THE RADIATION FROM AN ELECTRON MOVING IN A MAGNETOACTIVE PLASMA

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We consider the spectral and angular distribution of the energy radiated by an electron moving in a magnetoactive plasma, and find the polarization of the radiation.

1. The problem of radiation of electromagnetic energy by charged particles moving in a magnetic field is of considerable interest for some parts of astrophysics and radioastronomy. But in most papers on this subject, the treatment is limited to the case of motion of electrons in vacuum.[1-3]

The presence of the medium can, under certain conditions, strongly influence the character of the radiation and, for example, lead to a reduction in intensity of the low frequency radiation from discrete sources.[2,4]

The problem of the radiation from an electron moving in a magnetic field in an isotropic medium was treated by Tsytovich[5] and Razin.[4] However, the magnetic field which accelerates the electron also causes the medium in which the electron moves to become magnetoactive. The anisotropy which arises in this way can be large in the case of a plasma. Under these conditions we must solve the problem of the radiation from an electron moving in a gyrotropic medium. This problem is solved in the present paper using the Hamiltonian method.

We should mention that this method was used by Ginzburg,[6] and later by Kolomenskii,[7] to determine the energy of the radiation from charged particles in anisotropic media.

2. The field equations for a charge e, moving along a vortex line in a magnetoactive medium, have the following form:

$$\operatorname{curl}\mathbf{H} = \frac{4\pi}{c} e\mathbf{v}\delta(\mathbf{r}-\mathbf{r}_e) + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}, \quad \operatorname{div}\mathbf{H} = 0, \quad \operatorname{div}\mathbf{D} = 4\pi e\delta(\mathbf{r}-\mathbf{r}_e), \quad \operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \tag{1}$$

where the components of the radius vector $\mathbf{r}_e$ and the velocity $\mathbf{v}$ of the electron along the x, y, z axes are, respectively,

$$r_0\cos\Omega_0 t; \quad r_0 \sin\Omega_0 t; \quad v_2 t$$
$$\text{and } -v_1\sin\Omega_0 t; \quad v_2\cos\Omega_0 t; \quad v_2$$

$(v_1 = \Omega_0 r_0)$. Here it is assumed that the axis of the vortex line, which is parallel to the constant magnetic field $\mathbf{H}_0$, is along the z axis, while $\mathbf{v}_1$, $\mathbf{v}_2$ are the projections of the electron velocity on the xy plane and the z axis. The Fourier components of the electric induction $\mathbf{D}_\omega$ and the electric field intensity $\mathbf{H}_\omega$ are related by

$$\mathbf{D}_\omega = \{\varepsilon_{\alpha\beta}(\omega)\}\,\mathbf{E}_\omega,$$

where $\{\epsilon_{\alpha\beta}\}$ is the Hermitian dielectric tensor $(\epsilon_{\alpha\beta} = \epsilon^*_{\beta\alpha})$, whose components in the system of coordinates which we are using are given on p. 326 of Ref. 8. According to the Hamiltonian method,[6,7] the vector potential of the radiation field,

$$\mathbf{A} = \sqrt{4\pi}\, c \sum_{j\lambda} q_{j\lambda}\mathbf{a}_{j\lambda} e^{i\mathbf{k}_\lambda \mathbf{r}} \tag{2}$$

can be found by solving the system of oscillator equations

$$\ddot{q}_{j\lambda} + \omega^2_{j\lambda} q_{j\lambda} = \sqrt{4\pi}\,\frac{e\mathbf{v}}{n_{j\lambda}}\,\mathbf{a}^*_{j\lambda} e^{-i\mathbf{k}_\lambda \mathbf{r}_e}. \tag{3}$$

In these relations, $\mathbf{a}_{j\lambda}$ is the complex polarization vector (the index j = 1, 2 corresponds to the two normal waves propagating in the gyrotropic plasma). The wave vector $\mathbf{k}_\lambda$ is related to the frequency $\omega_{j\lambda}$ by $\omega^2_{j\lambda} = k^2_\lambda c^2/n^2_{j\lambda}$, where $n_{j\lambda}$ is the refractive index. If we introduce the parameters α_j and β_j through the relations

$$\alpha_j = K_j\cos\theta + \gamma_j\sin\theta, \quad \beta_j = \gamma_j\cos\theta - K_j\sin\theta, \quad iK_j = E_\theta/E_x; \quad i\gamma_j = E_{rj}/E_x, \tag{4}$$

the polarization vector $\mathbf{a}_{j\lambda}$ of the j-th normal mode will have the following components along the coordinate axes: $1/\sqrt{2}$, $i\alpha_j/\sqrt{2}$; $i\beta_j/\sqrt{2}$. The orientation of E_x, H_θ, E_r is clear from Fig. 1.

