THEORY OF CYCLOTRON RESONANCE

E. A. KANER

Radiophysics and Electronics Institute, Academy of Sciences, Ukrainian S.S.R.

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The possibility of cyclotron resonance is studied for a metal in a magnetic field which is inclined to the surface. The surface impedance for a metal in a magnetic field parallel to the surface is calculated for an arbitrary law of reflection of electrons from the surface.

1. INTRODUCTION

AZBEL' and the present author[1-3] worked out the theory of cyclotron resonance for the case of a metal in a magnetic field parallel to the surface, assuming the reflection of electrons from the surface to be purely diffuse. When the magnetic field is parallel and the dispersion law is quadratic, all the electrons in the top layer of the Fermi distribution contribute to the resonance, since they all have the same orbital revolution frequency Ω and they all pass many times within the skin depth. When the dispersion law has a more general character, only electrons with an extremal value of effective mass contribute to the resonance, and the relative depth of the resonance is then smaller than for a quadratic dispersion law. In an inclined magnetic field the great majority of electrons enter the skin depth only once, after which they disappear into the interior of the metal, and they therefore do not contribute to the resonance. However, Chambers[4] pointed out that, in a magnetic field inclined at a considerable angle Φ to the surface, a resonance can still be produced by the minority of electrons which have $\bar{v}_H \approx 0$ and therefore return to the skin layer many times without drifting along the direction of the field. The bar over $\bar{v}_H$ denotes an average along the path of an electron with $\epsilon(\mathbf{p}) = \zeta$, $p_H = \text{const}$. Here ϵ is the energy, $\mathbf{p}$ the wave vector, $\mathbf{v} = \partial\epsilon/\partial\mathbf{p}$ the velocity, ζ the chemical potential of an electron. p_H and v_H are the projections of $\mathbf{p}$ and $\mathbf{v}$ along the direction of the magnetic field.

We showed earlier[2,3] that to the lowest order in (δ_{eff}/r) the surface impedance Z of the metal is in general independent of the magnetic field. Field-dependent and resonant behavior occurs in general only in the next order in (δ_{eff}/r). Here δ_{eff} is the effective skin depth and r the orbital radius of an electron in the magnetic field. It is important to calculate the magnitude of resonant effects in an inclined magnetic field and to see whether such effects might be observed experimentally.

It is also interesting to see how the law of reflection of electrons from the metal surface influences the high-frequency surface impedance. If the surface were strictly regular the reflection of electrons should be specular. But even a single crystal always has surface irregularities of sizes comparable with interatomic spacings, though they may be small compared with the effective skin depth. The reflection of electrons from the surface will therefore be diffuse or almost diffuse. This means that the distribution function of the electrons reflected from the surface is uncorrelated with that of the incident electrons.

The present paper discusses cyclotron resonance in an inclined magnetic field, and investigates in the case of a parallel field the influence of the reflection coefficient of electrons at the metal surface on the impedance.

2. CYCLOTRON RESONANCE IN AN INCLINED MAGNETIC FIELD

The calculation of the complete surface impedance tensor

$$Z_{\mu\nu} \equiv R_{\mu\nu} + iX_{\mu\nu} = \partial E_\mu(0)/\partial I_\nu = (4\pi\omega/ic^2)\,\partial E_\mu(0)/\partial E'_\nu(0) \quad (\mu,\ \nu = x,\ y) \tag{2.1}$$

reduces to a simultaneous solution of Maxwell's equations

$$\operatorname{curl}\mathbf{E} = -i\omega\mathbf{H}/c; \qquad \operatorname{div}\mathbf{H} = \operatorname{div}\mathbf{E} = 0; \quad \operatorname{curl}\mathbf{H} = 4\pi\mathbf{j}/c; \qquad \mathbf{j} = -\frac{2e}{h^3}\int \mathbf{v} f_1\, d\mathbf{p} \tag{2.2}$$

and of the kinetic equation for the departure f_1 from an equilibrium Fermi distribution function $f_0(\epsilon)$

$$i\omega f_1 + v_z \frac{\partial f_1}{\partial z} + \Omega \frac{\partial f_1}{\partial \tau} + \frac{f_1}{t_0} = e\mathbf{E}\mathbf{v}\frac{\partial f_0}{\partial \varepsilon}. \tag{2.3}$$

