects, on the basis of the approximate method developed by Einstein's school. The starting point is a paper by Infeld,[10] who gives a relatively simple derivation of the non-Newtonian equations of translational motion for a non-rotating object represented by a singularity in the gravitational field. This derivation is so simple that Infeld, abandoning the tradition of the Einstein school of deriving the equations of motion from the free-space field equations, makes use, like Papapetrou,[8] of the vanishing of the divergence of the energy-momentum tensor, a result of Einstein's gravitational equations. At the same time he introduces a novelty in the form of Dirac delta-functions, corresponding to the infinitesimal spherically-symmetrical particles mentioned above, as part of the energy-momentum tensor. By "smearing out" the delta-functions he arrives at a continuous distribution of matter, e.g. at the spherically symmmetrical object of finite dimensions which is used repeatedly in his paper. Thus, this paper by Infeld,[10] serves in many ways as a link between the methods of Einstein and Fock.

1. As already mentioned, the present paper will make use of Infeld's method[10] for deriving the equations of motion. In this method we start essentially from the known equations for the graviational field

$$R^{\alpha\beta} - {}^1/_2\, g^{\alpha\beta} R = -8\pi T^{\alpha\beta}, \tag{1.1}$$

where Newton's constant $\gamma$ and the speed of light c have been set equal to unity, and the other quantities have their usual meaning.* Let us consider the motion of a system of n gravitating point particles. We assume here that the world lines of the particles do not intersect. Then Infeld takes as $\widetilde{T}^{\alpha\beta}$, for a system of n particles without angular momentum, the expression

$$\widetilde{T}^{\alpha\beta} \equiv \sqrt{-g}\, T^{\alpha\beta} = \sum_a \widetilde{T}_a^{\alpha\beta}, \qquad \widetilde{T}_a^{00} = m_a \delta_a, \quad \widetilde{T}_a^{0l} = m_a \frac{da^l}{dx^0}\delta_a, \quad \widetilde{T}_a^{ik} = m_a \frac{da^i}{dx^0}\frac{da^k}{dx^0}\delta_a. \tag{1.2}$$

Here the sum is to be taken over all the n particles of the system, which are distinguished by the small letters a, b, ...; there is no summation over repeated subscripts a, b, ...; $m_a$ is a function only of the time, $x^0$; $a^i$ are the coordinates of the a-th particle; $\delta_a = \delta(x^i - a^i)$ is the usual three-dimen-

*Greek indices $\alpha$, $\beta$, ... take the values 0, 1, 2, and 3, referring to both space and time coordinates. Latin indices i, k, $\ell$, ... take the values 1, 2, and 3, referring only to the spatial coordinates.

sional delta-function, referring to particle a. Expression (1.2) for $\tilde{T}^{\alpha\beta}$ is derived from the energy-momentum tensor density for a dust cloud

$$\tilde{T}^{\alpha\beta} = \sqrt{-g}\,\rho U^{\alpha} U^{\beta}, \quad U^{\alpha} = dx^{\alpha}/ds, \tag{1.3}$$

by substituting $\rho = \sum \rho_a = \sum m'_a \delta_a$ (i.e., each spherically symmetrical cloud of material, described by a function $\rho_a$, is replaced by a material point having the same mass and located at its center of gravity). Therefore, $m_a = \sqrt{-g}(dx^0/ds)^2 m'_a$. Integrating the relation

$$\tilde{T}^{\alpha\beta}_{;\beta} \equiv \sqrt{-g}T^{\alpha\beta}_{;\beta} \equiv \tilde{T}^{\alpha\beta}_{,\beta} + \Gamma^{\alpha}_{\beta\sigma}\tilde{T}^{\beta\sigma} = 0, \tag{1.4}$$

which comes from (1.1), over a three-dimensional volume which contains the a-th particle but no others, we obtain the 4n equations

$$\int\limits_{V_a} \tilde{T}^{\alpha\beta}_{;\beta} dV = 0, \tag{1.5}$$

from which the equations of translational motion can be derived.*

Note that although the equations in (1.5) are not covariant in the usual sense (this is not a four-vector) they are still "covariant" in the sense that they preserve their form and symmetry in any system of coordinates which permits an expansion in powers of $\lambda$.† This is a reasonable conclusion, in view of the presence of delta-functions in $\tilde{T}^{\alpha\beta}$. For, suppose the covariant expression

