In conclusion, the author wishes to express his gratitude to M. M. Agrest for valuable advice and discussion.

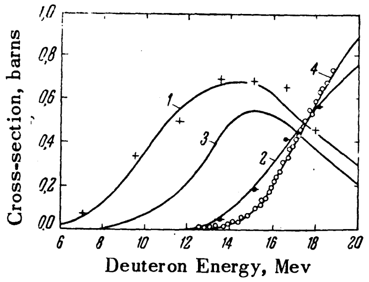

FIG. 3. Dependence of the Bi^{209}_{83} reaction cross-section on the deuteron energy: 1 — (d, 2n); 2 — (d, 3n); and of the I^{127}_{53} cross-section: 3 — (d, 2n); 4 — (d, 3n). Solid curves drawn according to Eq. (24); +, ● — data of Ref. 11; ○ — data of Ref. 3.

[1] J. Blatt and V. Weisskopf, Theoretical Nuclear Physics, Wiley, 1952.
[2] H. Bethe, Elementary Nuclear Physics, Wiley, N.Y. 1947.
[3] E. L. Kelly and E. Segrè, Phys. Rev. 75, 999 (1949).
[4] E. Segrè (editor), Experimental Nuclear Physics, vol 2, part II, Wiley, 1953.
[5] H. A. Bethe and E. I. Konopinski, Phys. Rev. 54, 130 (1938).
[6] K. I. Le Couter, Proc. Phys. Soc. A63, 259 (1950).
[7] I. Heidman and H. A. Bethe, Phys. Rev. 83, 237, 84, 274 (1951) Проблемы современной физики (Problems of Modern Physics) 8, 153 (1952).
[8] V. P. Weisskopf and D. H. Ewing, Phys. Rev. 57, 472 (1940).
[9] S. N. Choshal and T. N. Dave, Ind. J. Phys. 27, 213 (1953).
[10] E. E. Slutskii, (Tables of Γ and Probability Functions), Moscow, 1950.
[11] S. J. Balestrini, Phys. Rev. 95, 1502 (1954).

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SOME PROBLEMS OF MAGNETOGASDYNAMICS WITH ACCOUNT OF FINITE CONDUCTIVITY

G. S. GOLITSYN and K. P. STANIUKOVICH

Moscow State University

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It is shown that if finite conductivity is taken into account, the equations of magnetogasdynamics become parabolically degenerate. The set of equations is replaced by an approximate but completely hyperbolic set, for which the characteristics are found. It is shown that the equations of a stationary one-dimensional flow have a singularity where the flow velocity is equal to the local sound velocity. Conditions of the transition of the flow velocity through this critical value under the action of a magnetic field have been studied. Small oscillations in a conducting medium, shock waves, and the structure of the shock are investigated.

THE magnetogasdynamics of an ideally conducting medium have been sufficiently studied. Types of vibration,[1,2] shock waves,[3,4] and their structure[5-7] have been investigated; one-dimensional motions have been studied,[8] where the characteristics were found for a system of equations and the particular (Reinmann) solution was found for arbitrary isentropy.

Taking the finite conductivity into account greatly complicates the equation by introducing new nonlinearities, raising the order of the system, and changing its character. As will be shown below, the system of equations changes from completely hyperbolic to parabolically degenerate. Finally, account of the Joule heat makes the motion essentially non-isentropic.

Small vibrations were first studied;[9] then, the possible types of arbitrary vibrations of the medium[11,12] were treated on the basis of the general wave equation.[10] Recently, Staniukovich found the particular solution by a perturbation method[13] for the case in which the conductivity is very large but finite.

In the present paper, several problems of one-dimensional motion are analyzed for a medium possessing arbitrary conductivity.

1. EQUATIONS EMPLOYED

If $\mathbf{v} \perp \mathbf{H} \perp \mathbf{E}$, where $\mathbf{v} = (u, 0, 0)$, $\mathbf{H} = (0, H, 0)$, and $\mathbf{E} = (0, 0, E)$, then the system of equations of magnetogasdynamics has, for the one-dimensional case, the form:[13]

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0, \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{h}{\rho}\left(\frac{\partial h}{\partial x} - \frac{\partial e}{c dt}\right) = 0,$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \gamma p\frac{\partial u}{\partial x} = (\gamma - 1)\varkappa\left(\frac{\partial h}{\partial x} - \frac{\partial e}{c\partial t}\right)^2, \quad \frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} = \frac{\partial}{\partial x}\left[\varkappa\left(\frac{\partial h}{\partial x} - \frac{\partial e}{c\partial t}\right)\right], \quad \frac{\partial e}{\partial x} = \frac{\partial h}{c\partial t}. \tag{1}$$

Here, in addition to the usual notation, we have used

$$h = H/\sqrt{4\pi}; \quad e = E/\sqrt{4\pi}; \quad \gamma = c_p/c_v; \quad \varkappa = c^2/4\pi\sigma,$$

where c is the velocity of light, and σ is the conductivity of the medium, which we shall henceforth consider constant. The first equation of (1) is the equation of continuity, the second is the equation of motion, the third is the energy equation, the last two are Maxwell's equations with account of Ohm's law for a moving medium.