Here $R_{\mu\nu}$ and $X_{\mu\nu}$ are the resistive and reactive parts of the impedance tensor, $E_\mu(z)$ is the tangential component of electric field, I_ν is a component of the total current, ω is the applied frequency, $\Omega = eH/mc$ is the "cyclotron" frequency, $m = (1/2\pi)\,\partial S(\epsilon, p_H)/\partial\epsilon$, $S(\epsilon, p_H)$ is the area cut out from the surface $\epsilon(\mathbf{p}) = \epsilon$ by the plane $p_H = \text{const}$, τ is the dimensionless time of revolution of an electron in its orbit, $t_0(\mathbf{p})$ is the relaxation time, the z-axis is the inward normal to the metal surface, and the x-axis is the projection of the constant magnetic field H onto the surface. The boundary coniditon for the kinetic equation (2.3) is in the case of diffuse scattering

$$f_1 = 0 \text{ for } z = 0,\ v_z > 0. \tag{2.4}$$

We have found[2,3] that the surface impedance tensor $Z_{\mu\nu}$ is conveniently deduced from Eq. (2.2) and (2.3) by taking Fourier transforms with respect to z. After calculations precisely analogous to those made earlier[2,3], we obtain the following equations for the Fourier transforms.

$$-k^2\mathcal{E}_\mu(k) - 2E'_\mu(0) = (4\pi i\omega/c^2)\, j_\mu(k); \qquad j_\mu(k) = \sum_\nu \left\{K_{\mu\nu}(k)\,\mathcal{E}_\nu(k) - \int_0^\infty Q_{\mu\nu}(k;\,k')\,\mathcal{E}_\nu(k')\,dk'\right\}; \tag{2.5}$$

$$\mathcal{E}_\mu(k) = 2\int_0^\infty E_\mu(z)\cos kz\,dz; \quad j_\mu(k) = 2\int_0^\infty j_\mu(z)\cos kz\,dz,$$

$$K_{\mu\nu}(k) = -\frac{2e^2}{h^3}\int_0^\infty \frac{\partial f_0}{\partial\varepsilon}\,d\varepsilon \int_{p_H^{\min}}^{p_H^{\max}} \frac{m}{\Omega}\,dp_H \cdot \int_0^{2\pi} v_\mu(\tau)\,d\tau \int_{-\infty}^{\tau} v_\nu(\tau_1)\exp\left(\int_\tau^{\tau_1}\gamma\,d\tau_2\right)\cos\left(\frac{k}{\Omega}\int_\tau^{\tau_1} v_z\,d\tau_2\right)d\tau_1,$$

$$Q_{\mu\nu}(k;\,k') = -\frac{2e^2}{\pi h^3}\int_0^\infty \frac{\partial f_0}{\partial\varepsilon}\,d\varepsilon \int \frac{m}{\Omega^2}\,dp_H \int_0^{2\pi} v_\mu\,d\tau \int_{-\infty}^{\tau} |v_z(\tau_1)|\cos\left(\frac{k}{\Omega}\int_\tau^{\tau_1} v_z\,d\tau_2\right)d\tau_1 \int_{\varphi(\tau_1)}^{\tau_1} v_\nu(\xi)\exp\left(\int_\tau^{\xi} v_z\,d\tau_2\right)\cos\left(\frac{k'}{\Omega}\int_\xi^{\tau_1} v_z\,d\tau_2\right)d\xi,$$

$$\gamma = i\omega/\Omega + 1/\Omega t_0, \tag{2.6}$$

Here $\varphi(\tau)$ denotes the root immediately preceding τ of the equation

$$\int_{\varphi(\tau)}^{\tau} v_z\,d\tau_2 = 0.$$

If there is no such root, then $\varphi(\tau) = -\infty$. The quantity (kv/Ω), which appears in the argument of the cosines in Eq. (2.6), is of order $(r/\delta_{\text{eff}}) \gg 1$. Therefore the intégrals (2.6) can be evaluated by the method of steepest descent. Elementary but very lengthy calculation then leads to the following results.