For a magnetoactive plasma, the quantities K_j and n_j are given by the expressions

$$n_j^2 = 1 - 2V(1-V)/[2(1-V) - u\sin^2\theta \pm \sqrt{u^2\sin^4\theta + 4u(1-V)^2\cos^2\theta}];$$

$$K_j = \frac{-2\sqrt{u}(1-V)\cos\theta}{[u\sin^2\theta \pm \sqrt{u^2\sin^4\theta + 4u(1-V)^2\cos^2\theta}]}. \quad (5)$$

On the basis of the definition (4) we can also easily compute the parameter γ_j:

$$\gamma_j = \frac{-\sin\theta V\sqrt{u} + K_j uV\cos\theta\sin\theta}{1 - u - V(1 - u\cos^2\theta)}. \quad (6)$$

In formulas (5) and (6) we assume, as usual, that

$$V = 4\pi eN/m\omega^2; \quad \sqrt{u} = eH_0/mc\omega,$$

where m is the electron mass and N is the electron concentration in the plasma. Using the expres-

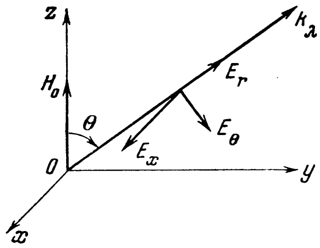

FIG. 1. $E_x \perp H_0$; E_θ, E_r and k_λ are in the yz plane; $E_x \perp k_\lambda$, $E_\theta \perp k_\lambda$.

sion for the components of the vector $\mathbf{a}_{j\lambda}$, and the expansion

$$\exp\{-ik_\lambda r_0\sin\theta\sin\Omega_0 t\} = \sum_{-\infty}^{\infty} J_s(k_\lambda r_0\sin\theta)\, e^{-is\Omega_0 t}$$

(J_s is the Bessel function of order s), we reduce Eq. (3) to the following form:

$$\ddot{q}_{j\lambda} + \omega_{j\lambda}^2 q_{j\lambda} = -\sqrt{4\pi}\frac{ei}{n_{j\lambda}}\sum_{-\infty}^{\infty} G_s(k_\lambda r_0\sin\theta)\exp\{-it[s\Omega_0 + v_2 k_\lambda\cos\theta]\}. \quad (7)$$

Here

$$G_s(k_\lambda r_0\sin\theta) = v_1 J_s'(k_\lambda r_0\sin\theta) + \left[\frac{\alpha s v_1}{k_\lambda r_0\sin\theta} + \beta v_2\right] J_s(k_\lambda r_0\sin\theta),$$

while J_s' is the derivative of the Bessel function with respect to its argument.

In the system of equations (7), those frequencies $\omega_{j\lambda}$ which satisfy the relation

$$\omega_{j\lambda} \approx s\Omega_0 + v_2 k_\lambda\cos\theta = p_\lambda, \quad (8)$$

give solutions of (7) which increase with time and correspond to radiation. The equations for frequencies given by (8) are:

$$\ddot{q}_{j\lambda} + \omega_{j\lambda}^2 q_{j\lambda} = -\sqrt{4\pi}\frac{ei}{n_{j\lambda}} G_s e^{-ip_\lambda t} = f(t). \quad (9)$$

The solution of this system for the initial conditions $t = 0$; $q_{j\lambda} = \dot{q}_{j\lambda} = 0$ is elementary:

$$q_{j\lambda} = \frac{i}{2\omega_{j\lambda}}\left[e^{-i\omega_{j\lambda}t}\int_0^t f(t)\, e^{i\omega_{j\lambda}t}\, dt - e^{i\omega_{j\lambda}t}\int_0^t f(t)\, e^{-i\omega_{j\lambda}t}\, dt\right].$$