If we neglect the displacement current, then the system (1) is simplified:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0, \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial x}\left(p + \frac{h^2}{2}\right) = 0,$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \gamma p\frac{\partial u}{\partial x} = (\gamma - 1)\varkappa\left(\frac{\partial h}{\partial x}\right)^2, \qquad \frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} = \varkappa\frac{\partial^2 h}{\partial x^2}. \tag{2}$$

2. GENERAL PROPERTIES OF THE SYSTEM OF EQUATIONS

We shall show that the set of equations (2), together with (1), is parabolically degenerate.* We write down the set (2) as a set of five first-order equations for five unknown functions:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{hz}{\rho} = 0,$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \gamma p\frac{\partial u}{\partial x} - (\gamma - 1)\varkappa z^2 = 0, \quad \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} - \varkappa\frac{\partial z}{\partial x} = 0, \quad \frac{\partial h}{\partial x} - z = 0. \tag{2a}$$

We convert the system (2a) to the characteristic form[14]

$$C(y) = \begin{vmatrix} y_1 + uy_2 & \rho & 0 & 0 & 0 \\ 0 & y_1 + uy_2 & 1/\rho & 0 & 0 \\ 0 & \gamma p & y_1 + uy_2 & 0 & 0 \\ 0 & h & 0 & y_1 + uy_2 & -\varkappa y_2 \\ 0 & 0 & 0 & y_2 & 0 \end{vmatrix}.$$

The system would be completely hyperbolic if the algebraic equation $C(y) = 0$ had five real, not necessarily different, roots. But we have the equation

$$y_2^2(y_1 + uy_2)[(y_1 + uy_2)^2 - \gamma p/\rho] = 0,$$

which has only three roots.

*The possibility of parabolic degeneracy was pointed out to us by B. L. Rozhdestvenskii.

In similar fashion, if we look at the set (1), we then obtain a set of six equations for the six unknown functions. The corresponding characteristic equation $C = 0$ has only five roots for y_1.

The parabolic degeneracy is unfortunate in that it deprives us of the possibility of studying the system by the method of characteristics. However, we can replace the set (2) by an approximate set of the hyperbolic type. Let $\partial^2 h/\partial x^2$ be bounded everywhere. We then write

$$(\partial h / \partial x)^2 = b_1 \partial h / \partial x, \quad \partial^2 h / \partial x^2 = b_2 \partial h / \partial x,$$

where

$$b_1 = \overline{\frac{\partial h}{\partial x}}, \quad b_2 = \overline{\frac{\partial}{\partial x}\left(\ln \frac{\partial h}{\partial x}\right)}.$$

After such an averaging, we can write the set (2) in the form

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = \rho \frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{h}{\rho}\frac{\partial h}{\partial x} = 0,$$

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + \gamma p \frac{\partial u}{\partial x} - (\gamma - 1) \varkappa b_1 \frac{\partial h}{\partial x} = 0, \quad \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} - \varkappa b_2 \frac{\partial h}{\partial x} = 0. \tag{2b}$$

It is not difficult to verify the fact that this system is completely hyperbolic. The characteristic relation for it has the form

$$\begin{vmatrix} u - \dot{x} & \rho & 0 & 0 \\ 0 & u - \dot{x} & 1/\rho & h/\rho \\ 0 & \gamma p & u - \dot{x} & -\varkappa b_1 (\gamma - 1) \\ 0 & h & 0 & u - \dot{x} - \varkappa b_2 \end{vmatrix} = 0.$$

Expanding the determinant, we first get

$$u - \dot{x} = 0; \quad dx/dt = u. \tag{3}$$

This is the equation of the flow line. Along it, the equation

$$dQ = T\,dS = (j^2/\rho\sigma)\,dt, \tag{4}$$

is satisfied, since $dQ/dt = j^2/\rho\sigma$ is the influx of Joule heat per unit time. Furthermore, we get

$$(u - \dot{x})^2 = a_m^2 + \varkappa \frac{b_2 [(u - \dot{x})^2 - a_0^2] + h b_1 (\gamma - 1)/\rho}{u - \dot{x}},$$

where

$$a_m^2 = \gamma p/\rho + h^2/\rho = a_0^2 + V^2.$$

In the zeroth approximation, $\sigma \to \infty$, i.e., $\kappa \to 0$, and we get the well known result[8]

$$dx/dt = u \pm a_m. \tag{5}$$

If we take into account terms of first order of smallness in κ, we can compute

$$dx/dt = u \pm a_m \mp (\varkappa / 2a_m^2)\,[b_2 V^2 + h b_1 (\gamma - 1)/\rho]. \tag{6}$$

The equation

$$d\left(p + \frac{h^2}{2}\right) = \pm \rho a_m\, du \mp \frac{\varkappa}{a_m^2}\left[b_2 V^2 + \frac{h}{\rho} b_1 (\gamma - 1)\right]\left(a_m \frac{dh}{h} + \frac{du}{2}\right), \tag{7}$$

which is computed in the same approximation, is satisfied, along this sytem of characteristics.

It is evident that the velocity of transfer of the excitation in the gas diminishes with increasing κ, i.e., decreasing conductivity.