To the lowest order in $(\Omega/kv)^{1/2} \sim (\delta_{\text{eff}}/r)^{1/2} \ll 1$, the surface impedance is independent of magnetic field and is equal to its value for $H = 0$. In the first and second higher orders, the dependence on H does not have a resonant behavior. A weakly-resonant dependence appears only in still higher terms in the expansion of the impedance in powers of $(\delta_{\text{eff}}/r)^{1/2}$. Also only the Z_{yy} component of the surface impedance tensor shows a resonance. Thus resonant absorption of power can occur only for electromagnetic waves incident on the metal with a definite polarization. The electric vector of the incident wave must be parallel to the y-axis (perpendicular both to H and to the normal to the surface).

The relative magnitude of the resonant contribution is

$$\frac{\Delta Z_{\text{res}}}{Z(0)} \sim A\left(\frac{\delta_{\text{eff}}}{r}\right)^{3/2}\ln\left[\frac{1}{1 - \exp(-2\pi i\omega/\Omega_0 - 2\pi\nu_0/\Omega_0)}\right], \tag{2.7}$$

At resonance (when $\omega \approx q\Omega_0$ with q an integer) this contribution has only a logarithmic singularity for $(\nu_0/\omega) \ll 1$. Here Ω_0, r, and $\nu_0 = \overline{(1/t_0)}$ are the Larmor frequency, the orbit radius, and the average along the orbit of the collision rate, for an electron with $\bar{v}_H = 0$. A is a complex coefficient independent of H,

$$\delta_{\text{eff}} = |c^2 Z(0)/4\pi\omega| \sim (\delta^2 l)^{1/3}; \quad \delta = (mc^2/2\pi ne^2)^{1/2},$$

and ℓ is the mean free path. The resonant effect is small because it is produced only by the small minority of electrons which return many times into the skin depth. Since the resonant contribution is so small we do not exhibit the exact formulae for it.

The results are correct so long as the angle Φ between the magnetic field and the metal surface is not close to zero or to $\frac{1}{2}\pi$, i.e., so long as $\sin 2\Phi \gg \delta_{\text{eff}}/r$.

The case of a perpendicular magnetic field ($\cos\Phi \ll \delta_{\text{eff}}/r$) is special, because in this case the electrons which give the main contribution to the current density have orbits lying entirely within the skin depth. We have then not cyclotron resonance ($\omega = q\Omega$) but diamagnetic resonance ($\omega = \Omega$) similar to the diamagnetic resonance in semiconductors. Azbel' and Kaganov[5] showed that, to the lowest order in (δ_{eff}/ℓ) [in a perpendicular field the expansion parameter is (δ_{eff}/ℓ) instead of (δ_{eff}/r)], the effective mean free path $\ell^* = \ell/[1 + i(\omega - \Omega)t_0]$ in general disappears from the formula for the surface impedance. The impedance is in this approximation independent of H. A resonance (i.e., an extremum with respect to H) appears only in a second-order term proportional to $(\delta_{\text{eff}}/\ell^*)^2 \ln(\ell^*/\delta_{\text{eff}})$.