The energy of the oscillator $q_{j\lambda}$ after time T is

$$W_{j\lambda} = \frac{1}{4}\left|\int_0^T f(t)\, e^{i\omega_{j\lambda}t}\, dt\right|^2 + \frac{1}{4}\left|\int_0^T f(t)\, e^{-i\omega_{j\lambda}t}\, dt\right|^2.$$

Summing over all oscillators $q_{j\lambda}$ in the frequency interval $d\omega_{j\lambda}$ and solid angle $d\Omega$, we find that the energy radiated within this range of frequency and angle for the s-th harmonic is given by the following expression:

$$W_{sj} = \frac{1}{2}|G_s|^2\frac{1 - \cos T[-s\Omega_0 + \omega_{j\lambda}(1 - \beta_2 n_{j\lambda}\cos\theta)]}{[-s\Omega_0 + \omega_{j\lambda}(1 - \beta_2 n_{j\lambda}\cos\theta)]^2}\rho_{j\lambda}d\omega_{j\lambda}d\Omega, \quad (10)$$

where

$$\beta_2 = v_2/c; \quad \rho_{j\lambda} = n_{j\lambda}^3\omega_{j\lambda}^2/(2\pi c)^3.$$

For $T \to \infty$, (10) becomes a δ function of the argument

$$-s\Omega_0 + \omega_{j\lambda}(1 - \beta_2 n_{j\lambda}\cos\theta).$$

This means that only the Doppler frequency ω_s, defined by the equation

$$-s\Omega_0 + \omega_s(1 - \beta_2 n_{js}\cos\theta) = 0, \quad (11)$$

is responsible for the harmonic $\Omega_0 s$. Using (10) it is easy to find the total energy of the radiation for the j-th normal mode, as a sum:

$$W_j = \sum_{-\infty}^{\infty} W_{sj}.$$

3. We should mention that, according to formula (10), radiation is also possible for $s = 0$ (zeroth harmonic). In this case the energy is radiated at the Cerenkov angle θ, which satisfies the equation:

$$1 - \beta_2 n_j(\omega; \theta_1)\cos\theta_1 = 0,$$

where, for simplicity, we have dropped the subscripts on the frequency $\omega_{j\lambda}$. Carrying out the integration over solid angle in (10), we find that the radiated energy in the j-th normal mode is ($s = 0$):

$$W_{0j}=\frac{Te^2}{2c^2v_2}\int\limits_{\beta_2 n_j(\omega)\geqslant 1}\frac{\omega d\omega\left\{v_1J_0'\left(\frac{v_1}{v_2}\frac{\omega}{\Omega_0}\sqrt{\beta_2^2n_j^2-1}\right)+\beta_j v_2 J_0\left(\frac{v_1}{v_2}\frac{\omega}{\Omega_0}\sqrt{\beta_2^2n_j^2-1}\right)\right\}^2}{|1-(\partial n_j/\partial\theta)/n_j\sqrt{\beta_2^2n_j^2-1}|}. \quad (12)$$

If the electron moves uniformly along the z axis ($v_1 = 0$; $r_0 = 0$), the preceding expression reduces to formula (4.5) of Ref. 7 for Cerenkov radiation:

$$W_{0j}=\frac{Te^2v_2}{2c^2}\int\limits_{\beta_2 n_j\geqslant 1}\frac{\beta^2\omega d\omega}{|1-(\partial n_j/\partial\theta)/n_j\sqrt{\beta_2^2n_j^2-1}|}. \quad (13)$$

If is necessary to remark that formulas (12) and (13) are invalid for those frequencies for which the refractive index $n_j(\omega)\to\infty$, since in this case the expression for W_{0j} diverges ($\beta_j\to\infty$ like n_j^2). In order to eliminate this divergence, we must either take account of collisions in the plasma (which will make $n_j(\omega)$ finite everywhere), or set some limiting value n_m for the refractive index n_j, if we are interested in the total loss from the moving particle. In this case, the limits of the domain of integration will be given by the following inequalities (for more details, cf. Ref. 9):

$$\beta_2 n_m \geqslant \beta_2 n_j(\omega)\geqslant 1.$$

This remark applies to all relations obtained later on, when $n_j(\omega, \theta)$ goes to infinity.