3. STATIONARY MOTIONS

In this case, the set (2) becomes a set of ordinary differential equations

$$\frac{d\rho u}{dx}=0,\quad \rho u\frac{du}{dx}+\frac{d}{dx}\left(p+\frac{h^2}{2}\right)=0,\quad u\frac{dp}{dx}+\gamma p\frac{du}{dx}=(\gamma-1)\varkappa\left(\frac{dh}{dx}\right)^2,\quad \frac{dhu}{dx}=\varkappa\frac{d^2h}{dx^2}. \tag{8}$$

This set has a series of integrals

$$\rho u=m,\quad mu+p+h^2/2=J,\quad u\,dp/dx+\gamma p\,du/dx=(\gamma-1)\varkappa(dh/dx)^2,\quad hu=\varkappa\,dh/dx+mb, \tag{9}$$

where m, J, and b are constant quantities. From (9) we can obtain a single second-order equation for h′:

$$h''=\frac{h'}{h}\left[h'+\frac{mb}{\varkappa}+\frac{(\gamma h'+mb/\varkappa)\,h^3}{\gamma h(J-h^2/2)-(\gamma+1)\,m\,(\varkappa h'+mb)}\right]. \tag{10}$$

It has a singularity where

$$\gamma h(J-h^2/2)-(\gamma+1)\,m\,(\varkappa h'+mb)=0.$$

Since

$$J-h^2/2=mu+p,\quad \varkappa h'+mb=hu,\quad m=\rho u,$$

then at the singularity $u^2=\gamma p/\rho$. Here too, as in ordinary gasdynamics, the flow velocity, which is equal to the local velocity of sound, has a critical value. It should be observed that this result is valid for any value of the conductivity, large or small, and also for values varying with distance, as is readily seen from (8) and (9). Morozov arrived at the same result by an entirely different method.[15]

The order of Eq. (10) can be lowered; it is not difficult to integrate it numerically, following all possible stationary flows. However, clarification of the behavior of quantities near the critical value of the velocity is also possible by direct consideration of the problem.

For stationary flow we can write down Bernoulli's equation

$$\frac{dp}{\rho}+\frac{du^2}{2}+\frac{dh^2}{2\rho}=0. \tag{11}$$

Let the pressure be a function of the density and entropy, $p=p(\rho, S)$. Then, using well known thermodynamic relations, we get

$$dp=a_0^2\left[d\rho-\frac{T}{c_p}\left(\frac{\partial\rho}{\partial T}\right)_p dS\right]. \tag{12}$$

Comparing (11) and (12), we get, considering that $\rho u=m=\text{const}$,

$$\rho\frac{du^2}{2}+\frac{dh^2}{2}=a_0^2\left[\frac{\rho}{u}du+\frac{T}{c_p}\left(\frac{\partial\rho}{\partial T}\right)_p dS\right].$$

Introducing $dQ=TdS$ and $M=u/a_0$, we obtain, after a number of transformations,

$$(M^2-1)\frac{du}{u}=\left(\frac{\partial\rho}{\partial T}\right)_p\frac{dQ}{\rho c_p}-\frac{hdh}{\rho a_0^2}. \tag{13}$$

In our case, $dQ=(j^2/\rho\sigma)\,dt$ and $dt=dx/u$. Maxwell's equation gives

$$j=\frac{c}{\sqrt{4\pi}}\frac{dh}{dx}.$$

As a result,

$$dQ=(\varkappa/m)(dh/dx)^2\,dx. \tag{14}$$

Equation (13) is the standard form of the Vulis law of the inversion of action,[16] which states that identical conditions produce in a flow opposite effects for subsonic and supersonic flow. For example, contraction of the jet accelerates the subsonic flow and retards the supersonic. In ordinary gasdynamics, this fact is associated with changeover of the flow under the action of the excitations propagated in it with the velocity of sound.[16] In a conducting medium in a magnetic field, the excitations are propagated with a velocity greater than $a_0 = \sqrt{\gamma p/\rho}$. Why is a_0 exactly the critical velocity in our case?

If $u = a_0$, i.e., $M = 1$, then the left side of (13) is equal to zero. For this to happen, we must have in the critical cross section

$$dh / dx = 0. \tag{15}$$

If this is so, this field will not affect the motion in precisely the critical region of flow, since the corresponding terms in the equations disappear in this region. Therefore the excitations there ought to be propagated with the ordinary velocity of sound.

The right side of (13) also vanishes for

$$h = h_0 \exp(-x/\rho a_0^2 \alpha), \text{ where } \alpha = -(\partial\rho/\partial T)_p \varkappa/m,$$

But such a special value of the field has only an accidental relation to the critical point.

Let us investigate the conditions for the possibility of going through the velocity of sound.

For $M < 1$ and an accelerating flow, the right side of (13) must be less than zero. Let us write it in the form

$$d\Sigma = \left[\left(\frac{\partial\rho}{\partial T}\right)_p \frac{\varkappa}{m}\left(\frac{dh}{dx}\right)^2 - \frac{h}{\rho a_0^2}\frac{dh}{dx}\right]dx = -\left[\alpha\left(\frac{dh}{dx}\right)^2 + \frac{h}{\rho a_0^2}\frac{dh}{dx}\right]dx.$$

In our case of an ideal gas, $(\partial\rho/\partial T)_p = -\rho/T < 0$, therefore $\alpha > 0$; $d\Sigma < 0$, if $dh/dx > 0$ and if $h/\rho a_0^2 > -\alpha dh/dx$. Thus, if the field increases with x, or falls more slowly than $h_0 \exp(-x/\rho a_0^2\alpha)$, then the subsonic flow is accelerated.