$$\tilde{T}^{\alpha\beta}_{;\beta} d\Omega \equiv \tilde{T}^{\alpha\beta}_{;\beta} dV dx^0 = 0$$

is integrated over an extended four-volume $\Omega_a$ containing the corresponding a-th particle at the world point $x^{\alpha}_a$. Then it is well known that upon going to the limit $\Omega_a \to 0$ we again obtain a covariant expression. Since $\tilde{T}^{\alpha\beta}$ contains delta-functions, we can make use of a three-dimensional volume $V_a$ with extended boundaries. The theorem of the mean can be applied to the integral over $dx^0$. All this leads to

$$\int\limits_{\Omega_a} \tilde{T}^{\alpha\beta}_{;\beta} d\Omega = \int\limits_{\Delta x^0} dx^0 \int\limits_{V_a} \tilde{T}^{\alpha\beta}_{;\beta} dV = \Delta x^0 \Big[ \int\limits_{V_a} \tilde{T}^{\alpha\beta}_{;\beta} dV + O(\Delta x^0) \Big] = 0,$$

where $O(\Delta x^0) \to 0$ as $\Delta x^0 \to 0$. Upon dividing the last equation by $\Delta x^0$ and then going to the limit as $\Delta x^0 \to 0$ (that is to say, in the limit as $\Omega_a \to 0$) we obtain Eqs. (1.5).‡

2. The equations of translational motion can be derived from (1.5) with the aid of an approximate method. The particular method which is to be used will depend fundamentally on the choice of an expansion parameter $\lambda$, chosen in the following way. Let $\varphi(x^{\alpha})$ be any of the functions occurring in (1.1). Assume that, in the system of units where the speed of light c is unity, $\partial\varphi/\partial x^0 \ll \partial\varphi/\partial x^i$, i.e., $\varphi$ varies rapidly with position in space, but changes slowly in time. (This assumption implies that the speed of the moving objects which create the gravitational field is small compared with the speed of light.) Hence we may say that $\partial\varphi/\partial x^0$ is an order of magnitude smaller than $\partial\varphi/\partial x^i$. In order to express this analytically, let us introduce the auxiliary time variable

$$\tau = \lambda x^0, \tag{2.1}$$

where the parameter $\lambda$ is chosen so that $\varphi_{,i} \equiv \partial\varphi/\partial x^i$ and $\varphi_{,0} \equiv \partial\varphi/\partial\tau$ are of the same order of smallness. We now expand all the functions occurring in (1.1) in powers of $\lambda$ in the following way. From

*Repeated indices are to be summed, as usual. Semicolons represent covariant differentiation, and commas represent the usual partial differentiation. g is the determinant consisting of the components of the covariant metric tensor $g_{\alpha\beta}$.

†Such coordinate systems are essentially restricted to those in which a unique solution of the equations of motion exists. (They are Galilean in the zeroth approximation and harmonic in the first.) The coordinate conditions which arise in this problem have been discussed many times in the literature,[5,9-14] and since they have no direct bearing here and require a special treatment, they will not be considered now.

‡Another argument for the "covariance" of (1.5) is the fact that the equations of motion derived from them by Infeld,[10] for non-rotating particles, agree exactly with the equations derived earlier.[3,5,7,8]

Newton's law of universal gravitation we deduce that the expansion of $m_a$ must begin with the second degree, since the acceleration $d^2a^i/dx^{02} = \lambda^2 d^2a^i/d\tau^2 \equiv \lambda^2 \ddot{a}^i$ is of the second degree. Therefore, according to Infeld,[10] we assume the expansion of $m_a$ to be of the form

$$m_a = \lambda^2 \underset{2}{m_a} + \lambda^4 \underset{4}{m_a} + \cdots, \tag{2.2}$$

which determines the expansions of all the other functions through the use of (1.1). For example,

$$\begin{aligned} \tilde{T}^{00} &= \lambda^2 \underset{2}{\tilde{T}^{00}} + \lambda^4 \underset{4}{\tilde{T}^{00}} + \ldots, \\ \tilde{T}^{0i} &= \lambda^3 \underset{3}{\tilde{T}^{0i}} + \lambda^5 \underset{5}{\tilde{T}^{0i}} + \ldots, \\ \tilde{T}^{ik} &= \lambda^4 \underset{4}{\tilde{T}^{ik}} + \lambda^6 \underset{6}{\tilde{T}^{ik}} + \ldots; \end{aligned} \tag{2.3}$$

$$\begin{aligned} g_{00} &= 1 + \lambda^2 \underset{2}{h_{00}} + \lambda^4 \underset{4}{h_{00}} + \ldots, \\ g_{0i} &= \qquad \lambda^3 \underset{3}{h_{0i}} + \ldots, \\ g_{ik} &= -\delta_{ik} + \lambda^2 \underset{2}{h_{ik}} + \ldots. \end{aligned} \tag{2.4}$$