4. For harmonics other than the zeroth, it is more convenient to carry out the integration over ω in (10). Then the energy radiated, at the frequency ω_s defined by (11), in time T into the solid angle $d\Omega$ is

$$W_{js}=\frac{Te^2\omega_s^2n_jd\Omega\left\{v_1J_s'(n_j\omega_s r_0\sin\theta/c)+[\alpha s v_1/k(\omega_s)r_0\sin\theta+\beta v_2]J_s(n_j\omega_s r_0\sin\theta/c)\right\}^2}{4\pi c^3|1-\beta_2\cos\theta(n_j+\omega_s\partial n_j(\omega_s)/\partial\omega)|}. \quad (14)$$

If we set $u = V = 0$ ($n_j = 1$; $\alpha_j = \pm\cos\theta$) in (11), (14) and (5), i.e., if we treat the case of motion of the electron along a vortex line in vacuum, the amount of energy radiated will be given by the following:

$$W_s=\sum_{j=1}^{2}W_{js}=\frac{Te^2(s\Omega_0)^2d\Omega}{2\pi c^3(1-\beta_2\cos\theta)^3}\left\{v_1^2\left[J_s'\left(\frac{s\beta_1\sin\theta}{1-\beta_2\cos\theta}\right)\right]^2+\left[\frac{c\cos\theta-v_2}{\sin\theta}\right]^2J_s^2\left(\frac{s\beta_1\sin\theta}{1-\beta_2\cos\theta}\right)\right\},$$

where $\beta_1 = v_1/c$.

If, in (14), we set $v_2 = 0$, which corresponds to motion of the electron in a circle, we get the analog of the Schott formula for a gyrotropic medium:

$$W_{js}=\frac{Te^2\Omega_0^2s^2n_jv_1^2}{4\pi c^3}\left[J_s'(sn_j\beta_1\sin\theta)+\frac{\alpha_j}{\sin\theta\beta_1 n_j}J_s(sn_j\beta_1\sin\theta)\right]^2d\Omega. \quad (15)$$

For the energy radiated in vacuum ($n_j = 1$, $\alpha_j = \pm\cos\theta$), expression (15) gives the Schott formula (cf., for example, Ref. 10, p. 216):

$$W_s=\sum_{j=1}^{2}W_{js}=\frac{Te^2\Omega_0^2s^2}{2\pi c^3}\left\{\cot^2\theta J_s^2(s\beta_1\sin\theta)+\beta_1^2J_s'^2\right\}d\Omega.$$

5. Let us now examine the character of the radiation corresponding to high harmonics

$$s=(\omega_s/\Omega_0)(1-\beta_2 n_j\cos\theta)\gg 1$$

(from now on we omit the subscript s on the frequency ω_s).

Using (11) and (14), we easily get the spectral density of the radiation, which is given by

$$W_{j\omega}d\omega=\frac{\pi T}{2\Omega_0}\left|G_s\left(\frac{\omega}{\Omega_0}\beta_1 n_j\sin\theta\right)\right|^2\rho_{j\omega}d\omega d\Omega.$$

If there are angles θ, within the particular frequency interval, for which $\beta_j n_j \sin\theta$ is of order one, the Bessel functions in the preceding expression can be transformed to Airy functions (cf., for example, Ref. 10, p. 217). We then find that

$$W_{j\omega}d\omega=\frac{Te^2\omega^2n_jd\omega d\Omega}{(2c)^3\pi^2\Omega_0}\left(\frac{2\Omega_0}{\omega\beta_1 n_j\sin\theta}\right)^{4/3}\left\{-\Phi'(x)+\left[\frac{v_1\alpha(1-\beta_2 n_j\cos\theta)}{\beta_1 n_j\sin\theta}+\beta v_2\right]\left(\frac{\omega\beta_1 n_j\sin\theta}{2\Omega_0}\right)^{1/3}\Phi(x)\right\}^2, \quad (16)$$

where the Airy function is

$$\Phi(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \cos\left(x\xi + \frac{\xi^3}{3}\right) d\xi; \qquad x = (2\omega^2 / \beta_1 n_j \Omega_0^2 \sin\theta)^{1/3} (1 - \beta_1 n_j \sin\theta - \beta_2 n_j \cos\theta).$$