For $M > 1$, $d\Sigma > 0$, and for an accelerating supersonic flow, the field must fall more rapidly than $h_0 \exp(-x/\rho a_0^2 \alpha)$. In the opposite case, the sign of the effect does not change, and in a stationary mode the flow velocity cannot be greater than a_0. Thus, by passing definite currents through the gas, we can accelerate the ionized gas, i.e., we can construct a magnetic jet of a kind.

It is not difficult to obtain expressions analogous to (13) for p, ρ and h, and to carry through other detailed studies just as for ordinary flows.[16] The behavior of all the quantities close to the critical value is given in Fig. 1.

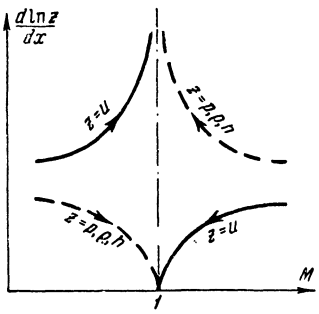

FIG. 1. Diagram of the current relationship. The direction of the process is shown by the arrows.

Although the critical velocity of flow, which is equal to the local sound velocity, does not depend explicitly on the field, it depends on the field implicitly, since ρ and h are connected with one another in a conducting medium. It is possible to find the dependence of the critical velocity a_* on the field for an ideally conducting medium and to estimate the dependence in our case also.

Let us transform the expression for the flow of energy (26a) [derived in Sec. 6] with the help of the relation $h/\rho = b = \text{const.}$ and the adiabatic of Poisson $\rho a_0^{-2/(\gamma-1)} = \text{const.} = A_1$. We get

$$\frac{u^2}{2} + \frac{a_0^2}{\gamma-1} + A_1 b^2 a_0^{2/(\gamma-1)} = \text{const} = \frac{a_{00}^2}{\gamma-1} + A_1 b^2 a_{00}^{2/(\gamma-1)}. \tag{16}$$

We introduce the parameter

$$\mu = A_1 b^2 a_{00}^{2(2-\gamma)/(\gamma-1)}.$$

For $\gamma = 5/3$, μ has the meaning of the square of the ratio of the Alfven velocity $V = h/\sqrt{\rho}$ to the sound velocity a_{00}, determined for the gas at rest. When the flow reaches the critical velocity, $u = a_*$. Introducing the ratio $x_* = a_*/a_{00}$, we rewrite (16) as

$$\frac{x_*^2}{2} + \frac{x_*^2}{\gamma-1} + \mu x_*^{2/(\gamma-1)} = \frac{1}{\gamma-1} + \mu. \tag{16'}$$

This equation determines the dependence of the critical velocity on the parameter μ, which characterizes the field. Let us represent (16′) in the form

$$\mu\left(1 - x_*^{2/(\gamma-1)}\right) + \frac{1}{\gamma-1}\left(1 - \frac{\gamma+1}{2}x_*^2\right) = 0. \tag{16''}$$

μ	x_*
0	0.866
0.0404	0.87
0.2854	0.89
0.6339	0.91
1.1494	0.93
2.1382	0.95
4.3726	0.97
15.506	0.99
∞	1.00

As μ changes from 0 to ∞, x_* changes from $\sqrt{2/(\gamma+1)}$ to 1. The dependence $x_* = x_*(\mu)$ for $\gamma = 5/3$ is shown in the table. For $\gamma = 1.4$, this dependence is still weaker: x_* changes from $\sqrt{5/6} = 0.9129$ to 1.

4. SMALL EXCITATIONS

Let u be a small quantity, and let $\rho = \rho_0 + \rho'$, $p = p_0 + p'$, $h = h_0 + h'$, where small quantities are denoted by a prime. In what follows, we shall omit the primes. The system (2) is linearized in the usual fashion:

$$\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial t} + \frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{h_0}{\rho_0}\frac{\partial h}{\partial x} = 0, \quad \frac{\partial h}{\partial t} + h_0 \frac{\partial u}{\partial x} - \varkappa \frac{\partial^2 h}{\partial x^2} = 0. \tag{17}$$

Since the Joule losses are neglected in the linearization, the energy equation is equivalent to the adiabatic equation $p = \text{const} \cdot \rho^\gamma$. In this case, it is possible to make the following transformation:

$$\frac{\partial p}{\partial x} = \frac{dp}{d\rho}\frac{\partial \rho}{\partial x} = a_0^2 \frac{\partial \rho}{\partial x}.$$

We look for all quantities in (16) proportional to $\exp\{i(kx - \omega t)\}$. Eliminating $\rho/\rho_0 = ku/\omega$, we get

$$(k^2 a_0^2/\omega - \omega)\, u + (kh_0/\rho_0)\, h = 0, \quad kh_0 u - (\omega + ik^2\varkappa)\, h = 0.$$

As the condition for nontrivial solution of this system, we get the dispersion equation

$$k^4 - k^2\left(\frac{\omega^2}{a_0^2} + \frac{i\omega}{\varkappa}\frac{a_m^2}{a_0^2}\right) + \frac{i\omega}{\varkappa a_0^2} = 0. \tag{18}$$

Let us investigate the solution of Eq. (18) in two limiting cases:

(1) $\omega \gg a_m^2/\kappa$ — the case of a weakly conducting gas (κ large);

(2) $\omega \ll a_m^2/\kappa$ — the case of a strongly conducting gas (κ small).