The expansion of $g^{\alpha\beta}$ is found from the conditions $g_{\alpha\beta}g^{\beta\sigma} = \delta^{\sigma}_{\alpha}$; as shown by Infeld,[15] successive terms in these series differ by two orders.

3. We now have to choose $\tilde{T}^{\alpha\beta}$ for the case of rotating particles. For this purpose, consider a number of objects $a$ of finite size, whose centers of gravity move in a Euclidean space with velocities $\mathbf{v}_a(\tau)$ while at the same time they rotate rigidly around their centers of gravity with angular velocities $\boldsymbol{\omega}_a(\tau)$. Then the velocity of any point in an object is given by the sum $\mathbf{v}'_a = \mathbf{v}_a + \boldsymbol{\omega}_a \times \mathbf{r}_a$, where $\mathbf{r}_a \equiv \mathbf{r} - \mathbf{a}$ is the radius vector from the object's center of gravity to the point in question. The following two questions now arise: (1) To what does the additional term $\boldsymbol{\omega}_a \times \mathbf{r}_a$ reduce when the object shrinks to a point at its center of gravity; and (2) in what way is this to be included in $\tilde{T}^{\alpha\beta}$?

Using an analogy from the hydrodynamics of vortical motion and from the determination of the vector field in classical field theory[16] (particles with spin correspond to our rotating, infinitesimal, spherically symmetrical objects) we assume that $\boldsymbol{\omega}_a \times \mathbf{r}_a$ can be replaced by

$$1/2\,[\boldsymbol{\sigma}_a \times \nabla\delta_a], \tag{3.1}$$

where $\boldsymbol{\sigma}_a = \boldsymbol{\sigma}_a(\tau)$ is a pseudovector function of $\tau$, and $\nabla \equiv \mathbf{e}_i\,\partial/\partial x^i$ operates on $\delta_a$.

To find $\tilde{T}^{\alpha\beta}$ we shall again start from (1.3), but this time the four-velocity $U^{\alpha}$ must include the additional effect of rotation, (3.1). To see how this additional effect can be introduced into $U^{\alpha}$, consider an arbitrary object $a$ of finite dimensions, the position of whose center of gravity is described by a point $M_a(x^{\alpha}_a)$ in a Riemann space-time with a metric determined by the system of $n$ particles which we are considering. If a point $M(x^{\alpha})$ is infinitely close to the point $M_a$, i.e., $x^{\alpha} = x^{\alpha}_a + \Delta x^{\alpha}$, then its four-velocity is (noting that $x^i_a = a^i$, and separating the space and time components)

$$U = \left\{\frac{dx^i}{ds};\ \frac{dx^0}{ds}\right\} = \left\{\frac{dx^0}{ds}\frac{dx^0_a}{dx^0}\left(\frac{da^i}{dx^0_a} + \frac{d\Delta x^i}{dx^0_a}\right);\ \frac{dx^0}{ds}\right\}, \tag{3.2}$$

where the quantity $d\Delta x^i/dx^0_a$ describes the velocity of the point $x^i$ relative to the point $x^i_a = a^i$, measured at the time $x^0_a$. Upon going to the limit (shrinking the object to a point at its center of gravity) this leads to (3.1), as expected. As for the ratio $dx^0_a/dx^0$, it is easy to expand it in $\lambda$:

$$q_a \equiv dx^0_a / dx^0 = d\tau_a / d\tau = 1 + \lambda^4 \underset{4}{q_a} + \ldots \tag{3.3}$$