3. CYCLOTRON RESONANCE IN A PARALLEL MAGNETIC FIELD WITH ARBITRARY REFLECTION LAW OF ELECTRONS AT THE METAL SURFACE

To study the dependence of the surface impedance on the reflection coefficient of electrons at the metal surface, we have to change the boundary condition (2.4). We suppose that a fraction ρ of the electrons is reflected specularly (i.e., without any change in their distribution function), while the rest are distributed after reflection in the equilibrium distribution $f_0(\epsilon)$. This gives the condition

$$f(0;\ v_x,\ v_y,\ v_z) = \rho f(0;\ v_x,\ v_y,\ -v_z) + (1-\rho) f_0(\varepsilon) \text{ for } v_z > 0$$

For the departure from equilibrium $f_1 = f - f_0(\epsilon)$, the condition becomes

$$f_1(0;\ v_x,\ v_y,\ v_z) = \rho f_1(0;\ v_x,\ v_y,\ -v_z), \quad v_z > 0. \tag{3.1}$$

We showed earlier[2] that the Fourier transform of the current density can be determined if we know the relation between $\partial\psi_-(0;\mathbf{v})/\partial z$ and $\psi_-(0;\mathbf{v})$, where $\psi_\pm(z;\mathbf{v}) = f_1(z;\mathbf{v}) \pm f_1(z;-\mathbf{v})$. Equation (2.3) implies

$$\partial\psi_-(z;\ \mathbf{v})/\partial z = -\{i\omega + 1/t_0(\mathbf{p}) + \Omega\partial/\partial\tau\}\psi_+(z;\ \mathbf{v})/v_z \tag{3.2}$$

using also the fact that $\epsilon(\mathbf{p})$ is an even function. We need therefore only find the relation between $\psi_+(0;\mathbf{v})$ and $\psi_-(0;\mathbf{v})$. The boundary condition (3.1) gives

$$\psi_+(0;\ \mathbf{w},\ v_z) = \rho\psi_+(0;\ \mathbf{w},\ -v_z) + \operatorname{sgn} v_z \cdot \{\rho\psi_-(0;\ \mathbf{w};\ -v_z) - \psi_A(0;\ \mathbf{w},\ v_z)\}, \tag{3.3}$$

where $\mathbf{w}$ is the two-dimensional vector with components (v_x, v_y), and $\operatorname{sgn} x$ denotes the sign of x. Changing v_z into $-v_z$ in Eq. (3.3) and eliminating $\psi_+(0;\mathbf{w},-v_z)$, we find

$$\psi_+(0;\ v_x,\ v_y,\ v_z) = -\frac{\operatorname{sgn} v_z}{1-\rho^2}\{(1+\rho^2)\psi_-(0;\ v_x,\ v_y,\ v_z) - 2\rho\psi_-(0;\ v_x;\ v_y,\ -v_z)\}. \tag{3.4}$$

When $\rho \to 0$ we obtain the boundary condition used in Ref. 2. When $\rho \to 1$ we find

$$\psi_-(0;\ \mathbf{w};\ v_z) \to \psi_-(0;\ \mathbf{w};\ -v_z).$$

The Fourier transform of the current density can be found from Eq. (3.4). For simplicity we write down the formula for $\mathbf{j}(\mathbf{k})$ only for the case of an isotropic quadratic dispersion law $\epsilon(\mathbf{p}) = p^2/2m$, where m is the effective mass, assuming a relaxation time t_0 independent of $\mathbf{p}$ (the resistance being in the residual range), and taking the radio-frequency and constant magnetic fields perpendicular to each other. Under these conditions we have

$$j_\mu(k) = K(k)\mathcal{E}_\mu(k) - \int_0^\infty Q(k;\ k')\mathcal{E}_\mu(k')\,dk'; \quad K(k) = \frac{3\sigma\gamma}{\pi}\int_0^{\pi/2}\cos^2\vartheta\sin\vartheta\,d\vartheta\cdot\int_0^\infty e^{-\gamma x}dx\cdot\int_0^\pi\cos[kr\sin\vartheta(\cos\tau - \cos(\tau - x))]\,d\tau;$$

$$Q(k;\ k') = \frac{3\sigma\gamma r}{\pi^2}\int_0^{\pi/2}\cos^2\vartheta\sin^2\vartheta\, d\vartheta\int_0^{\infty} e^{-\gamma x}dx\cdot I(\vartheta;\ x);$$