For motion of the electron in a circle $(v_2 = 0)$, we obtain from (16):

$$W_{j\omega} d\omega = \frac{Te^2 v_1^2 \omega^2 n_j d\Omega d\omega}{(2c)^3 \pi^2 \Omega_0} \left(\frac{2\Omega_0}{\omega\beta_1 n_j \sin\theta}\right)^{4/3} \left\{-\Phi'(x) + \frac{\alpha_j}{\beta_1 n_j \sin\theta} \left(\frac{\omega\beta_1 n_j \sin\theta}{2\Omega_0}\right)^{1/3} \Phi(x)\right\}^2; \quad x = (2\omega^2 / \beta_1 n \Omega_0^2 \sin\theta)^{1/3} (1 - \beta_1 n \sin\theta). \tag{17}$$

Let us consider our result in a little more detail. To do this we give the asymptotic expression for the Airy function, whose behavior for positive arguments is essentially different from its behavior for negative arguments. Thus, if $x \gg 1$,

$$\Phi(x) = x^{-1/4} \exp\{-2x^{3/2}/3\},$$

whereas for $x < 0$; $|x| \gg 1$:

$$\Phi(x) = |x|^{-1/4} \sin\left(\frac{2}{3}|x|^{3/2} + \frac{\pi}{4}\right).$$

From this it follows that the character of the angular distribution of the energy radiated at frequency ω depends mainly on the nature of the roots of the equation

$$1 - \beta_1 n(\omega, \theta) \sin\theta = 0. \tag{18}$$

For example, for an isotropic medium ($n(\theta)$ = const > 1), Eq. (18) has two roots:

$$\sin\theta_1 = 1/\beta_1 n, \qquad \sin\theta_2 = \sin(\pi - \theta_1) = 1/\beta_2 n.$$

In this case the main part of the radiation is at angles θ satisfying the inequalities

$$\theta_1 \leqslant \theta \leqslant \theta_2, \tag{19}$$

and for these values of θ, the directivity pattern is multi-lobed, with the main lobes directed at angles close to θ_1 and θ_2. For values of θ outside the interval defined by (19), the radiation intensity drops exponentially as we move away from the angles θ_1 and θ_2.

For a gyrotropic plasma, $n(\theta)$ is given by the quite complicated expression (5), so that it is convenient to solve (18) graphically by finding the points of intersection of the functions $n(\theta)$ and $1/\beta_1 \sin\theta$. The following two cases are possible for the angular distribution of the energy:

(1) if

$$dn(\theta_1)/d\theta > -\cos\theta_1 / \beta_1 \sin^2\theta_1,$$

then for

$$\theta_1 \leqslant \theta \leqslant \pi - \theta_1 \tag{20}$$

$x < 0$, while for

$$\theta_1 > \theta, \qquad \theta > \pi - \theta_1, \tag{21}$$

$x > 0$, i.e. the radiation is mainly contained within the cone defined by the inequalities (19);

(2) if

$$dn(\theta_1)/d\theta < -\cos\theta_1 / \beta_1 \sin^2\theta_1,$$

then the radiation is concentrated at the angles corresponding to the inequalities (21).

Figure 2 shows the graph of the function

$$W_{1\omega}(\theta) \approx n_1 \left(\frac{2\Omega_0}{\omega\beta_1 n_1 \sin\theta}\right)^{4/3} \left\{-\Phi'(x) + \frac{\alpha_j}{\beta_1 n_j \sin\theta} \left(\frac{\omega\beta_1 n_1 \sin\theta}{2\Omega_0}\right)^{1/3} \Phi(x)\right\}^2 = W(\theta),$$

defined by (17), for the following values of the parameters:

$$u = 0.25; \quad V = 0.8; \quad \beta_1 = 0.4; \quad \omega/\Omega_0 = 20.$$

As we see from the graph, the second case occurs here.