(the equation $U^{\alpha}U_{\alpha} = 1$ must be solved for $dx^0_a/dx^0$ by approximations or by Landau's Eq. (82.7). In the absence of rotation, this reduces to unity. Substituting (3.2) into (1.3), taking account of (3.1) and (3.3), and introducing the delta-functions, we obtain

$$\tilde{T}^{\alpha\beta} = \sum_a \tilde{T}^{\alpha\beta}_a,$$

$$\tilde{T}^{00}_a = m_a\delta_a, \qquad \tilde{T}^{0i}_a = m_a q_a\,(\lambda\dot{a}^i + 1/2\,[\sigma_a\times\nabla]^i)\delta_a, \qquad \tilde{T}^{ik}_a = m_a q_a^2\,(\lambda\dot{a}^i + 1/2\,[\sigma_a\times\nabla]^i)\,(\lambda\dot{a}^k + 1/2\,[\sigma_a\times\nabla]^k)\,\delta_a. \tag{3.4}$$

Here $m_a$ has an expansion in the form of (2.2), but is not the same as the $m_a$ in (1.2). The pseudovector $\sigma_a$ has the expansion

$$\sigma_a = \lambda \underset{1}{\sigma}_a + \lambda^3 \underset{3}{\sigma}_a + \dots, \tag{3.5}$$

since we naturally require that the tensor density $\tilde{T}^{\alpha\beta}$ in (3.4) have an expansion of the form (2.3) in $\lambda$. $\dot{a}^i \equiv da^i/d\tau$, and $\nabla$ operates on $\delta_a = \delta(x^i - a^i)$. When $\sigma_a = 0$, (3.4) reduces to (1.2). Notice also that in the following derivation of the equations of motion, in the Newtonian approximation, we make use of $\underset{5}{\tilde{T}}{}^{0i}$ and $\underset{6}{\tilde{T}}{}^{ik}$ but no terms of higher order, so that in (3.4) we may put $q_a = 1$.

Once $\tilde{T}^{\alpha\beta}$ has been found, we can obtain the equations of motion from (1.5). But now, in addition to the 4n unknown functions $m_a(\tau)$ and $a^i(\tau)$, which in the absence of rotation are completely determined by the 4n ordinary differential equations (1.5), another 3n unknown functions $\sigma_a^i(\tau)$ make their appearance. Equations (1.5) must therefore be supplemented by the 3n equations

$$\int_{V_a} (x^i \tilde{T}^{k\alpha}_{;\alpha} - x^k \tilde{T}^{i\alpha}_{;\alpha})\, dV = 0, \tag{3.6}$$

from which the equations of rotational motion are to be derived, and which also arise from Einstein's gravitational equation (1.1). The derivation of equation (3.6) is completely analogous to the derivation of (1.5) if we start with the covariant expression

$$(x^\alpha \tilde{T}^{\beta\nu}_{;\nu} - x^\beta \tilde{T}^{\alpha\nu}_{;\nu})\, d\Omega = 0.$$

4. The derivation of the equations of motion requires a knowledge of the metric, to the corresponding order of approximation, which means that $\underset{2}{h}_{00}$, $\underset{2}{h}_{ik}$, $\underset{3}{h}_{0i}$, and $\underset{4}{h}_{00}$ have to be determined from (1.1).

In our case the equations for determining $\underset{2}{h}_{00}$ and $\underset{2}{h}_{ik}$ are identical with the corresponding equations of Infeld,[10] and therefore they have the identical solution

$$\underset{2}{h}_{00} = -2\sum_a \underset{2}{m}_a / r_a, \quad r_a \equiv |\mathbf{r} - \mathbf{a}|, \tag{4.1}$$

$$\underset{2}{h}_{ik} = \delta_{ik} \underset{2}{h}_{00}. \tag{4.2}$$

However, the equations which determine $\underset{3}{h}_{0i}$ are no longer the same as Infeld's, but are of the form

$$\underset{3}{h}_{0i,ss} - \underset{3}{h}_{0s,si} = -2\underset{2}{h}_{00,\underset{1}{0}i} - 16\pi \sum_a \underset{2}{m}_a \dot{a}^i \delta_a - 8\pi \sum_a \underset{2}{m}_a [\underset{1}{\sigma}_a \times \nabla \delta_a]^i. \tag{4.3}$$

We write the solution to this in the form

$$\underset{3}{h}_{0i} = 4\sum_a \frac{\underset{2}{m}_a \dot{a}^i}{r_a} - 2\sum_a \underset{2}{m}_a \left[\underset{1}{\sigma}_a \times \nabla \frac{1}{r_a}\right]^i, \tag{4.4}$$

which differs from the corresponding expression in Infeld's paper[10] by the presence of the second term. Comparing the latter with (100-7) [sic!] in Ref. 14, we are led to an explanation of the physical significance of $\sigma_a$; this is nothing but the specific angular momentum for the a-th particle: $M_a = m_a \sigma_a$.