With accuracy up to terms of first order of smallness, we have

$$k_1^+ = (\omega/a_0) + (iV_0^2/2a_0\varkappa); \qquad k_1^- = (1+i)\sqrt{\omega/2\varkappa},$$
$$k_2^+ = (1+i)\sqrt{\omega/2\varkappa}\,(a_m/a_0); \qquad k_2^- = (\omega/a_m) + (i\omega^2\varkappa/2a_m^3). \tag{19}$$

The wave numbers k_1^+ and k_2^- correspond to the ordinary acoustic wave and a wave propagated with the effective magnetogasdynamical sound velocity[2,3,8]. The quantities k_1^- and k_2^+ correspond to waves of the skin type, and are rapidly damped. The indices ± correspond to the sign chosen in front of the inner root of the solution of the biquadratic equation (18).

It is interesting that the ordinary acoustic wave and its analogue in the case of infinite conductivity belong to different frequency modes and never cross each other upon corresponding increase or decrease of the conductivity, but degenerate into waves of the skin type. It is true that for such a change of the magnitude of the conductivity, we pass through regions of strong absorption, where the wave and its velocity lose meaning, so that this result takes on, perhaps, a somewhat formal character.

An equation of the type (18) was earlier obtained by Anderson;[17] however, he assumed that a_m is close to a_0 (weak field), therefore, he did not obtain from his approximate solution that a_m and a_b belong to different frequency modes.

5. ANOTHER METHOD OF INVESTIGATION OF SMALL EXCITATIONS

By linearization of the system (17), we can obtain a single equation for the velocity potential φ, which is introduced by the equation $u = \partial\varphi/\partial x$:

$$\varkappa \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 \varphi}{\partial t^2} - a_0^2 \frac{\partial^2 \varphi}{\partial x^2}\right) = \frac{\partial}{\partial t}\left(\frac{\partial^2 \varphi}{\partial t^2} - a_m^2 \frac{\partial^2 \varphi}{\partial x^2}\right). \tag{20}$$

We note that the form of Eq. (20) is strongly reminiscent of the equation of thermal condüctivity, which again bears witness to the parabolic degeneracy.

We seek the solution by the method of separation of variables:

$$\varphi = X(x)\,T(t).$$

Let

$$X^{IV} + k^2X'' = 0, \quad X'' + k^2X = 0.$$

Thence

$$X = B_1 \cos kx + B_2 \sin kx.$$

Then we get for T from Eq. (20):

$$\dddot{T} + a_m^2k^2\dot{T} + k^2\varkappa(\ddot{T} + a_0^2k^2T) = 0.$$

We seek a solution in the form $T = \exp(\alpha t)$. For the determination of α, we obtain the cubic equation

$$\alpha^3 + k^2\varkappa(\alpha^2 + a_0^2k^2) + a_m^2k^2\alpha = 0. \tag{21}$$

This equation, for arbitrary values of κ, has two complex and one real root.

We first consider the case of small κ. With accuracy up to terms of second order of smallness in κ,

$$\alpha_{1,2} = -\frac{\varkappa k^2}{2a_m^2}V_0^2 \pm ia_mk\left[1 - \frac{\varkappa^2k^2}{8a_m^4}V_0^2\left(1 + \frac{3a_0^2}{2a_m^2}\right)\right] = -\varkappa M \pm iN, \qquad \alpha_3 = -\varkappa a_0^2k^2/a_m^2 = -\varkappa R$$

(the notation is self evident). As a result,

$$T = e^{-\varkappa Mt}(A_1 \cos Nt + A_2 \sin Nt) + A_3e^{-\varkappa Rt}.$$

We can always write down a certain particular solution of Eq. (20) in the form

$$\overline{\varphi} = \varphi_0 \cos[N(t - \tau)] \cos[k(x - l)]. \tag{22}$$

To find k, we must solve some sort of actual boundary problem, for which we can construct a general solution in the form of the superposition of standing waves, summing all possible particular solutions of type (22). However, this is not necessary for the determination of the phase (or group) velocity of propagation of a small vibration. For example, we get for the phase velocity $v = N/k = \omega/k$,

$$v = a_m\left[1 - \frac{\varkappa^2k^2}{8a_m^4}V_0^2\left(1 + \frac{3a_0^2}{2a_m^2}\right)\right]. \tag{23}$$

In the case $\sigma \to \infty$ we get $v \to a_m$, as was to have been expected.

In the limit $\sigma \to 0$, we have $\kappa \to \infty$ and from Eq. (21) [with accuracy up to terms of the second order of smallness in $1/\kappa$] we get $\alpha_{1,2} = \pm ia_0k$, whence the phase velocity is $v = a_0$.

We shall not carry out any further investigation for obtaining skin waves.

6. SHOCK WAVES

For simplicity, let us consider only a rectilinear shock propagated with a velocity u_1 in a motionless, conducting gas located in a magnetic field h_1. The wavefront is parallel to the field.

The equations of stationary one-dimensional flow have the three integrals (9) of mass flow, momentum flux, and magnetic field flux:

$$\rho u = m, \quad mu + p + h^2/2 = J, \quad uh = \varkappa dh/dx + mb. \tag{24}$$

The Maxwell equations yield

$$e = \text{const} = -\frac{u_1h_1}{c}; \quad j = \frac{c}{\sqrt{4\pi}}\frac{dh}{dx}. \tag{25}$$

The equation of energy flux has the form:

$$m\frac{d}{dx}\left(i + \frac{u^2}{2}\right) = \sqrt{4\pi}je,$$

where i is the specific enthalpy. Substituting Eq. (25) and integrating, we get the integral of energy flux:

$$m(i+u^2/2) = mA - u_1 h_1 h, \text{ where } A = \text{const}. \tag{26}$$

We can rewrite Eq. (25) in the form

$$i + (u^2/2) + bh = A. \tag{26a}$$

The set of equations (24) and (26) completely determine the parameters of stationary flow.