$$I(\vartheta;\ x) = \int_0^{\pi} d\lambda\cdot\sin\lambda\,\frac{\cos\left[kr\sin\vartheta(\cos\lambda - \cos(\lambda + x))\right] - \rho\cos\left[kr\sin\vartheta(\cos\lambda - \cos(\lambda - x))\right]}{e^{\gamma\lambda} - \rho e^{-\gamma\lambda}}$$

$$\times\int_0^{\lambda}\cosh\gamma\eta\cdot\cos\left[k'r\sin\vartheta(\cos\lambda - \cos\eta)\right]d\eta;\quad \sigma = \frac{ne^2 l}{mv};\quad \gamma = \frac{r}{l};\quad r = \frac{mvc}{eH};\quad l = \frac{vt_0}{1 + i\omega t_0};\quad v = \left(\frac{2\zeta}{m}\right)^{1/2};$$

$$\mathbf{v} = v(\cos\vartheta,\ \sin\vartheta\cos\tau,\ \sin\vartheta\sin\tau);\ n = 8\pi p^3/3h^3. \tag{3.5}$$

When the skin effect is anomalous $kr \sim (r/\delta_{eff}) \gg 1$. The method of steepest descent then gives the asymptotic formula

$$K(k) = \frac{3\pi\sigma}{4l}\frac{1 + \exp(-2\pi\gamma)}{1 - \exp(-2\pi\gamma)}\frac{1}{k},$$

$$Q(k;\ k') = \frac{3\sigma}{4l}\frac{\exp(-2\pi\gamma)}{1 - \exp(-2\pi\gamma)}\frac{1 - \rho}{1 - \rho\exp(-2\pi\gamma)}\frac{1}{\sqrt{kk'}} \tag{3.6}$$

$$\times\left[\pi\delta(k - k') + \frac{\cosh^2\pi\gamma}{k + k'}\right] + \frac{3(1 + \rho)\sigma}{4\pi l}\left\{\frac{\cosh\pi\gamma\cdot\exp(-\pi\gamma)}{1 - \rho\exp(-2\pi\gamma)} + \left[1 - \rho + \frac{\sqrt{2}\gamma}{\pi B(3/4,\ 3/2)}\frac{\ln(k/k')}{\sqrt{kr} - \sqrt{k'r}}\right]^{-1}\right\}\frac{\ln(k/k')}{k^2 - k'^2},$$

where $B(p, q)$ is the Eulerian integral of the second kind. Equation (3.6) may be regarded as an interpolation formula, since it gives the correct result both for

$$1 - \rho \gg |\gamma\sqrt{\delta_{eff}/r}| \text{ and for } 1 - \rho \ll |\gamma\sqrt{\delta_{eff}/r}|.$$

Detailed calculation shows that, for all values of $\rho \neq 1$ [more precisely, for $1 - \rho \gg |\gamma|(\delta_{eff}/r)^{1/2}$], the surface impedance near to resonance ($\omega \approx q\Omega$) or in the strong field regime ($|\gamma| \ll 1$) is independent of the reflection coefficient ρ. The formulae for Z which we obtained earlier[1-3] remain valid.

The case of specular reflection is special, because in this case the main contribution to the current density comes from electrons which collide repeatedly with the surface. To the lowest order in $(\delta_{eff}/r)^{1/2}$ the impedance has a non-resonant dependence on magnetic field. Resonance appears only in a higher-order term of order (δ_{eff}/r).

In strong fields ($|\gamma| \ll 1$) the field-dependence of the impedance is quite different for $\rho = 1$ and for $\rho \neq 1$. We find

$$Z(H) \sim Z(0)\gamma^{1/6}(\delta/l)^{2/15} \sim H^{-1/6}\quad (\rho = 1);\qquad Z(H) \sim Z(0)\gamma^{1/3} \sim H^{-1/3}\quad (1 - \rho \gg |\gamma|(\delta_{eff}/r)^{1/2}). \tag{3.7}$$

The exact formulae for the impedance with $\rho = 1$ will not be reproduced here, since this case is of purely theoretical interest.

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