Knowing $\underset{2}{h}_{00}$, $\underset{2}{h}_{ik}$, and $\underset{3}{h}_{0i}$, we can derive the third, fourth, and fifth order equations of translational motion, as well as the fourth order rotational equations, from Eqs. (1.5) and (3.6). In the third order, the equations of translation (1.5) lead to the relation

$$\underset{2}{m}_a = \text{const}; \tag{4.5}$$

in the fourth order, to the Newtonian equations of motion

$$\underset{2}{m}_a \ddot{a}^i = \sum_{b \neq a} (\underset{2}{m}_a \underset{2}{m}_b / r_{ab})_{,a^i}; \tag{4.6}$$

and in the fifth order, to a value of $\underset{4}{m}_a$ identical with that of Infeld:[10]

$$\underset{4}{m}_a = {}^1/_2 \underset{2}{m}_a \dot{a}^s \dot{a}^s + \sum_{b \neq a} \underset{2}{m}_a \underset{2}{m}_b / r_{ab} + \underset{4}{C}_a, \tag{4.7}$$

where $\underset{4}{C}_a$ is a constant, and $r_{ab} \equiv |a - b|$. So far there has been no effect due to the rotation — all the integrals containing $\sigma$ vanish — and our results (4.5) to (4.7) agree with Infeld's. The manipulations required to derive (4.1) to (4.7) are completely analogous to his[10] and need not be discussed in detail.

The equations of rotational motion (3.6) in the fourth (lowest) order, after a short, straightforward development, lead to the simple relations

$$\underset{3}{\dot{M}}{}_a^{ik} = 0, \quad \underset{3}{M}{}_a^{ik} = \text{const}; \text{ for } \underset{1}{\dot{\sigma}}_a = 0, \quad \underset{1}{\sigma}_a = \text{const}, \tag{4.8}$$

where

$$\underset{3}{M}{}_a^{ik} \equiv \delta_{ikl} \underset{2}{m}_a \underset{1}{\sigma}{}_a^l \tag{4.9}$$

($\delta_{ikl} = \pm 1$, $\delta_{123} = 1$ is a completely antisymmetric pseudotensor). Thus in this Newtonian approximation the angular momentum of each spherically symmetrical object is individually conserved, as expected.

5. Derivation of the sixth-order rotational equation of motion requires a knowledge of $\underset{4}{h}_{00}$. Infeld[10] carried out a detailed derivation of those terms in $\underset{4}{h}_{00}$ which affect the sixth-order equations of motion without rotation. Therefore in this paper we shall consider only the additional terms $\underset{4}{h}{}^*_{00}$ due to the rotation. By methods similar to Infeld's, we arrive at the following equations for $\underset{4}{h}{}^*_{00}$:

$$\underset{4}{h}{}^*_{00,\,ss} = 8\pi \sum_a \underset{2}{m}_a \left(\dot{a}^s\, [\underset{1}{\sigma}_a \times \nabla\delta_a]^s + \frac{1}{4}\, [\underset{1}{\sigma}_a \times \nabla]^s\, [\underset{1}{\sigma}_a \times \nabla\delta_a]^s\right). \tag{5.1}$$

The solution of this takes the form

$$\underset{4}{h}{}^*_{00} = 2 \sum_a \underset{2}{m}_a \left(\dot{a}^s \left[\underset{1}{\sigma}_a \times \nabla \frac{1}{r_a}\right]^s - \frac{1}{4}\, [\underset{1}{\sigma}_a \nabla]^s \left[\underset{1}{\sigma}_a \times \nabla \frac{1}{r_a}\right]^s\right). \tag{5.2}$$