It is now easy to write down the conditions on the front of the discontinuity:

$$\begin{aligned} \rho_1 u_1 &= \rho_2 u_2, \\ \rho_1 u_1^2 + p_1 + h_1^2/2 &= \rho_2 u_2^2 + p_2 + h_2^2/2, \\ i_1 + (u_1^2/2) + bh_1 &= i_2 + (u_2^2/2) + bh_2, \\ u_1 h_1 - (\varkappa dh/dx)_1 &= u_2 h_2 - (\varkappa dh/dx)_2. \end{aligned} \tag{27}$$

The index 1 denotes the parameters in front of the shock, the index 2 those behind the shock. If the conductivity is infinite, then $h_1/\rho_1 = h/\rho = b = \text{const}$, $\kappa = 0$, and the conditions (27) coincide with the known expressions on the front of the shock.[3]

It follows from the first two equations of (27) that

$$u_1^2 = \frac{\rho_2}{\rho_1}\left[\frac{p_2 - p_1}{\rho_2 - \rho_1} + \frac{h_2^2 - h_1^2}{2(\rho_2 - \rho_1)}\right], \quad u_2^2 = \frac{\rho_1}{\rho_2}\left[\frac{p_2 - p_1}{\rho_2 - \rho_1} + \frac{h_2^2 - h_1^2}{2(\rho_2 - \rho_1)}\right]. \tag{28}$$

For $p_2 = p_1 + \Delta p$, $\rho_2 = \rho_1 + \Delta\rho$, $h_2 = h_1 + \Delta h$, we shall have

$$u_1^2 = u_2^2 = a_*^2 = (\Delta p/\Delta\rho) + (h\Delta h/\Delta\rho), \tag{29}$$

where a_* is the effective sound velocity in the medium under consideration. From the last equation of the system (24),

$$\frac{h}{\rho} = b + \frac{\varkappa}{m}\frac{dh}{dx}, \tag{30}$$

whence

$$\frac{\Delta h}{\Delta\rho} = b + \frac{\varkappa}{m}\frac{\Delta}{\Delta\rho}\left(\rho\frac{dh}{dx}\right); \quad a_*^2 = a_0^2 + \left(b + \frac{\varkappa}{m}\frac{dh}{dx}\right)^2\rho + \frac{\varkappa}{m}\rho^2\left(b + \frac{\varkappa}{m}\frac{dh}{dx}\right)\frac{\Delta}{\Delta\rho}\left(\frac{dh}{dx}\right).$$

This expression, with the help of (30), can be put in the form

$$a_*^2 = a_0^2 + \frac{h^2}{\rho} + \frac{\varkappa}{m}h\rho\frac{\Delta}{\Delta\rho}\left(\frac{dh}{dx}\right). \tag{31}$$

Since the flow takes place from right to left, then $dh/dx < 0$ and

$$a_*^2 < a_m^2 = a_0^2 + h^2/\rho. \tag{32}$$

We estimate the value of the contribution connected with the conductivity, relative to h^2/ρ. We have

$$\varkappa = c^2/4\pi\sigma = c^2 m_e/4\pi n e^2 \tau.$$

As will be shown later, change of the field takes place at a distance of the order of the mean free path $\ell = m_e a_1 \sigma/ne^2$. Then

$$\frac{h\rho}{m}\frac{\Delta}{\Delta\rho}\left(\frac{dh}{dx}\right) \approx -\frac{h^2}{u\rho l}.$$

The ratio in which we are interested will have a value of order $1/\Pi_e$, where $\Pi_e = (e^2/m_e c^2)n\ell^2$ is a dimensionless quantity, the so called "linear" electron.

Let us compute the change in the entropy for a weak discontinuity. In this case,

$$u_1 = u_2 = a_*; \quad h_2 - h_1 = \Delta h = (\varkappa/a_*)\,dh/dx.$$

From (14) we get

$$dS = \frac{dQ}{T} = \frac{\varkappa}{mT}\left(\frac{dh}{dx}\right)^2 dx; \quad S_2 - S_1 = \int_0^L \frac{\varkappa}{mT}\left(\frac{dh}{dx}\right)^2 dx.$$

Obtaining the mean values T and dh/dx over the region of change of the quantities (0 to L), we get

$$S_2 - S_1 = \Delta S = (a_*^2 L / mT\varkappa)(\Delta h)^2. \tag{33}$$

Thus, the change in the entropy is a quantity of second order of smallness in comparison with the change in the field.

7. STRUCTURE OF THE SHOCK WAVE

Although this question has already been studied in detail by Marshall[6,7] with consideration of viscosity and thermal conductivity, it occurred to us that it is of methodological and some practical interest to consider the weakening of the front under the action of conductivity alone.