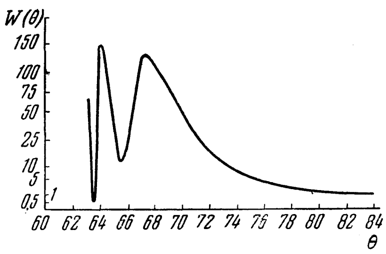

FIG. 2

We note that expression (17) for the energy W_j diverges for $n_j \to \infty$, since α_j tends to infinity like n_j^2 (in our example this occurs at $\theta = 60°$); whereas in an isotropic medium, the amount of energy radiated, W_ω, remains finite even for $n \to \infty$ [cf. the remark concerning divergences which we made with respect to forumlas (12) and (13)].

We now give the approximate value of (17) for the case where the given frequency interval contains angles θ for which

$$1 > \beta_1 n_j \sin\theta \approx 1.$$

It is easy to see that then $x > 0$ and $\Phi(x)$ has a sharp maximum for angles satisfying the relation

$$\beta_1 n_j \sin\theta \approx 1. \tag{22}$$

If we make use of the approximate equality (22), we get the following expression for the energy radiated during one revolution of the electron:

$$W_{\omega j} d\omega \approx (2e^2 v_1^2 / \pi c^3)(\omega / 2\Omega_0)^{2/3} n_j d\omega d\Omega \{-\Phi'(x) + \alpha_j (\omega / 2\Omega_0)^{1/3} \Phi(x)\}^2,$$

$$x = (2\omega^2 / \Omega_0^2)^{1/3} (1 - \beta_1 n_j \sin\theta).$$

6. In conclusion, let us consider the case where

$$n_j \omega_s r_0 \sin\theta / c \ll 1. \qquad (23)$$

It is easy to see that in this case the greatest in-intensity occurs for the zeroth and first harmonics. For the zeroth harmonic ($s = 0$), the expression for the radiated energy, when we satisfy the inequality (23), is identical with the Cerenkov term (13). In this connection we note that the picture of the radiation of an electron, moving along a vortex line in a medium with $n > 1$, as the radiation from a pair of perpendicular oscillating dipoles is incorrect (since it does not give the Cerenkov effect).

We now determine the energy corresponding to the first harmonic ($s = 1$). To do this, we substitute in (14) the first terms of the expansion of the Bessel function and its derivative. We then find that

$$W_{1j} = \frac{Te^2 \omega_1^2 n_j d\Omega \, [v_1 (1 + \alpha_j) + \beta \omega_1 n_j r_0 \beta_2 \sin\theta]^2}{16\pi c^3 [1 - \beta_2 \cos\theta (n_j + \omega_1 \partial n_j(\omega_1) / \partial\omega)]}. \qquad (24)$$

From this formula we see that only for motion of the electron in an isotropic non-dispersive dielectric ($n(\theta, \omega) = \text{const}$, $\alpha_j = \pm\cos\theta$; $\beta_j = \pm\sin\theta$) do we get the familiar expression for the energy radiated by a pair of mutually perpendicular dipoles oscillating with a phase shift of $\pi/2$ (cf., for example, Refs. 11, 12):

$$W_1 = \sum_{j=1}^{2} W_{1j} = \frac{Te^2 p_0^2 n d\Omega}{8\pi c^3 |1 - \beta_2 n \cos\theta|^5} \{(1 - \beta_2 n \cos\theta)^2 + (\cos\theta - \beta_2 n)^2\},$$

where $p_0^2 = e^2 r_0^2$ is the dipole moment.

If the electron moves in a circle, (24) becomes the following simple expression:

$$W_{1j} = (Te^2 \Omega_0^2 \beta_1^2 n_j d\Omega / 16\pi c)(1 + \alpha_j)^2. \qquad (25)$$

From (25) it is easy to determine the ratio of the energy radiated in the extraordinary wave to the energy of the ordinary wave:

$$W_{11} / W_{12} = n_1 (1 + \alpha_1)^2 / n_2 (1 + \alpha_2)^2.$$

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