Now by virtue of the linearity of the original differential equations, up to terms involved in the sixth-order equations of motion for the a-th object, $\underset{4}{h}_{00}$ may be expressed in the form

$$\underset{4}{h}_{00} \sim 2\left(\sum_{b \neq a} \frac{\underset{2}{m}_b}{r_b}\right)^2 + \sum_{b \neq a} \left(- \underset{2}{m}_b \underset{11}{r_{b,\,00}} - 3\dot{b}^s\dot{b}^s \frac{\underset{2}{m}_b}{r_b} + 2\frac{\underset{2}{m}_a \underset{2}{m}_b}{r_{ab} r_b}\right) + \sum_{b \neq a} \sum_{b \neq c\, c \neq a} \frac{\underset{2}{m}_b \underset{2}{m}_c}{r_{bc} r_c} + \underset{4}{h}{}^*_{00}, \tag{5.3}$$

All the terms on the right hand side, with the exception of $\underset{4}{h}{}^*_{00}$, are already available. They describe the field for non-rotating particles to the indicated degree of approximation.

The sixth-order translational equations of motion (or the second approximation, according to V. A. Fock's terminology) for body a is given by the expression

$$\int_{V_a} (\underset{6}{\tilde{T}}{}^{i\alpha}_{;\alpha})\, dV \equiv \int_{V_a} \Big[\underset{5}{\tilde{T}}{}^{i0}_{,\underset{1}{0}} + \underset{6}{\tilde{T}}{}^{is}_{,s} + \frac{1}{2} \underset{2}{h}_{00;i} \underset{4}{\tilde{T}}{}^{00} + \frac{1}{2} (\underset{4}{h}_{00,\,i} + \underset{2}{h}_{00} \underset{2}{h}_{00;i} - 2\underset{3}{h}_{0i,\underset{1}{0}})\, \underset{2}{\tilde{T}}{}^{00}$$
$$+ (\underset{3}{h}_{0s,i} - \underset{3}{h}_{0i,s})\, \underset{3}{\tilde{T}}{}^{0s} - \underset{2}{h}_{00,\underset{1}{0}}\, \underset{3}{\tilde{T}}{}^{0i} - \underset{2}{h}_{00,s} \underset{4}{\tilde{T}}{}^{si} + \frac{1}{2} \underset{2}{h}_{00,i} \underset{4}{\tilde{T}}{}^{ss}\Big]\, dV = 0. \tag{5.4}$$

If into this we substitute (4.1), (4.4), and (5.3) without the rotational terms, we obtain the well-known non-Newtonian equations of translation given, for example, by Papapetrou[8] and reducing to the expressions given by Einstein[3,5] and Petrova[7] for the case of two particles. A detailed derivation, using delta-functions, is given for the two-body problem by Infeld.[10] Since the n-body problem introduces nothing fundamentally new, we limit ourselves here to a derivation of the additional rotational terms $D_a^i$ in the equations of motion which do not appear in the detailed calculations mentioned above. If into (5.4) we substitute $\underset{2}{h}_{00}$, $\underset{3}{h}_{0i}$, and $\underset{4}{h}_{00}$, as well as the value of $\tilde{T}^{\alpha\beta}$ to the required approximation, and take out the terms containing $\sigma$, we note that some of the integrals vanish identically while others vanish because the integrand is an odd function, and we finally obtain the simple and relatively short expression

$$D_a^i = \sum_{b\neq a} \frac{\gamma m_a m_b}{c^2} \Big\{ (\dot{a}^s - \dot{b}^s) \left[ \sigma_a \times \nabla \left(\frac{1}{r_{ab}}\right)_{,a^s} \right]^i - 2(\dot{a}^s - \dot{b}^s) \left[ \sigma_b \times \nabla \left(\frac{1}{r_{ab}}\right)_{,a^s} \right]^i + (2\dot{a}^s - \dot{b}^s) \left[ \sigma_b \times \nabla \left(\frac{1}{r_{ab}}\right)_{,a^i} \right]^s$$

$$+ (2\dot{b}^s - \dot{a}^s) \left[ \sigma_a \times \nabla \left(\frac{1}{r_{ab}}\right)_{,a^i} \right]^s - [\sigma_a \times \nabla]^s \left[ \sigma_b \times \nabla \left(\frac{1}{r_{ab}}\right)_{,a^i} \right]^s + \frac{1}{4} [\sigma_a \times \nabla]^s \left[ \sigma_a \times \nabla \left(\frac{1}{r_{ab}}\right)_{,a^i} \right]^s + \frac{1}{4} [\sigma_b \times \nabla]^s \left[ \sigma_b \times \nabla \left(\frac{1}{r_{ab}}\right)_{,a^i} \right]^s \Big\} \quad (5.5)$$