Let us introduce the dimensionless variables

$$\nu = u/a_1; \quad \eta = p/p_1; \quad \chi = h/h_1, \text{ where } a_1 = \sqrt{\gamma p_1/\rho_1}.$$

We choose $\xi = x/\lambda$ as the dimensionless unit of length, where $\lambda = c^2/4\pi\sigma a_1 = \kappa/a_1$. The conditions (27) become in the non-dimensional form,

$$k\nu + \eta + Q_1\chi^2 = J', \quad \frac{\gamma}{1-\gamma}\nu\eta + k\frac{\nu^2}{2} + 2\nu_1 Q_1\chi = A', \quad \frac{d\chi}{d\xi} = \nu\chi - \nu_1, \tag{34}$$

where use is made of the fact that $i = (\gamma p/\rho)/(\gamma - 1)$ for an ideal gas; $k = ma_1/p_1$; $J' = J/p_1$; $A' = Am/p_1a_1$; $Q_1 = h_1^2/2p_1$. From the first two equations of (34), we have

$$\nu = \nu(\chi) = \frac{\gamma}{(\gamma+1)k}(J' - Q_1\chi^2) + \left\{\left[\frac{\gamma}{(\gamma+1)k}(J' - Q_1\chi^2)\right]^2 - \frac{2(\gamma-1)}{k(\gamma+1)}(A' - 2\nu_1 Q_1\chi)\right\}^{1/2}. \tag{35}$$

As computations have shown, the minus sign in front of the root has no meaning. From the last equation of (34), we now find $\xi - \xi_0$ by quadrature. This integral was computed for the case of monatomic deuterium at T = 20,000°K, p_1 = 20,000 g/cm-sec^2 (about 15 mm Hg), H_1 = 689 oersted, and $\nu_1 = 2$. As is seen from Fig. 2, the change in all the quantities takes place over a distance on the order of two units. If we assume that in this case, $\sigma \approx 10^{14}$ cgs units, then the change will take place over a distance of approximately two mean free path lengths.

If $\lambda \to 0$, then $x = \xi\lambda \to 0$. Thus, in the case of very high conductivity, we get a sharp discontinuity. It should be pointed out that in fact a large thermal conductivity also corresponds to very high electric conductivity; therefore the jump will be strongly diffused by reason of the thermal conductivity.[7]

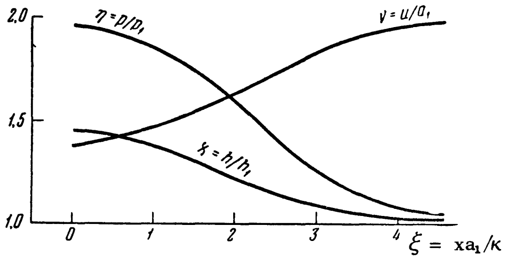

FIG. 2. Profile of the jump. The shock wave moves from left to right.

If the conductivity is low, the field will change slowly at large distances, on the order of κ/a_1. If the jump is strong, then it is clear that the field should have no effect on the shock wave. We shall analyze this in detail. We write down the condition for conservation of momentum for the specific volume $V = 1/\rho$:

$$p + m^2V + h^2/2 = p_1 + m^2V_1 + h_1^2/2. \tag{36}$$

We differentiate this equation with respect to V:

$$\left(\frac{dp}{dT}\right)_V \frac{dT}{dV} + \left(\frac{\partial p}{\partial V}\right)_T + m^2 + h\frac{dh}{dV} = 0. \tag{37}$$

$(\partial p/\partial T)_V > 0$ for all gasses; $dT/dV < 0$, since the gas in the shock wave is heated in the compression; $dh/dV < 0$, since $dh/d\rho > 0$. On side 1 we have

$$m^2 > -\left[\left(\frac{\partial p}{\partial T}\right)_V \frac{dT}{dV} + \left(\frac{\partial p}{\partial V}\right)_T\right]_1, \quad (38)$$

since $u_1 > a_1$, and to satisfy Eq. (37), we must have $dh/dV < 0$, which is to be expected. If the intensity of the shock wave is so large that

$$m^2 < -\left[\left(\frac{\partial p}{\partial T}\right)_V \frac{dT}{dV} + \left(\frac{\partial p}{\partial V}\right)_T\right]_2, \quad (39)$$

then to satisfy condition (37), we must have $dh/dV > 0$; this would mean that upon compression from V_1 to V_2 the field passes through a maximum, and thereafter falls to the value h_2; this cannot be. The real picture is the following; at first, in the compression from the volume V_1 to some intermediate volume V', a slow continuous change of the quantities takes place. Here $dh/dv < 0$, and the field changes from h_1 to h_2. After the shock, a compression takes place from V' to V_2. The field in this case does not change, while the velocity, pressure, and temperature undergo a jump. If condition (39) is not satisfied, then there will be a slow continuous change of all quantities to a distance of the order κ/a_1. This phenomenon is very much like an isothermal jump;[18] therefore, by analogy, it ought to be called an isomagnetic jump.

The condition (39), as is easily seen, is equivalent to $u_2 < a_2$. It is possible to show[6] that it reduces to

$$Q_1\left\{2(2-\gamma)\frac{u_1}{u_2} + 2\gamma - \gamma(\gamma-1)\left(\frac{u_1}{u_2}-1\right)^3\right\} < \gamma(\gamma+1)\left(\frac{u_1}{u_2}-1\right). \quad (40)$$

If $u_1/u_2 > 2.64$ for $\gamma \simeq 5/3$, then the left side of (40) will be less than zero and the jump will always occur. For smaller intensities the fact as to whether a discontinuity is approached or not will depend on the magnitude of the field.