(here $\nabla \equiv e_i \partial/\partial a^i$). The relativistic equations of translation can now be written in the form:

$$m_a \ddot{a}^i = \sum_{b\neq a} \left( \frac{\gamma m_a m_b}{r_{ab}} \right)_{,a^i} + F_a^i + D_a^i, \quad (5.6)$$

where $F_a^i$ are the terms to be added to the Newtonian forces if rotational effects are ignored, and which are given, for instance, by Papapetrou.[8] Expressions (5.5) and (5.6) are written in the usual c.g.s. system of units. To transform to this system from the original system of units, we first eliminate the parameter $\lambda$ by introducing new units of time and mass according to the formulae: old $\tau$ = new $\tau \times \lambda$; old mass = new mass $\times \lambda^{-2}$. We then introduce the usual values of Newtonian constant $\gamma$ and the speed of light c. To do this we must replace m by $\gamma m$, insert the factor $c^{-2}$ on the right hand side in $D_a^i$, and remember that when taking time derivatives (represented by dots over the variables) the vector components are to be differentiated with respect to the usual time variable t. Then $m_a$ and $m_b$ are the Newtonian masses, and $\sigma_a$ and $\sigma_b$ are the Newtonian angular momenta of the objects (if the preceding terms $\underset{2}{m_a}$, $\underset{2}{m_b}$, $\underset{1}{\sigma_a}$, and $\underset{1}{\sigma_b}$ are constants). As to the structure of the $D_a^i$, notice that one group of terms (the first four terms) describes a "spin-orbit" interaction between the particles, while another group (the remaining terms) describes a "spin-spin" interaction. These interactions lead to some additional forces which will effect the orbital motions of the particles.

6. We now can derive the sixth-order equations of rotational motion. From (3.6) we have

$$\int_{V_a} [x^i (\underset{6}{\tilde{T}^{k\alpha}_{;\alpha}}) - x^k (\underset{6}{\tilde{T}^{i\alpha}_{;\alpha}})] dV = \int_{V_a} [(x^i - a^i)(\underset{6}{\tilde{T}^{k\alpha}_{;\alpha}}) - (x^k - a^k)(\underset{6}{\tilde{T}^{i\alpha}_{;\alpha}})] \, dV = 0 \quad (6.1)$$

The expressions for $\underset{6}{\tilde{T}^{i\alpha}_{;\alpha}}$ are given in (5.4). Relatively simple and straightforward calculations show that some of the integrals vanish,* and that the remainer lead to the following non-Newtonian equations of rotational motion (again we have transformed to the c.g.s. system):

$$\dot{M}_a^{ik} = \frac{\gamma}{c^2} \sum_{b\neq a} m_b \Big\{ 2M_{a(0)}^{ik} (\dot{b}^s - \dot{a}^s) \left(\frac{1}{r_{ab}}\right)_{,a^s} + M_{a(0)}^{is} \left[ (\dot{a}^s - 2\dot{b}^s) \left(\frac{1}{r_{ab}}\right)_{,a^k} - (\dot{a}^k - 2\dot{b}^k) \left(\frac{1}{r_{ab}}\right)_{,a^s} \right]$$

$$- M_{a(0)}^{ks} \left[ (\dot{a}^s - 2\dot{b}^s) \left(\frac{1}{r_{ab}}\right)_{,a^i} - (\dot{a}^i - 2\dot{b}^i) \left(\frac{1}{r_{ab}}\right)_{,a^s} \right] + M_{a(0)}^{is} \Big( \frac{1}{2} \left[ \sigma_a \times \nabla \left(\frac{1}{r_{ab}}\right)_{,a^s} \right]^k - \left[ \sigma_b \times \nabla \left(\frac{1}{r_{ab}}\right)_{,a^s} \right]^k - \frac{1}{2} \left[ \sigma_a \times \nabla \left(\frac{1}{r_{ab}}\right)_{,a^k} \right]^s$$

$$+ \left[ \sigma_b \times \nabla \left(\frac{1}{r_{ab}}\right)_{,a^k} \right]^s \Big) - M_{a(0)}^{ks} \Big( \frac{1}{2} \left[ \sigma_a \times \nabla \left(\frac{1}{r_{ab}}\right)_{,a^s} \right]^i - \left[ \sigma_b \times \nabla \left(\frac{1}{r_{ab}}\right)_{,a^s} \right]^i - \frac{1}{2} \left[ \sigma_a \times \nabla \left(\frac{1}{r_{ab}}\right)_{,a^i} \right]^s + \left[ \sigma_b \times \nabla \left(\frac{1}{r_{ab}}\right)_{,a^i} \right]^s \Big) \Big\}. \quad (6.2)$$

Here, as in (5.5), $m_a$, $m_b$, ... and $\sigma_a$, $\sigma_b$, ... represent the Newtonian (i.e., constant) values of mass and specific angular momentum of the objects a, b, ...; $M_{a(0)}^{ik}$ denotes the Newtonian (constant) angular momentum of the a-th object, obtained by expressing (4.9) in the c.g.s. system. The left-hand side of (6.2) is $\dot{M}_a^{ik} \equiv dM_a^{ik}/dt$, the time rate of change of angular momentum of the a-th object in the first non-Newtonian approximation (including the effect of mass changes due to velocity). An expression for the angular momentum can be obtained from

$$M_a^{ik} = \lambda^3 \underset{3}{M_a^{ik}} + \lambda^5 \underset{5}{M_a^{ik}} \equiv \lambda^3 \delta_{ikl} \underset{2}{m_a} \underset{1}{\sigma_a^l} + \lambda^5 \delta_{ikl} (\underset{4}{m_a} \underset{1}{\sigma_a^l} + \underset{2}{m_a} \underset{3}{\sigma_a^l}) \quad (6.3)$$

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*In particular, it is found that

$$\int_{V_a} [(x^i - a^i) \underset{4}{h_{00,k}} - (x^k - a^k) \underset{4}{h_{00,i}}] \underset{2}{\tilde{T}^{00}} dV = 0,$$

where $\underset{4}{h_{00}}$ does not refer to the approximation of (5.3), but to its exact value. Hence we may conclude generally that $\underset{4}{h_{00}}$ does not affect the sixth-order equations of rotation.

by transforming to c.g.s. units.

The relativistic equations of rotational motion (6.2), thus developed, contain two fundamentally distinct groups of terms. One group (the first three terms) describes the "spin-orbit" interaction, while all the remaining terms describe the "spin-spin" interaction of the objects. We shall see that the existence of these interactions alters the angular momentum of spherically symmetrical objects even in the first non-Newtonian approximation.

7. It is not the object of this paper to make a detailed study of the equations of motion which have just been derived. Nevertheless, we must satisfy ourselves that they do not conflict with any known facts. For example, it is well known (cf. Landau and Lifshitz,[17] pp. 291 and 341) that in the gravitational field of an object b of mass $m_b$ and angular momentum $M_b$, a light non-rotating particle a of mass $m_a$ is acted on by a force, analogous to a Coriolis force, equal to

$$m_a [\dot{\mathbf{a}} \times [\nabla \times \mathbf{h}_b]], \text{ where } \mathbf{h}_b = \frac{-2\gamma m_b}{c^2}\left[\boldsymbol{\sigma}_b \times \nabla \frac{1}{r_{ab}}\right]. \quad (7.1)$$

To compare (5.5) with (7.1) we have to put $\dot{b}^s = 0$, $\sigma_a = 0$, and omit the summation sign. A few transformations applied to the resulting expression does in fact lead to (7.1), plus the last term of (5.5), which has not previously been derived.

It is also interesting to compare our results, in a general way, with those obtained by Fock.[9] If Fock's extra translational and rotational terms are applied to infinitesimal spherical particles, we obtain a group of terms corresponding to the first four terms of (5.5). (We naturally ignore those terms in Fock's equations which arise from the internal structure of the objects, since this has no meaning in the present case.) Fock's equations contain no terms quadratic in $\sigma$ and none of the fourth order in $1/r_{ab}$, since his method uses not only a development in the parameter v/c but also a development in the parameter L/R (L being a characteristic linear dimension of the objects and R the distance between them) and the development is broken off at terms of the third order in L/R. For large separations (as in systems of astronomical bodies) the terms in $r_{ab}^{-4}$ are not essential; but for systems such as double stars (close, massive pairs) these terms could possibly make a significant contribution.

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