When Marshall, by numerical integration of the differential equations for a certain example, obtained this picture, it appeared strange to him that all the change of the field took place in front of the jump. We see here that it could not be otherwise. In this lies the essence of the isomagnetic jump.

In conclusion, we express our deep appreciation to Academician M. A. Leontovich for his constant interest in the work and for many useful discussions.

[1] K. Alfven, Cosmical Electrodynamics (Oxford, 1950).

[2] H. Herlofson, Nature, **165**, 1020 (1950).

[3] F. Hoffman and E. Teller, Проблемы современной физики (Problems in Modern Physics), **2**, 47 (1954) (Russian Translation).

[4] H. L. Helfer, Astrophys. J. **119**, 34 (1954).

[5] H. Sen, Phys. Rev. **1021**, 5 (1956).

[6] W. Marshall, Proc. Roy. Soc. (London) **238A**, 367 (1955).

[7] W. Marshall, Phys. Rev. **103**, 1900 (1956).

[8] S. A. Kaplan and K. P. Staniukovich, Dokl. Akad. Nauk SSSR **95**, 769 (1954).

[9] G. van de Holst, Problems in Cosmic Aerodynamics (Russian translation, 1953), Ch. VI.

[10] A. Banos, Phys. Rev. **97**, 1435 (1955).

[11] A. Banos, Proc. Roy. Soc. (London) **233A**, 300 (1955).

[12] A. Banos, Phys. Rev. **104**, 306 (1956).

[13] Staniukovich, Baum, and Kaplan, Введение в космическую газовую динамику Introduction to Cosmic Gas Dynamics, Gostekhizdat, 1958.

[14] R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. 2, Springer, Berlin 1937.

[15] A. I. Morozov, J. Exptl. Theoret. Phys. (U.S.S.R.) **32**, 305 (1957); Soviet Phys. JETP **5**, 215 (1957).

[16] L. A. Vulis, Thermodynamics of Gas Currents (Gosenergoizdat, 1950).

[17] N. S. Anderson, J. Accoust. Soc. Amer. **25**, 529 (1953).

[18] L. D. Landau and E. M. Lifshitz, Механика сплошных сред (Mechanics of Continuous Media) (Gostekhizdat, 1954).

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up the crystals as infinitely long unidimensional or two-dimensional atom complexes, bound together by "small" forces of one nature, whereas in the complex itself the atoms are bound by "big" forces of another nature.

6. The difference between the typical molecular crystals (e.g., the CH_4 or C_6H_6 crystals) and the heteropolar molecular crystals (such as the NaCl, $HgCl_2$ or PbS crystals) lies: (1) in the degree of molecularity β; (2) in the nature of the forces in the molecules; (3) in the nature of intermolecular forces. The quantity β is defined as the ratio of the intramolecular energy $U^a \cong D$ (D is the energy of dissociation of the diatomic molecule into ions) to the intermolecular energy U^e per bond. For the substances for which β is given below, it is possible to take $U^e \approx 2S/l$. Example:

$\beta = 300(CH_4)$, $200(HCl)$, $22(HgCl_2)$, $10(NaCl)$ taking $l = 12$ in all four cases.

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ERRATA

Volume 5

Page	Line	Reads	Should Read
1043	Eq. (4)		$W = y^2 a_{14}^2 \sin 2\phi/2\rho\,(a_{11} a_{44} - \alpha_{14}^2 \sin^2 3\phi)$ The coefficient k_2 equals $0.185 \times 10^{-3}\,cm^{-1}$.
1044	3 from bottom (l.h.)	$\Delta y = 2.87 \times 10^{-3}$ cm	$\Delta y = 3.18 \times 10^{-3}$ cm
	4 from top (r.h.)	$\Delta\vartheta_\Sigma = 7.2 \times 10^{-5}$ radians	$\Delta\vartheta_\Sigma = 5.9 \times 10^{-5}$ radians

Volume 6

Page	Line	Reads	Should Read
1090	4 and 5 from top	2—(d, 3n); and of the I_{53}^{127} cross section, 3—(d, 2n); 4—(d, 3n)	2—(d, 3n) on I_{53}^{127} and 3—(d, 3n); 4—(d, 3n) on Bi_{83}^{209}
1091	6 from bottom expression for determinant C(y)	ρ, γp, h, $1/\rho$	ρy_2, $\gamma p y_2$, $h y_2$, y_2/ρ
1094	7 from bottom	For $\gamma = 5/3$, μ has . . .	Here μ has . . .

Volume 7

Page	Line	Reads	Should Read
55	16 from bottom	Correct submittal date is April 5, 1957	
169	17 from bottom	Delete "Joint Institute for Nuclear Research"	
215	Table	Add: Note. Columns 2—9 give the number of counts per 10^6 monitor counts	
215	Table, column headings	1, 2, 3, 4-7, 8	1, 2, 3, 4, 8-7
312	Eq. (8)	. . . $(1 \pm \mu/2M)^2$	. . . $(1 \mp \mu/2M)^2$
313	2, r.h. col.	$\alpha_{33} = 0.235$	$a_{33} = 0.235$
692	Eq. (5)	$m_B/M_B = \ldots \mp [1 + \ldots]$	$m_B/M_B = \mp [1 + \ldots]$
461	Title	. . . Elastically Conducting	. . . Electrically Conducting